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# APPLICATION OF BISHOP-PHELPS THEOREM IN THE APPROXIMATION THEORY

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ABSTRACT. In this paper we apply the Bishop-Phelps Theorem to show that if $X$ is a Banach space and $G \subseteq X$ is a maximal subspace so that $G^{\perp} = \{x^* \in X^* | x^*(y) = 0; \ \forall y \in G\}$ is an $L-summand$ in $X^*$, then $L^1(\Omega, G)$ is contained in a maximal proximinal subspace of $L^1(\Omega, X)$.

## 1. INTRODUCTION

To follow the note we need some definitions and notations which are following. Let $(\Omega, \Sigma, \mu)$ be a measure space with nonnegative complete $\sigma$-finite measure $\mu$ and $\sigma$-algebra $\Sigma$ of $\mu$-measurable sets. We denote by $L^p(\Omega, \Sigma, \mu : X) = L^p(\Omega, X)$ the Banach space of all equivalence classes of all Bochner integrable functions $f : \Omega \to X$ with norm

$$\|f\| = (\int_{\Omega} \|f(t)\|^p d\mu)^{\frac{1}{p}}; 1 \le p < \infty,$$

$$\|f\|_{\infty} = ess \sup_{t \in \Omega} \|f(t)\|; p = \infty.$$

A subset $A \subseteq X$ is decomposable if for any two elements $f$ , $g$ in $A$ and $E \subseteq \Sigma$, we get $\chi_E f + \chi_{X \setminus E} g \in A$. Where $\chi_A$ is the characteristic function. Let $X$ be a real or complex Banach space and $C$ be a closed convex subset of $X$. The set of *support points* of $C$, is the collection of all points $z \in C$ for which there exists nontrivial $f \in X^*$ such that $\sup_{x \in C} |f(x)| = |f(z)|$. Such an $f$ is called support functional. The support point $z$ is said to be exposed, if $Ref(x) < Ref(z)$, for

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$x(\neq z) \in C$. We denote by $SuppC$ and $\Sigma C$ the set of support points and support functionals, respectively. Bishop and Phelps [1, 7] have shown that if $C$ is a closed convex and bounded subset of $X$ then $SuppC$ is dense in the boundary of $C$ and $\Sigma C$ is dense in $X^*$. The complex case of the Bishop-Phelps Theorem is also studied in [6, 8] and some results are given.

Let $X$ be a Banach space and $G$ a closed subspace of $X$. The subspace $G$ is called proximinal in $X$ if for every $x \in X$ there exists at least one $y \in G$ such that

$$\|x - y\| = inf\{\|x - z\| : z \in G\}.$$

A linear projecton $P : X \longrightarrow Y$ is called an $L-projecton$ if

$$\|x\| = \|Px\| + \|x - Px\|; \quad \forall x \in X.$$

A closed subspace $Y \subset X$ is called an $L - summand$ if it is the range of an $L - projection$. The natural question is that, whether or not $L^1(\Omega, G)$ is proximinal in $L^1(\Omega, X)$ if $G$ is proximinal in $X$ [4]. We will show that if $G^\perp$ is an $L - summand$ then $L^1(\Omega, G)$ is contained in a maximal proximinal subspace of $L^1(\Omega, X)$.

## 2. The main results

**Theorem 2.1.** [5] *If $X$ is a Banach space and $T \in X^*$, then $kerT$ is a proximinal set in $X$ if and only if $T$ supports some points of the closed unit ball of $X$.*

**Lemma 2.2.** *Let $X$ be a Banach space and $G$ a support set in $X$. Suppose $L^1(\Omega, G)$ is a decomposable set. Then each constant function of $L^1(\Omega, G)$ is a support point for $L^1(\Omega, G)$.*

*Proof.* Let $g_0 \in L^1(\Omega, G)$ be a constant function, then there exists a point $x_0 \in G$ such that $g_0(t) = x_0$. Since $G$ is a support set, we have

$$\exists T_0 \in X^* \ \ s.t. \ \ inf_G T_0 = T_0(x_0).$$

We define $F_0 : L^1(\Omega, X) \to R$ as follows:

$$F_0(g) = \int_\Omega T_0(g(t))d\mu.$$

It is obvious that $F_0 \in L^1(\Omega, X)^*$, because if

$$g_n \to g \quad (\|g_n - g\| \to 0),$$

then

$$\begin{aligned} |F_0(g_n) - F_0(g)| &= |\int_\Omega T_0(g_n(t) - g(t))d\mu| \\ &\leq \int_\Omega |T_0(g_n(t) - g(t))|d\mu \\ &\leq \int_\Omega \|T_0\|\|g_n(t) - g(t)\|d\mu \\ &= \|T_0\|\|g_n - g\| \to 0. \end{aligned} \tag{2.1}$$

hence $F_0(g_n) \to F_0(g)$ therefore $F_0 \in L^1(\Omega, X)^*$.
Now by Theorem 2.2 [3], we have

$$\begin{aligned} inf_{L^1(\Omega,G)}F_0 &= inf_{L^1(\Omega,G)}\int_\Omega T_0(g(t))d\mu \\ &= \int_\Omega T_0(x_0)d\mu = T_0(x_0). \end{aligned} \tag{2.2}$$

Note that the middle equality is true, because $L^1(\Omega, G)$ is a decomposable set. By letting $g_0(t) = x_0$ we get that $g_0 \in L^1(\Omega, G)$, and the required result follows:

$$inf_{L^1(\Omega,G)}F_0 = F_0(g_0) = T_0(x_0) = inf_G T_0.$$

Therefore, $g_0 \in L^1(\Omega, G)$ is a support point for $L^1(\Omega, G)$. □

**Theorem 2.3.** (See Proposition 1.1 of [2]). *Let $G$ be a subspace of a Banach space $X$ such that $G^\perp = \{x^* \in X^* | x^*(y) = 0;\ \forall y \in G\}$ be an $L-summand$ in $X^*$, then $G$ is proximinal in $X$.*

By applying the above results we will have the following theorem.

**Theorem 2.4.** *Let $X$ be a Banach space and $G \subset X$ be a maximal subspace such that $G^\perp = \{x^* \in X^* | x^*(y) = 0;\ \forall y \in G\}$ be an $L-summand$ in $X^*$, then $L^1(\Omega, G)$ is contained in a maximal proximinal subspace of $L^1(\Omega, X)$.*

*Proof.* Since $G^\perp$ is an $L-summand$ in Banach space $X^*$ then by theorem 2.3, $G$ is proximinal in $X$. On the other hand $G$ is a maximal subspace, so there exists $T \in X^*$ such that $kerT = G$. Applying Theorem 2.1, there exists a point $x_0$ in the closed unit ball of $X$ such that $T$ supports $x_0$. It is trivial that

$$F(g) = \int_\Omega T(g(t))d\mu$$

is a continuous linear functional on $L^1(\Omega, X)$. Since $T$ is a support functional by the proof of Lemma 2.2, that $F$ is also a support functional for the closed unit ball of $L^1(\Omega, X)$ (by choosing $g_0(t) = x_0$), therefore $kerF$ is proximinal in $L^1(\Omega, X)$ It is obvious that $L^1(\Omega, G) \subseteq kerF$ and $kerF$ is a maximal subspace, so $L^1(\Omega, G)$ is contained in maximal proximinal subspace of $L^1(\Omega, X)$. □

**Remark 2.5.** It is easy to see that if $x_0$ is a support point for a closed convex subset $C$ of a Banach space $(X, \|.\|_1)$ then it may not be a support point for $C \subseteq (X, \|.\|_2)$ even when $\|.\|_2$ is equivalent norm to $\|.\|_1$. Now from above results we conclude that the proximinality of a subset of a Banach space does not hold with two equivalent norm in general.

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