

SOLVABILITY OF NONLINEAR HAMMERSTEIN QUADRATIC INTEGRAL EQUATIONS

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This paper is dedicated to Professor Hemant Kumar Nashine

ABSTRACT. We are concerning with a nonlinear Hammerstein quadratic integral equation. We prove the existence of at least one positive solution $x \in L_1$ under Carathèodory condition. Secondly we will make a link between Peano condition and Carathèodory condition to prove the existence of at least one positive continuous solution. Finally the existence of the maximal and minimal solutions will be proved.

1. INTRODUCTION AND PRELIMINARIES

Quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. The quadratic integral equation can be very often encountered in many applications (see[1]-[4] and [8]-[11]).

Let $I = [0, 1]$, $L_1 = L_1[0, 1]$ be the space of Lebesgue integrable functions on I and $C = C[0, 1]$ be the space of continuous functions defined on I .

Recently, the existence of a solution $x \in L_1$ for the nonlinear quadratic integral equation

$$x(t) = a(t) + g(t, x(t)) \int_0^t k(t, s) f(s, x(s)) ds, \quad t \in [0, 1]$$

was studied in [9] by using Lusin and Dragoni theorems and applying Schauder-Tychonoff fixed point Theorem.

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Here we are concerning with the nonlinear Hammerstein quadratic integral equation

$$x(t) = a(t) + g(t, x(t)) \int_0^1 k(t, s) f(s, x(s)) ds \quad t \in [0, 1] \quad (1.1)$$

by using the same assumptions assumed in [9]. Firstly, the existence of at least one L_1 -positive solution of the nonlinear quadratic integral equation (1.1) will be proved where the functions f and g satisfy Carathèodory condition.

Secondly, the existence of at least one positive continuous solution for the quadratic integral equation (1.1) will be proved where g is continuous and f satisfies Carathèodory condition.

The following theorems will be needed in our investigations (see [6],[7]and [14]).

Theorem 1.1. Tychonov's fixed-point Theorem. *Suppose B is a complete, locally convex linear space and S is a closed convex subset of B . Let the mapping $T : B \rightarrow B$ be continuous and $T(S) \subset S$. If the closure of $T(S)$ is compact, then T has a fixed-point in S .*

Theorem 1.2. Schauder fixed-point Theorem. *Let S be a convex subset of a Banach space B , let the mapping $T : S \rightarrow S$ be compact, continuous. Then T has at least one fixed-point in S .*

Theorem 1.3. Arzela-Ascoli Theorem. *Let E be a compact metric space and $C(E)$ the Banach space of real or complex valued continuous functions normed by*

$$\| f(t) \| = \max_{t \in E} | f(t) |.$$

If $A = \{f_n\}$ is a sequence in $C(E)$ such that f_n is uniformly bounded and equi-continuous. Then the closure of A is compact.

Theorem 1.4. Lusin Theorem. *Let $m : [0, 1] \rightarrow R$ be a measurable function. For any $\epsilon > 0$ there exists a closed subset A_ϵ of $[0, 1]$, $meas.(A_\epsilon^c) < \epsilon$, such that m restricted to A_ϵ is continuous.*

Theorem 1.5. Scorza Dragoni Theorem. *Let $k : [0, 1] \times [0, 1] \rightarrow R$ be a function satisfying Carathèodory condition (i.e. measurable in t for all $s \in [0, 1]$ and continuous in s for all $t \in [0, 1]$). For any $\epsilon > 0$ there exists a closed subset A_ϵ of $[0, 1]$, $meas.(A_\epsilon^c) < \epsilon$, such that k restricted to $A_\epsilon \times [0, 1]$ is continuous.*

2. L_1 -POSITIVE SOLUTION

Let $I = [0, 1]$, and consider the assumptions:

- (i) $a : I \rightarrow R_+ = [0, +\infty)$ is integrable on I ;
- (ii) $f, g : I \times R_+ \rightarrow R_+$ satisfy Carathèodory condition (i.e. measurable in t for all $x \in R_+$ and continuous in x for all $t \in [0, 1]$) and there exist two functions $m_1, m_2 \in L_1$ such that

$$g(t, x) \leq m_1(t), \quad f(t, x) \leq m_2(t) \quad \forall (t, x) \in I \times R_+;$$

- (iii) $k : [0, 1] \times [0, 1] \rightarrow R$ satisfies Carathèodory condition (i.e. measurable in t for all $s \in [0, 1]$ and continuous in s for all $t \in [0, 1]$).

For the existence of at least one L_1 -positive solution of the nonlinear quadratic integral equation (1.1) we have the following theorem.

Theorem 2.1. *Let the assumptions (i)-(iii) be satisfied. Then the nonlinear quadratic integral equation (1.1) has at least one L_1 -positive solution x .*

Proof. Consider the set $Q \subset L_1$ such that

$$Q = \{ x \in L_1, |x(t)| \leq x_0(t) \text{ a.e. } \}$$

where

$$x_0(t) = a(t) + m_1(t) \int_0^1 k(t, s) m_2(s) ds. \quad (2.1)$$

The set Q can be shown to be nonempty, bounded, closed and convex in L_1 . Let H be the operator defined by

$$(Hx)(t) = a(t) + g(t, x(t)) \int_0^1 k(t, s) f(s, x(s)) ds, \quad t \in I. \quad (2.2)$$

We shall prove that $H : Q \rightarrow Q$. For that let $x \in Q$, then

$$|(Hx)(t)| \leq |a(t)| + m_1(t) \int_0^1 k(t, s) m_2(s) ds = x_0(t),$$

so $Hx \in Q$ and hence $HQ \subset Q$.

To apply Schauder fixed-point Theorem, we shall prove that HQ is relatively compact in L_1 .

By using Lusin and Scorza Dragoni Theorems, we can find a closed subset A_n of $[0, 1]$, with $meas.(A_n^c) < \frac{1}{n}$ such that $a(t)$, $m_1(t)$, $k|_{A_n \times [0, 1]}$ and $g|_{A_n \times Q}$ are uniformly continuous on A_n .

Assume that x_h is any sequence in Q , then for $t_1, t_2 \in A_n$, we have

$$\begin{aligned} (Hx_h)(t_2) - (Hx_h)(t_1) &= a(t_2) - a(t_1) \\ &+ g(t_2, x_h(t_2)) \int_0^1 k(t_2, s) f(s, x_h(s)) ds - g(t_1, x_h(t_1)) \int_0^1 k(t_1, s) f(s, x_h(s)) ds \\ &= a(t_2) - a(t_1) + g(t_2, x_h(t_2)) \int_0^1 k(t_2, s) f(s, x_h(s)) ds \\ &- g(t_1, x_h(t_1)) \int_0^1 k(t_1, s) f(s, x_h(s)) ds + g(t_1, x_h(t_1)) \int_0^1 k(t_2, s) f(s, x_h(s)) ds \\ &\quad - g(t_1, x_h(t_1)) \int_0^1 k(t_2, s) f(s, x_h(s)) ds \\ &\leq a(t_2) - a(t_1) + [g(t_2, x_h(t_2)) - g(t_1, x_h(t_1))] \int_0^1 k(t_2, s) f(s, x_h(s)) ds \\ &\quad + g(t_1, x_h(t_1)) \int_0^1 \{k(t_2, s) - k(t_1, s)\} f(s, x_h(s)) ds \end{aligned}$$

Then we get

$$\begin{aligned} |(Hx_h)(t_2) - (Hx_h)(t_1)| &\leq |a(t_2) - a(t_1)| + m_1(t_1) \int_0^1 |k(t_2, s) - k(t_1, s)| m_2(s) ds \\ &\quad + |g(t_2, x_h(t_2)) - g(t_1, x_h(t_1))| \int_0^1 k(t_2, s) m_2(s) ds \end{aligned}$$

This means that the sequence $\{Hx_h\}$ is sequence of equi-continuous functions on A_n and we can prove that this sequence is uniformly bounded.

Now

$$\begin{aligned} |(Hx_h)(t)| &= |a(t)| + m_1(t) \int_0^1 k(t, s) m_2(s) ds \\ &\leq M_1 + M_2 K \int_0^1 m_2(s) ds \end{aligned}$$

where $|a(t)|_{A_n} \leq M_1$, $|m_1(t)|_{A_n} \leq M_2$ and $k|_{A_n \times [0,1]} \leq K$.

Hence by Arzela-Ascoli Theorem Hx_h is relatively compact subset of $C(A_n)$ and this can be done for each $n \in N$. this implies the existence of convergent subsequence $\{x_{h_j}\}$ of $\{x_h\}$ in each $C(A_n)$. Given $\epsilon > 0$ and choose $n_1 \in N$ so that $meas(A_{n_1}) < \epsilon$, then

$$\begin{aligned} \int_0^T |Hx_{h_j} - Hx_{h_l}| dt &= \int_{A_{n_1}^c} |Hx_{h_j} - Hx_{h_l}| dt \\ &\quad + \int_{A_{n_1}} |Hx_{h_j} - Hx_{h_l}| dt. \end{aligned}$$

Since $C(A_n)$ is complete metric space, hence this subsequence is a Cauchy sequence in each $C(A_n)$, $n = 1, 2, 3, \dots$

That is for given $\epsilon > 0$ and j, l are arbitrary large we have

$$\|Hx_{h_j} - Hx_{h_l}\|_{C(A_n)} < \epsilon. \tag{2.3}$$

But we want to prove that the set HQ is relatively compact in L_1 , that is \overline{HQ} is compact in L_1 .

To do this, we will prove that the sequence $\{Hx_h\}$ is convergent in L_1 , since L_1 is complete metric space, then it is sufficient to prove that the subsequence $\{Hx_{h_j}\}$ is a Cauchy sequence in L_1 .

i.e. $\forall \eta > 0, \exists N(\eta)$ and $\int_{A_n} x_0(t) dt < \eta/4$ such that

$$\|Hx_{h_j} - Hx_{h_l}\|_{L_1} < \eta, j, l > N(\eta).$$

Now from (2.1) and (2.2) we have

$$\begin{aligned} \int_0^T |Hx_{h_j} - Hx_{h_l}| dt &= \int_{A_n^c} |Hx_{h_j} - Hx_{h_l}| dt \\ &\quad + \int_{A_n} |Hx_{h_j} - Hx_{h_l}| dt \\ &\leq \int_{A_n^c} \{|Hx_{h_j}| + |Hx_{h_l}|\} dt + \|Hx_{h_j} - Hx_{h_l}\|_{C(A_n)} \end{aligned}$$

$$\leq \eta/4 + \eta/4 + \|Hx_{h_j} - Hx_{h_l}\|_{C(A_n)}.$$

Choose N such that $l, j > N$, then (2.3) implies that $\|Hx_{h_j} - Hx_{h_l}\|_{C(A_n)} \leq \eta/2$. This means that the subsequence $\{Hx_{h_j}\}$ is a Cauchy sequence in L_1 which implies that HQ is relatively compact in L_1 . Then H has at least one fixed point. Hence there exists at least one solution $x \in L_1$ of (1.1). Since all conditions of Schauder's fixed-point Theorem hold, then H has a fixed point in Q . \square

3. CONTINUOUS SOLUTIONS

Let $I = [0, 1]$, and consider the assumptions:

- (i) $a : I \rightarrow R_+ = [0, +\infty)$ is continuous on I ;
- (ii) $f : I \times R_+ \rightarrow R_+$ satisfies Carathéodory condition (i.e. measurable in t for all $x \in R_+$ and continuous in x for all $t \in [0, 1]$) and there exists function $m \in L_1$ such that

$$f(t, x(t)) \leq m(t) \quad \forall (t, x) \in I \times R_+;$$

- (iii) $g : I \times R_+ \rightarrow R_+$ is continuous in t, x and $|g(t, x)| \leq M$;
- (iv) $k : [0, 1] \times [0, 1] \rightarrow R$ satisfies Carathéodory condition (i.e. measurable in t for all $s \in [0, 1]$ and continuous in s for all $t \in [0, 1]$).

Now for the existence of at least one positive continuous solution of the nonlinear quadratic integral equation (1.1) we have the following theorem.

Theorem 3.1. *Let the assumptions (i)-(iv) be satisfied. Then the nonlinear quadratic integral equation (1.1) has at least one positive solution $x \in C(I)$.*

Proof. We shall use Tychonov's fixed point Theorem to prove this theorem. It can be verified that [7] C is complete locally convex linear space. Define a subset S of C by

$$S = \{x \in C : |x(t)| \leq M_2\}, \quad t \in [0, 1],$$

where M_2 is a positive constant. It is clear that the set S is closed and convex. Let H be an operator defined by

$$(Hx)(t) = a(t) + g(t, x(t)) \int_0^1 k(t, s) f(s, x(s)) ds, \quad \forall x \in S.$$

Assumptions (ii) and (iii) imply that $H : S \rightarrow C$ is continuous operator in x . We shall prove that $HS \subset S$.

For every $x \in S$ we have

$$\begin{aligned} |(Hx)(t)| &\leq |a(t)| + M \int_0^1 k(t, s) m(s) ds \quad t \in [0, 1] \\ &\leq M_1 + M \int_0^1 k(t, s) m(s) ds = M_2, \end{aligned}$$

where $|a(t)| \leq M_1$. Then, $Hx \in S$ and hence $HS \subset S$. Also for t_1 and $t_2 \in [0, 1]$ we can have

$$\begin{aligned} (Hx)(t_2) - (Hx)(t_1) &= a(t_2) - a(t_1) + g(t_2, x(t_2)) \int_0^1 k(t_2, s) f(s, x(s)) ds \\ &\quad - g(t_1, x(t_1)) \int_0^1 k(t_1, s) f(s, x(s)) ds + g(t_1, x(t_1)) \int_0^1 k(t_2, s) f(s, x(s)) ds \\ &\quad - g(t_1, x(t_1)) \int_0^1 k(t_2, s) f(s, x(s)) ds \\ &= a(t_2) - a(t_1) + [g(t_2, x(t_2)) - g(t_1, x(t_1))] \int_0^1 k(t_2, s) f(s, x(s)) ds \\ &\quad + g(t_1, x(t_1)) \int_0^1 [k(t_2, s) - k(t_1, s)] f(s, x(s)) ds \end{aligned}$$

Using assumptions (ii) (iii) then, we have

$$\begin{aligned} |(Hx)(t_2) - (Hx)(t_1)| &\leq |a(t_2) - a(t_1)| + M \int_0^1 |k(t_2, s) - k(t_1, s)| m(s) ds \\ &\quad + |g(t_2, x(t_2)) - g(t_1, x(t_1))| \int_0^1 k(t_2, s) m(s) ds \end{aligned}$$

This means that the functions of HS are equi-continuous on $[0, 1]$, then by Arzela-Ascoli Theorem the closure of HS is compact.

Hence, all conditions of Tychonov fixed-point Theorem hold, then H has a fixed point in S . \square

4. MAXIMAL AND MINIMAL SOLUTIONS

Definition 4.1. [13] Let $q(t)$ be a solution of the nonlinear Hammerstein quadratic integral equation (1.1). Then $q(t)$ is said to be a maximal solution of (1.1) if every solution $x(t)$ of (1.1) satisfies the inequality $x(t) < q(t)$. A minimal solution $s(t)$ can be defined by similar way by reversing the above inequality i.e. $x(t) > s(t)$.

We shall use the following lemma to prove the existence of the maximal and minimal solutions.

Lemma 4.2. Let $a(t)$ is continuous function on I and $k(t, s)$ satisfies the assumption (iv) of Theorem 3.1. Let $f(t, x), g(t, x) \in L_1$ and $x(t), y(t)$ are continuous functions on $[0, 1]$ satisfying

$$\begin{aligned} x(t) &\leq a(t) + g(t, x(t)) \int_0^1 k(t, s) f(s, x(s)) ds, \\ y(t) &\geq a(t) + g(t, y(t)) \int_0^1 k(t, s) f(s, y(s)) ds \end{aligned}$$

and one of them is strict. If $f(t, x), g(t, x)$ are monotonic nondecreasing in x , then

$$x(t) < y(t), t > 0. \quad (4.1)$$

Proof. Let the conclusion (4.1) be false, then there exists t_1 such that

$$x(t_1) = y(t_1), \quad t_1 > 0$$

and

$$x(t) < y(t), \quad 0 < t < t_1.$$

From the monotonicity of $f(t, x)$, $g(t, x)$ in x , we get

$$\begin{aligned} x(t_1) &\leq a(t_1) + g(t_1, x(t_1)) \int_0^1 k(t_1, s) f(s, x(s)) ds, \\ &< a(t_1) + g(t_1, y(t_1)) \int_0^1 k(t_1, s) f(s, y(s)) ds \\ &< y(t_1), \end{aligned}$$

which contradicts the fact that $x(t_1) = y(t_1)$, then $x(t) < y(t)$. \square

Theorem 4.3. *Let the assumptions of Theorem 3.1 be satisfied and if $f(t, x)$, $g(t, x)$ are nondecreasing in x on I . Then there exist maximal and minimal solutions of the nonlinear quadratic integral equation (1.1).*

Proof. Firstly we shall prove the existence of the maximal solution of (1.1). Let $\epsilon > 0$ be given. Now consider the quadratic integral equation

$$x_\epsilon(t) = a(t) + g_\epsilon(t, x_\epsilon(t)) \int_0^1 k(t, s) f_\epsilon(s, x_\epsilon(s)) ds \quad t \in [0, 1], \quad (4.2)$$

where

$$\begin{aligned} f_\epsilon(t, x_\epsilon(t)) &= f(t, x_\epsilon(t)) + \epsilon, \\ g_\epsilon(t, x_\epsilon(t)) &= g(t, x_\epsilon(t)) + \epsilon, \end{aligned}$$

Clearly the functions $f_\epsilon(t, x_\epsilon)$ and $g_\epsilon(t, x_\epsilon)$ satisfy assumptions (ii), (iii) of theorem 3.1 and therefore equation (4.2) at least a positive solution $x_\epsilon(t) \in C(I)$.

Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then

$$\begin{aligned} x_{\epsilon_2}(t) &= a(t) + g_{\epsilon_2}(t, x_{\epsilon_2}(t)) \int_0^1 k(t, s) f_{\epsilon_2}(s, x_{\epsilon_2}(s)) ds \\ x_{\epsilon_2}(t) &= a(t) + (g(t, x_{\epsilon_2}(t)) + \epsilon_2) \int_0^1 k(t, s) (f(s, x_{\epsilon_2}(s)) + \epsilon_2) ds \quad (4.3) \end{aligned}$$

$$\begin{aligned} x_{\epsilon_1}(t) &= a(t) + (g(t, x_{\epsilon_1}(t)) + \epsilon_1) \int_0^1 k(t, s) (f(s, x_{\epsilon_1}(s)) + \epsilon_1) ds \\ x_{\epsilon_1}(t) &> a(t) + (g(t, x_{\epsilon_1}(t)) + \epsilon_2) \int_0^1 k(t, s) (f(s, x_{\epsilon_1}(s)) + \epsilon_2) ds. \quad (4.4) \end{aligned}$$

Applying Lemma 4.2 on (4.3) and (4.4), we have

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t) \text{ for } t \in I.$$

As shown before the family of functions $x_\epsilon(t)$ is equi-continuous and uniformly bounded.

Hence by Arzela-Ascoli Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon \rightarrow 0$ as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} x_{\epsilon_n}(t)$ exists uniformly in I and

denote this limit by $q(t)$. From the continuity of the functions $f_\epsilon(t, x_\epsilon)$ and $g_\epsilon(t, x_\epsilon)$ in the second argument, we get

$$q(t) = \lim_{n \rightarrow \infty} x_{\epsilon_n}(t) = a(t) + g(t, q(t)) \int_0^1 k(t, s) f(s, q(s)) ds$$

implies $q(t)$ as a solution of (1.1).

Finally, we shall show that $q(t)$ is the maximal solution of (1.1). To do this, let $x(t)$ be any solution of (1.1). Then

$$\begin{aligned} x_\epsilon(t) &= a(t) + g_\epsilon(t, x_\epsilon(t)) \int_0^1 k(t, s) f_\epsilon(s, x_\epsilon(s)) ds \\ &= a(t) + (g(t, x_\epsilon(t)) + \epsilon) \int_0^1 k(t, s) (f(s, x_\epsilon(s)) + \epsilon) ds \\ &> a(t) + g(t, x_\epsilon(t)) \int_0^1 k(t, s) f(s, x_\epsilon(s)) ds. \end{aligned}$$

Also

$$x(t) = a(t) + g(t, x(t)) \int_0^1 k(t, s) f(s, x(s)) ds$$

implies

$$x(t) < x_\epsilon(t) \text{ for } t \in I.$$

from the uniqueness of the maximal solution (see[13]), it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $t \in I$ as $\epsilon \rightarrow 0$.

By similar way as done above we can prove the existence of the minimal solution. \square

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