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ON KANNAN FIXED POINT PRINCIPLE IN GENERALIZED METRIC SPACES

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ABSTRACT. The concept of a generalized metric space, where the triangle inequality has been replaced by a more general one involving four points, has been recently introduced by Branciari. Subsequently, some classical metric fixed point theorems have been transferred to such a space. The aim of this note is to show that Kannan's fixed point theorem in a generalized metric space is a consequence of the Banach contraction principle in a metric space.

1. INTRODUCTION AND PRELIMINARIES

The following notion of generalized metric space has been introduced by Branciari in [3]:

Definition 1.1. ([3]) Let X be a set and $d : X^2 \longrightarrow \mathbb{R}$ be a mapping. The pair (X, d) is called a *generalized metric space* (in the sense of Branciari) if, for all $x, y \in X$ and for all distinct points$z, w \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq d(z, z) + d(z, w) + d(w, y)$.

Any metric space is a generalized metric space, but the converse is not true ([3]). A generalized metric space is a topological space with neighborhood basis given by

$$\mathcal{B} = \{B(x, r),\ x \in X,\ r > 0\}$$

where $B(x, r) = \{y \in X, d(x, y) < r\}$.

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Let(X,d) be a generalized metric space. A sequence$\{x_n\}$ in X is said to be *Cauchy* if for any $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}, n \geq n_\epsilon$ one has $d(x_n, x_{n+m}) < \epsilon$. The space (X,d) is called *complete* if every Cauchy sequence in X is convergent in X. Let $T : X \to X$ be a mapping. The space (X,d) is said to be T*-orbitally complete* if every Cauchy sequence which is contained in $O(x,\infty) := \{T^n x, n \in \mathbb{N} \cup \{0\}\}$ for some $x \in X$, converges in X.

Starting with the paper of Branciari [3], some classical metric fixed point theorems have been transferred to generalized metric spaces, see e.g., [2], [1], [6], [5], [7]. Following an idea in [9], in this short note we show that Kannan's fixed point theorem [8] in such a space is a consequence of the following Banach contraction principle in a metric space:

Theorem 1.2. ([4]) *Let (X,ρ) be a metric space and $T : X \to X$ be a mapping such that*

$$\rho(Tx,Ty) \leq q\rho(x,y) \forall x,y \in X$$

where $0 \leq q < 1$. If X is T-orbitally complete then T has a unique fixed point in X.

2. Main results

We begin by recalling the fixed point theorem of Kannan in a generalized metric space, as stated in [5].

Theorem 2.1. *(Kannan fixed point principle in a generalized metric space) Let (X,d) be a generalized metric space and $T : X \to X$ be a mapping such that*

$$(K)\ d(Tx,Ty) \leq \beta[d(x,Tx) + d(y,Ty)] \quad (x,y \in X)$$

where $0 < \beta < \frac{1}{2}$. If X is T-orbitally complete then T has a unique fixed point in X.

We note that the fact that T has at most one fixed point easily follows from (K). In the following we show that the existence of a fixed point for a Kannan contraction in a orbitally complete generalized metric space is actually a consequence of Theorem 1.4.

In our proof we use the following lemma, which can immediately be proved by induction on n, without involving the triangle inequality:

Lemma 2.2. *If (X,d) is a generalized metric space and $T : X \to X$ is a mapping such that, for some $0 < \beta < \frac{1}{2}$,*

$$d(Tx,Ty) \leq \beta[d(x,Tx) + d(y,Ty)]\ \forall x,y \in X$$

then

$$d(T^n x, T^{n+1}x) \leq (\frac{\beta}{1-\beta})^n d(x,Tx) \quad (n \in \mathbb{N})$$

for every $x \in X$.

Proof. From

$$d(Tx,T^2x) \leq \beta[d(x,Tx) + d(Tx,T^2x)]$$

it follows that

$$d(Tx, T^2x) \leq \frac{\beta}{1-\beta} d(Tx, T^2x).$$

Next, from (K), $d(T^{n+1}x, T^{n+2}x) \leq \beta d(T^n x, T^{n+1}x) + \beta(T^{n+1}x, T^{n+2}x)$ so, from

$$(T^n x, T^{n+1}x) \leq (\frac{\beta}{1-\beta})^n d(x, Tx)$$

we obtain

$$d(T^{n+1}x, T^{n+2}x) \leq \frac{\beta}{1-\beta} d(T^n x, T^{n+1}x) \leq (\frac{\beta}{1-\beta})^{n+1} d(x, Tx).$$

□

Let us now suppose, with the aim to reach to a contradiction, that T has no fixed point.

We note that if $m, n, m \neq n$ are two positive integer numbers, then $T^m x \neq T^n x \ \forall x \in X$, for if $T^m x = T^n x$ for some $x \in X$ then $y = T^n x$ is a fixed point for T. Indeed, from $T^m x = T^n x$ it follows $T^{m-n}(T^n x) = T^n x$, i.e. $T^k y = y$, where $k = m - n \geq 1$ and therefore

$$d(y, Ty) = d(T^k y, T^{k+1} y) \leq (\frac{\beta}{1-\beta})^k d(y, Ty).$$

Since $0 < \frac{\beta}{1-\beta} < 1$, we obtain that $d(y, Ty) = 0$, that is, $y = Ty$.

Define

$$\rho(x, y) = \begin{cases} d(x, Tx) + d(y, Ty), & x \neq y; \\ 0, & x = y. \end{cases}$$

Since

$$\begin{aligned} \rho(x, y) &= d(x, Tx) + d(y, Ty) \\ &\leq d(x, Tx) + 2d(z, Tz) + d(y, Ty) = \rho(x, z) + \rho(z, y), \end{aligned}$$

for all $x, y \in X, x \neq y$, ρ is a metric on X.

Also,

$$\begin{aligned} \rho(Tx, Ty) &= d(Tx, T^2x) + d(Ty, T^2y) \\ &\leq \beta[d(x, Tx) + d(Tx, T^2x)] + \beta[d(y, Ty) + d(Ty, T^2y)] \\ &= \beta[d(Tx, T^2x) + d(Ty + T^2y)] = \beta\rho(x, y) + \beta\rho(Tx, Ty), \end{aligned}$$

that is,

$$\rho(Tx, Ty) \leq q\rho(x, y) \ \forall x, y \in X,$$

where $q = \frac{\beta}{1-\beta} \in (0, 1)$.

We show that

$$d(T^n x, T^m x) \leq 2\rho(T^n x, T^m x) \quad (m \geq n).$$

This inequality is obvious if $m = n$. It is also immediate if $m = n + 1$, because

$$d(T^n x, T^{n+1}x) \leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) = \rho(T^n x, T^{n+1}x).$$

If $m > n + 1$, then

$$\begin{aligned} d(T^n x, T^m x) &\leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{m+1}x) + d(T^m x, T^{m+1}x) \\ &= [d(T^n x, T^{n+1}x) + d(T^m x, T^{m+1}x)] + d(T^{n+1}x, T^{m+1}x) \end{aligned}$$

$$\leq (1+\beta)\rho(T^n x, T^m x) \leq 2\rho(T^n x, T^m x)$$

(note that if $m > n+1$, then $T^m x, T^{m+1}x, T^n x, T^{n+1}x$ are four distinct points in X).

Next, we prove that (X, ρ) is T-orbitally complete. We know that there is $x \in X$ such that for every d-Cauchy sequence $\{x_n\}$ contained in $O(x, \infty)$ there exists $u \in X$ such that $d(x_n, u) \to 0$. Let $\{x_n\}$ be a ρ-Cauchy sequence contained in $O(x, \infty)$. From the just proven inequality it follows that $\{x_n\}$ is also d-Cauchy, so $d(u, x_n) \to 0$ for some $u \in X$. We may assume that $x_n \neq u$ for some n, for otherwise $\rho(x_n)$ converges to u and we have nothing to prove. Then u, x_n, Tu, Tx_n are four distinct points of X. For otherwise, $T^k x = Tu$ or $T^k x = Tu$ for some $k \in \mathbb{N}$, which would imply $lim_{n\to\infty} T^n u = u$, and so, by letting $n \to \infty$ in $d(T^{n+1}u, Tu) \leq \beta[d(T^n u, T^{n+1}u) + d(u, Tu)] \quad (n \in \mathbb{N})$, we would obtain

$$d(u, Tu) \leq \beta d(u, Tu).$$

Since $\beta < 1$, $d(u, Tu)$ must be 0, that is, $u = Tu$, contradicting the fact that T is a fixed point free mapping.

Now, since $x_n \neq x_{n'}$ for some $n^{'} > n$, we have

$$\begin{aligned} \rho(u, x_n) &= d(u, Tu) + d(x_n, Tx_n) \\ &\leq [d(u, x_n) + d(x_n, Tx_n) + d(Tx_n, Tu)] + d(x_n, Tx_n)] \\ &\leq d(u, x_n) + 2d(x_n, Tx_n) + 2d(x_{n'}, Tx_{n'}) + \beta\rho(x_n, u) \\ &= d(u, x_n) + 2\rho(x_n, x_{n'}) + \beta\rho(x_n, u). \end{aligned}$$

It follows that

$$(1-\beta)\rho(u, x_n) \leq d(u, x_n) + 2\rho(x_n, x_{n'}),$$

that is, $\rho(u, x_n) \to 0$.

Thus, (X, ρ) is T-orbitally complete. From Theorem 1.4 it follows that T has a fixed point, contradicting our assumption. This completes the proof.

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