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CONTRACTIONS OVER GENERALIZED METRIC SPACES

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ABSTRACT. A generalized metric space (g.m.s) has been defined as a metric space in which the triangle inequality is replaced by the 'Quadrilateral inequality', $d(x, y) \leq d(x, a) + d(a, b) + d(b, y)$ for all pairwise distinct points x, y, a and b of X. (X, d) becomes a topological space when we define a subset A of X to be open if to each a in A there corresponds a positive number r_a such that $b \in A$ whenever $d(a, b) < r_a$. Cauchyness and convergence of sequences are defined exactly as in metric spaces and a g.m.s (X, d) is called complete if every Cauchy sequence in (X, d) converges to a point of X. A.Branciari [1] has published a paper purporting to generalize Banach's Contraction principle in metric spaces to g.m.s. In this paper we present a correct version and proof of the generalization.

1. MAIN RESULT

In what follows $\mathbb{N}$ denotes the set of natural numbers. The basic terms are already defined in the abstract. We denote $\{y \in X : d(x, y) < r\}$ for x in a g.m.s (X, d) by $B_r(x)$. In [1], the following were taken for granted and used:

(1) $\{B_r(x) : r > 0, x \in X\}$ is a basis for a topology on X
(2) d is continuous in each of the coordinates and
(3) a g.m.s is a Hausdorff space.

The following examples shows that (1), (2)and (3) are false.

Example 1.1. Let $A = \{0, 2\}$, $B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, $X = A \cup B$. Define d on $X \times X$ as follows: $d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ if $x \neq y$ and $\{x, y\} \subseteq A$ or $\{x, y\} \subseteq B$, $d(x, y) = d(y, x) = y$ if $x \in A$ and $y \in B$. Then (X, d) is a complete g.m.s in which

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(a) the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ converges to both 0 and 2 and it is not a Cauchy sequence,

(b) there does not exist $r > 0$ such that $B_r(0) \cap B_r(2) = \emptyset$,

(c) $B_{\frac{2}{3}}(\frac{1}{3}) = \{0, 2, \frac{1}{3}\}$ and there does not exist $r > 0$ such that $B_r(0) \subseteq B_{\frac{2}{3}}(\frac{1}{3})$,

(d) $\lim d(\frac{1}{n}, \frac{1}{2}) \neq d(0, \frac{1}{2})$.

Remark 1.2. Even Though the sets $B_r(x)$ do not form an open basis for a topology on a g.m.s X, the subsets A of X satisfying the following condition form a topology on X : To each a in A there corresponds $r_a > 0$ such that $B_{r_a}(a) \subseteq A$.

Theorem 1.3. *(Banach's Contraction principle in a g.m.s). Let (X, d) be a Hausdorff and complete g.m.s and let $f\colon X \to X$ be a mapping and $0 < \lambda < 1$ satisfying the inequality $d(fx, fy) \leq \lambda d(x, y)$ for all x, y in X(such a mapping is called a contraction mapping on X and λ is called the contractive constant of f). Then there is a unique point x in X satisfying $f(x) = x$ (such a point is called a fixed point of f).*

Proof. Let $x \in X$, $a_n = f^n(x)$ for $n \geq 0$ and $c = \inf S$ where $S = \{d(a_{n-1}, a_n) : n \in \mathbb{N}\}$. We claim that $c = 0$, If $c \neq 0$ then $c < \frac{c}{\lambda}$ and hence there is a positive integer n such $d(a_{n-1}, a_n) < \frac{c}{\lambda}$ so that $\lambda d(a_{n-1}, a_n) < c$. By Contractive property of f we have $d(f^n x, f^{n+1}x) < c$ a contradiction to the minimality of c. Hence $c = 0$. The monotonically decreasing property of the sequence $d(a_n, a_{n+1})$ implies that $d(a_n, a_{n+1})$ converges to 0 ....$(*)$.
We claim that f has a periodic point. Suppose, to obtain a contradiction, f has no periodic point. Then $\{a_n\}$ is a sequence of distinct points and for $m > n + 1$, we have

$$\begin{aligned} d(a_n, a_m) = d(f^n x, f^m x) &\leq d(f^n x, f^{n+1}x) + d(f^{n+1}x, f^{m+1}x) + d(f^{m+1}x, f^m x) \\ &\leq (\lambda^n + \lambda^m) d(x, fx) + \lambda d(f^n x, f^m x) \\ &\qquad (By\ \ Quadrilateral\ inequality) \end{aligned}$$

which implies $(1 - \lambda) d(a_n, a_m) \leq (\lambda^n + \lambda^m) d(x, fx)$ and hence $\{a_n\}$ is a Cauchy sequence in (X, d) (in view of $(*)$). By Completeness, $a_n \to a$ for some a in X. Also $d(fa_n, fa) \leq \lambda d(a_n, a)$ and $d(a_n, a) \to 0$. So $d(fa_n, fa) = d(a_{n+1}, fa) \to 0$. Hence $a_n \to a$ and $a_{n+1} \to fa$. Since (X, d) is Hausdorff it follows that $a = fa$, a contradiction to the assumption that f has no periodic point. Thus f has a periodic point say a of period n. Suppose if possible $n > 1$. Then $d(a, fa) = d(f^n a, f^{n+1} a) < \lambda^n d(a, fa)$, a contradiction. So $n = 1$ and a is a fixed point of f. If a, b are fixed points of f then $d(a, b) = d(fa, fb) \leq \lambda d(a, b)$. Since $0 < \lambda < 1$, we have $a = b$. $\square$

Remark 1.4. Several publications attempting to generalize fixed point theorems in metric spaces to g.m.s are plagued by the use of (1), (2), and (3) above (see for example [2],[3],[4],[5] and [6]). Valid proofs for many of them can be offered as in theorem1.3 which will be communicated soon by the authors for publication.

Further general topological properties of a g.m.s have been extensively studied by us and will be communicated in a forthcoming paper.

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