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B.Y. CHEN INEQUALITIES FOR BI-SLANT SUBMANIFOLDS IN GENERALIZED COMPLEX SPACE FORMS

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Abstract. The aim of the present paper is to study Chen inequalities for slant, bi-slant and semi-slant submanifolds in generalized complex space forms.

1. Introduction

In [7] B.Y. Chen recalls one of the basic problems in submanifold theory as to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. In [5] he established a sharp inequality for the sectional curvature of a submanifold in a real space forms in terms of the scalar curvature and squared mean curvature. Afterward several geometers [16],[20],[23] obtained similar inequalities for submanifolds in generalized complex space forms. Many geometers also studied contact version of above inequalities [1],[13],[15]. In this article, we establish Chen inequalities for bi-slant and semi-slant submanifolds in generalized complex space forms.

2. Preliminaries

Let $\tilde{M}$ be an almost Hermitian manifold with an almost complex structure J and Riemannian metric g. If J is integrable, i.e. the Nijenhuis tensor $[J, J]$ of J vanishes, then $\tilde{M}$ is called a Hermitian manifold. The fundamental 2-form Ω of $\tilde{M}$ is defined by

$$\Omega(X, Y) = g(X, JY), \text{ for all, } X, Y \in T\tilde{M}. \tag{2.1}$$

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An almost Hermitian manifold $\tilde{M}$ is called an almost Kaehler manifold if the fundamental 2-form Ω is closed and it becomes Kaehler manifold if $\tilde{\nabla}J = 0$, where $\tilde{\nabla}$ denotes the operator of covariant differentiation with respect to g on $\tilde{M}$.

If an almost complex structure J satisfies

$$(2.2)\qquad (\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0,$$

for any vector fields X and Y on $\tilde{M}$, then the manifold is called a nearly Kaehler manifold.

A. Gray [14] introduced the notion of constant type for a nearly Kaehler manifold, which led to the definition of RK-manifolds. An RK-manifold $\tilde{M}$ is an almost Hermitian manifold for which the curvature tensor $\tilde{R}$ is J-invariant, i.e.

$$(2.3)\qquad \tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W),$$

for all vector fields $X, Y, Z, W \in T\tilde{M}$.

An almost Hermitian manifold $\tilde{M}$ is said to have (pointwise) constant type if for each $p \in \tilde{M}$ and for all vector fields $X, Y, Z \in T_p\tilde{M}$ such that

$$(2.4)\qquad \begin{aligned} g(X,Y) = g(X,Z) = g(X,JY) = g(X,JZ) = 0,\\ g(Y,Y) = 1 = g(Z,Z), \end{aligned}$$

we have

$$(2.5)\qquad \tilde{R}(X,Y,X,Y) - \tilde{R}(X,Y,JX,JY) = \tilde{R}(X,Z,X,Z) - \tilde{R}(X,Z,JX,JZ).$$

An RK-manifold $\tilde{M}$ has (pointwise) constant type if and only if there is a differentiable function α on $\tilde{M}$ such that

$$(2.6)\qquad \tilde{R}(X,Y,X,Y) - \tilde{R}(X,Y,JX,JY) = \alpha\{g(X,X)g(Y,Y) - g^2(X,Y) - g^2(X,JY)\},$$

for all vector fields $X, Y \in T\tilde{M}$.

Furthermore, $\tilde{M}$ has global constant type if α is constant. The function α is called the constant type of $\tilde{M}$. An RK-manifold of constant holomorphic sectional curvature c and constant type α is called a generalized complex space form, denoted by $\tilde{M}(c,\alpha)$. The curvature tensor $\tilde{R}$ of $\tilde{M}(c,\alpha)$ has the following expression:

$$(2.7)\qquad \begin{aligned} \tilde{R}(X,Y,Z,W) = {} & \tfrac{c+3\alpha}{4}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\}\\ & + \tfrac{c-\alpha}{4}\{g(JX,Z)g(JY,W) - g(JX,W)g(JY,Z)\\ & + 2g(X,JY)g(Z,JW)\}, \end{aligned}$$

for all vector fields $X, Y, Z, W \in T\tilde{M}$.

If $c = \alpha$, then $\tilde{M}(c, \alpha)$ is a space of constant curvature. A complex space form $\tilde{M}(c)$ (i.e., a Kaehler manifold of constant holomorphic sectional curvature c) belongs to the class of almost Hermitian manifold $\tilde{M}(c, \alpha)$ (with constant type zero).

Let M be a Riemannian manifold and $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM,\ p \in M$.

For any orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_pM, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{2.8}$$

We denote by

$$(\inf K)(p) = \inf \{K(\pi) :\ \pi \subset T_pM,\ dim\ \pi = 2\}. \tag{2.9}$$

The first Chen invariant $\delta_M(p)$ is given by

$$\delta_M(p) = \tau(p) - (\inf K)(p). \tag{2.10}$$

Let L be a subspace of T_pM of dimension $k \geq 2$ and $\{e_1, ..., e_k\}$ an orthonormal basis of L. Define $\tau(L)$ be the scalar curvature of the k-plane section L by

$$\tau(L) = \sum_{i<j} K(e_i \wedge e_j),\ i, j = 1, ..., k. \tag{2.11}$$

Given an orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_pM, we denote by $\tau_{1.....k}$ the scalar curvature of k-plane section spanned by $e_1, ..., e_k$. The scalar curvature $\tau(p)$ of M at p is the scalar curvature of the tangent space of M at p. If L is a 2-plane section, then $\tau(L)$ reduces to the sectional curvature $K(L)$ of the plane section L. If $K(\pi)$ is the sectional curvature of M for a plane section π in $T_pM,\ p \in M$, then scalar curvature $\tau(p)$ at p is given by

$$\tau(p) = \sum_{i<j} K_{ij}, \tag{2.12}$$

where $\{e_1, ..., e_n\}$ is an orthonormal basis for T_pM and K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j at $p \in M$.

We recall the following Lemma of Chen [6].

Lemma 2.1. *Let $n \geq 2$ and $a_1, ..., a_n, b$ be (n+1)-real numbers, such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right). \tag{2.13}$$

Then $2a_1a_2 \geq b$ with equality holding if and only if

$$a_1 + a_2 = a_3 = ... = a_n.$$

Let M be an n-dimensional submanifold of a 2m-dimensional generalized complex space form $\tilde{M}(c, \alpha)$ and we denote by h, ∇ and $\nabla^{\perp}$ the second fundamental form of M, the induced connection on M and the normal bundle $T^{\perp}M$. Then, the Gauss and Weingarten formulae are given respectively

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.14}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.15}$$

for all vector fields X, Y tangent to M and vector field V normal to M, where A_V is the shape operator in the direction of V. The second fundamental form and the shape operator are related by

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.16}$$

Let R be the Riemannian curvature tensor of M, then the equation of Gauss is given by,

$$\begin{aligned} \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) &+ g(h(X, W), h(Y, Z)) \\ &- g(h(X, Z), h(Y, W)), \end{aligned} \tag{2.17}$$

for any vector fields X, Y, Z, W tangent to M.

Let $p \in M$ and $\{e_1, ..., e_n\}$ an orthonormal basis of the tangent space T_pM. We denote by $H(p)$ the mean curvature vector at p, that is

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \tag{2.18}$$

Also, we set

$$h^r_{ij} = g(h(e_i, e_j), e_r), \ i, j \in \{1, ..., n\}, \ r \in \{n + 1, ..., 2m\}, \tag{2.19}$$

and

$$||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.20}$$

For any $p \in M$ and $X \in T_pM$, we put

$$JX = PX + FX, \tag{2.21}$$

where PX and FX are the tangential and normal components of JX respectively.

Let us denote

$$||P||^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j). \tag{2.22}$$

Now, we recall that for a submanifold M in a Riemannian manifold, the relative null space of M at a point p is defined by

$$N_p = \{X \in T_pM |\ h(X,Y) = 0, \text{ for all } Y \in T_pM\}.$$

Definition(2.1)[2]. A differential distribution D on M is called a slant distribution if for each $p \in M$ and each non-zero vector $X \in D_p$, the angle $\theta_D(X)$ between JX and the vector subspace D_p is constant, which is independent of the choice of $p \in M$ and $X \in D_p$. In this case, the constant angle θ_D is called the slant angle of the distribution D.

Definition(2.2)[2]. A submanifold M is said to be a slant submanifold if for any $p \in M$ and $X \in T_pM$, the angle between JX and T_pM is constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM$. The angle $\theta \in [0, \frac{\pi}{2}]$ is called the slant angle of M in $\tilde{M}$.

Invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

Definition(2.3)[3]. A submanifold M is called a bi-slant submanifold of $\tilde{M}$ if there exist two orthogonal distributions D_1 and D_2 on M, such that

(i) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2$,
(ii) for any $i = 1, 2,\ D_i$ is slant distribution with slant angle θ_i.

On the other hand, CR-submanifolds of $\tilde{M}$ are bi-slant submanifolds with $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$.

Let $2d_1 = \dim D_1$ and $2d_2 = \dim D_2$.

If either d_1 or d_2 vanishes, the bi-slant submanifold is a slant submanifold. Thus, slant submanifolds are particular cases of bi-slant submanifolds.

Definition(2.4)[3]. A submanifold M is said to be a semi-slant submanifold of $\tilde{M}$ if there exist two orthogonal distributions D_1 and D_2 on M, such that

(i) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2$,
(ii) the distribution D_1 is an invariant distribution, that is, $J(D_1) = D_1$,
(iii) the distribution D_2 is slant with angle $\theta \neq 0$.

The invariant distribution of a semi-slant submanifold is a slant distribution with zero slant angle. Thus, it is obvious that, semi-slant submanifolds are particular cases of bi-slant submanifolds. However if $2d_1 = \dim D_1$ and $2d_2 = \dim D_2$
(a) $d_2 = 0$, then M is an invariant submanifold.
(b) $d_1 = 0$ and $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.
(c) $d_1 = 0$ and $\theta \neq \frac{\pi}{2}$, then M is a proper slant submanifold, with slant angle θ.

A semi-slant submanifold is proper if $d_1 d_2 \neq 0$ and $\theta \neq \frac{\pi}{2}$.

3. B.Y. Chen inequalities

In this section, we establish Chen inequalities for proper bi-slant submanifolds in a generalized complex space form. We consider a plane section π invariant by P and denote $\dim D_1 = 2d_1$ and $\dim D_2 = 2d_2$.

Theorem 3.1. *Let M be an n-dimensional proper bi-slant submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(c,\alpha)$. Then*

(I) For any plane section π invariant by P and tangent to D_1,

$$\delta_M \leq \tfrac{n-2}{2}\{\tfrac{n^2}{n-1}||H||^2 + \tfrac{c+3\alpha}{4}(n+1)\} + \tfrac{(c-\alpha)}{4}\{3(d_1-1)\cos^2\theta_1 + 3d_2\cos^2\theta_2\} \tag{3.1}$$

and

(II) For any plane section π invariant by P and tangent to D_2,

$$\delta_M \leq \tfrac{n-2}{2}\{\tfrac{n^2}{n-1}||H||^2 + \tfrac{c+3\alpha}{4}(n+1)\} + \tfrac{c-\alpha}{4}\{3d_1\cos^2\theta_1 + 3(d_2-1)\cos^2\theta_2\}. \tag{3.2}$$

The equality case of inequalities (3.1) *and* (3.2) *hold at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, ..., e_{2m}\}$ of $T_p^{\perp}M$ such that the shape operators of M in $\tilde{M}(c,\alpha)$ at p have the following forms*:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & ..... & 0 \\ 0 & b & 0 & ..... & 0 \\ 0 & 0 & \mu & ..... & 0 \\ .. & .. & .. & ..... & \\ 0 & 0 & 0 & ..... & \mu \end{pmatrix}, \qquad a+b=\mu, \tag{3.3}$$

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & ..... & 0 \\ h_{12}^r & -h_{11}^r & 0 & ..... & 0 \\ 0 & 0 & 0 & ..... & 0 \\ .. & .. & .. & ..... & \\ 0 & 0 & 0 & ..... & 0 \end{pmatrix} \tag{3.4}$$

where

$$A_r = A_{e_r}, \;\; r = n+1, ..., 2m. \tag{3.5}$$

$$h_{ij}^r = g(h(e_i,e_j),e_r), \;\; r = n+1, ..., 2m. \tag{3.6}$$

Proof. The Gauss equation for the submanifold M is given by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \tag{3.7}$$

for all vector fields $X, Y, Z, W \in TM$, where $\tilde{R},\ R$ denote the curvature tensors of $\tilde{M}(c,\alpha)$ and M respectively.

The curvature tensor $\tilde{R}$ of $\tilde{M}(c,\alpha)$ has the following expression [20]:

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = &\tfrac{c+3\alpha}{4}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\}\\ &+ \tfrac{c-\alpha}{4}\{g(JX,Z)g(JY,W) - g(JX,W)g(JY,Z)\\ &+ 2g(X,JY)g(Z,JW)\},\end{aligned} \tag{3.8}$$

for any vector fields $X, Y, Z, W \in TM$.

Let $p \in M$, we choose an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, ..., e_{2m}\}$ of $T_p^{\perp}M$. By substituting $X = Z = e_i,\ Y = W = e_j$ in equation (3.8), we have

$$\begin{aligned}\tilde{R}(e_i,e_j,e_i,e_j) = &\tfrac{c+3\alpha}{4}\{n^2 - n\}\\ &+ \tfrac{c-\alpha}{4}\{-g(Je_i,e_j)g(Je_j,e_i) + 2g(e_i,Je_j)g(e_i,Je_j)\\ = &\tfrac{c+3\alpha}{4}\{n^2 - n\} + \tfrac{c-\alpha}{4}\{3\sum_{i,j=1}^{n} g^2(Je_i,e_j)\}.\end{aligned} \tag{3.9}$$

Let M be a proper bi-slant submanifold of $\tilde{M}(c,\alpha)$ and dim $M = n = 2d_1 + 2d_2$. We consider an adapted bi-slant orthonormal frames

$$\begin{aligned}&e_1, e_2 = \tfrac{1}{\cos\theta_1}Pe_1, ..., e_{2d_1-1}, e_{2d_1} = \tfrac{1}{\cos\theta_1}Pe_{2d_1-1},\\ &e_{2d_1+1}, e_{2d_1+2} = \tfrac{1}{\cos\theta_2}Pe_{2d_1+1},\\ &...,\\ &...,\\ &e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \tfrac{1}{\cos\theta_2}Pe_{2d_1+2d_2-1}.\end{aligned} \tag{3.10}$$

Obviously, we have

$$\begin{aligned}g^2(Je_i, e_{i+1}) &= \cos^2\theta_1, \text{ for } i \in \{1, ..., 2d_1 - 1\} \text{ and}\\ &= cos^2\theta_2, \text{ for } i \in \{2d_1 + 1, ..., 2d_1 + 2d_2 - 1\}.\end{aligned} \tag{3.11}$$

Then, we have

$$\sum_{i,j=1}^{n} g^2(Je_i,e_j) = 2(d_1\cos^2\theta_1 + d_2\cos^2\theta_2). \tag{3.12}$$

Substituting (3.12) into (3.9), we have

$$\tilde{R}(e_i,e_j,e_i,e_j) = \tfrac{c+3\alpha}{4}\{n^2 - n\} + \tfrac{c-\alpha}{4}\{6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2)\}. \tag{3.13}$$

The equation (3.7) gives

$$\tilde{R}(e_i,e_j,e_i,e_j) = 2\tau + ||h||^2 - n^2||H||^2. \tag{3.14}$$

By using equations (3.13) and (3.14), we get

(3.15) $$2\tau = n^2||H||^2-||h||^2+\tfrac{c+3\alpha}{4}\{n(n-1)\}+\tfrac{c-\alpha}{4}\{6(d_1\cos^2\theta_1+d_2\cos^2\theta_2)\}.$$

If we set

(3.16) $$\epsilon = 2\tau - \tfrac{n^2}{n-1}(n-2)||H||^2 - \tfrac{c+3\alpha}{4}\{n(n-1)\} - \tfrac{c-\alpha}{4}\{6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2)\},$$

in equation (3.15), we get

(3.17) $$n^2||H||^2 = (n-1)(\epsilon + ||h||^2).$$

Let $p \in M$, $\pi \subset T_pM$, $\dim\pi = 2$ and π invariant by P.

Now, we consider two cases:

Case (a): The plane section π is tangent to D_1.

We may assume that $\pi = sp\{e_1, e_2\}$. We choose $e_{n+1} = \frac{H}{||H||}$.

From the equation (3.17) becomes,

(3.18) $$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\{\sum_{r=n+1}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \epsilon\}.$$

The above equation implies

(3.19) $$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\{\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \epsilon\}.$$

Using the Lemma (2.1) and equation (3.19), we obtain

(3.20) $$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \epsilon.$$

From the Gauss equation for $X = Z = e_1$ and $Y = W = e_2$, we get

(3.21) $$\begin{aligned} K(\pi) &= \tfrac{c+3\alpha}{4} + 3\tfrac{c-\alpha}{4}\cos^2\theta_1 + \sum_{r=n+1}^{2m}[h_{11}^r h_{22}^r - (h_{12}^r)^2] \\ &\geq \tfrac{c+3\alpha}{4} + 3\tfrac{c-\alpha}{4}\cos^2\theta_1 + \tfrac{1}{2}[\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \epsilon] \\ &\quad + \sum_{r=n+2}^{2m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m}(h_{12}^r)^2 \\ &= \tfrac{c+3\alpha}{4} + 3\tfrac{c-\alpha}{4}\cos^2\theta_1 + \tfrac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \tfrac{1}{2}\sum_{r=n+2}^{2m}\sum_{i,j>2}(h_{ij}^r)^2 \\ &\quad + \tfrac{1}{2}\sum_{r=n+2}^{2m}(h_{11}^r + h_{22}^r)^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \tfrac{\epsilon}{2} \\ &\geq \tfrac{c+3\alpha}{4} + 3\tfrac{c-\alpha}{4}\cos^2\theta_1 + \tfrac{\epsilon}{2}. \end{aligned}$$

From the equations (3.16), (3.21) and (2.9), it follows that

$$\inf K \geq \tfrac{c+3\alpha}{4} + 3\tfrac{c-\alpha}{4}\cos^2\theta_1 + \tau - \tfrac{n^2}{2(n-1)}(n-2)||H||^2 - \tfrac{c+3\alpha}{8}\{n(n-1)\} - \tfrac{c-\alpha}{8}\{6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2)\}. \tag{3.22}$$

From the equations (3.22) and (2.10), we get

$$\delta_M \leq \tfrac{n-2}{2}\{\tfrac{n^2}{n-1}||H||^2 + \tfrac{c+3\alpha}{4}(n+1)\} + \tfrac{c-\alpha}{4}\{3(d_1-1)\cos^2\theta_1 + 3d_2\cos^2\theta_2\}, \tag{3.23}$$

where δ_M is Chen invariant. This proves the inequality (3.1).

Case (b): The plane section π is tangent to D_2.

From the equation (3.17), we have

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\{\sum_{r=n+1}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \epsilon\}.$$

The above equation implies

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\{\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \epsilon\}. \tag{3.24}$$

Using the Lemma (2.1) and equation (3.24), we obtain

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \epsilon. \tag{3.25}$$

From the Gauss equation for $X = Z = e_1$ and $Y = W = e_2$, we get

$$\begin{aligned} K(\pi) &= \tfrac{c+3\alpha}{4} + 3\tfrac{c-\alpha}{4}\cos^2\theta_2 + \sum_{r=n+1}^{2m}[h_{11}^r h_{22}^r - (h_{12}^r)^2] \\ &\geq \tfrac{c+3\alpha}{4} + 3\tfrac{c-\alpha}{4}\cos^2\theta_2 + \tfrac{1}{2}[\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \epsilon] \\ &\quad + \sum_{r=n+2}^{2m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m}(h_{12}^r)^2 \\ &= \tfrac{c+3\alpha}{4} + 3\tfrac{c-\alpha}{4}\cos^2\theta_2 + \tfrac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \tfrac{1}{2}\sum_{r=n+2}^{2m}\sum_{i,j>2}(h_{ij}^r)^2 \\ &\quad + \tfrac{1}{2}\sum_{r=n+2}^{2m}(h_{11}^r + h_{22}^r)^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \tfrac{\epsilon}{2} \\ &\geq \tfrac{c+3\alpha}{4} + 3\tfrac{c-\alpha}{4}\cos^2\theta_2 + \tfrac{\epsilon}{2}. \end{aligned} \tag{3.26}$$

From the relations (3.16), (3.26) and (2.9), it follows that

(3.27) $$\inf K \geq \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4}\cos^2\theta_2 + \tau - \frac{n^2}{2(n-1)}(n-2)||H||^2$$
$$- \frac{c+3\alpha}{8}\{n(n-1)\} - \frac{c-\alpha}{8}\{6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2)\}.$$

From the equations (3.27) and (2.10), we get

(3.28) $$\delta_M \leq \frac{n-2}{2}\{\frac{n^2}{n-1}||H||^2 + \frac{c+3\alpha}{4}(n+1)\},$$
$$+ \frac{c-\alpha}{4}\{3d_1\cos^2\theta_1 + 3(d_2-1)\cos^2\theta_2\}.$$

This proves the inequality (3.2).

The equality case at a point p holds, if and only if equality holds in each of inequalities (3.20), (3.23) and (3.28) and Lemma (2.1). So we have

$$h^{n+1}_{ij} = 0, \ \forall\, i \neq j, \ i,j > 2,$$

$$h^{r}_{ij} = 0, \ \forall\, i \neq j, \ i,j > 2, \ r = n+1, \ldots.., 2m,$$

$$h^{r}_{11} + h^{r}_{22} = 0, \ \forall\, r = n+2, \ldots..., 2m,$$

$$h^{n+1}_{1j} = h^{n+1}_{2j} = 0, \ \forall\, j > 2,$$

$$h^{n+1}_{11} + h^{n+1}_{2} = h^{n+1}_{33} = \ldots... = h^{n+1}_{nn}.$$

We may choose $\{e_1, e_2\}$ such that $h^{n+1}_{12} = 0$ and we denote by $a = h_{11}$, $b = h^r_{22}$, $\mu = h^{n+1}_{33} = \ldots.... = h^{n+1}_{nn}$. Then the shape operators take the desired forms.

Now, we can state the following:

Corollary 3.2. *Let M be an n-dimensional proper semi-slant submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(c,\alpha)$. Then*

(I) For any plane section π invariant by P and tangent to D_1,

(3.29) $$\delta_M \leq \frac{n-2}{2}\{\frac{n^2}{n-1}||H||^2 + \frac{c+3\alpha}{4}(n+1)\}$$
$$+ \frac{(c-\alpha)}{4}\{3(d_1-1) + 3d_2\cos^2\theta\}$$

and

(II) For any plane section π invariant by P and tangent to D_2,

(3.30) $$\delta_M \leq \frac{n-2}{2}\{\frac{n^2}{n-1}||H||^2 + \frac{c+3\alpha}{4}(n+1)\}$$
$$+ \frac{c-\alpha}{4}\{3d_1 + 3(d_2-1)\cos^2\theta\}.$$

The equality case of inequalities (3.29) *and* (3.30) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \ldots.., e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots.., e_{2m}\}$ of $T^{\perp}_pM$ such that the shape operators of M in $\tilde{M}(c,\alpha)$ at p have the forms* (3.3) *and* (3.4).

Corollary 3.3. *Let M be an n-dimensional θ-slant submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(c, \alpha)$. Then*

$$\delta_M \leq \tfrac{n-2}{2}\{\tfrac{n^2}{n-1}||H||^2 + \tfrac{c+3\alpha}{4}(n+1) + 3\tfrac{c-\alpha}{4}\cos^2\theta\}. \quad (3.31)$$

The equality case of the inequality (3.31) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, ..., e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, ..., e_{2m}\}$ of $T_p^\perp M$ such that the shape operators of M in $\tilde{M}(c, \alpha)$ at p have the forms* (3.3) *and* (3.4).

Corollary 3.4. *Let M be an n-dimensional invariant submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(c, \alpha)$. Then*

$$\delta_M \leq \tfrac{n-2}{2}\{\tfrac{n^2}{n-1}||H||^2 + \tfrac{c+3\alpha}{4}(n+1) + 3\tfrac{c-\alpha}{4}\}. \quad (3.32)$$

The equality case of the inequality (3.32) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, ..., e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, ..., e_{2m}\}$ of $T_p^\perp M$ such that the shape operators of M in $\tilde{M}(c, \alpha)$ at p have the forms* (3.3) *and* (3.4).

Corollary 3.5. *Let M be an n-dimensional anti-invariant submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(c, \alpha)$. Then*

$$\delta_M \leq \tfrac{n-2}{2}\{\tfrac{n^2}{n-1}||H||^2 + \tfrac{c+3\alpha}{4}(n+1)\}. \quad (3.33)$$

The equality case of the inequality (3.33) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, ..., e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, ..., e_{2m}\}$ of $T_p^\perp M$ such that the shape operators of M in $\tilde{M}(c, \alpha)$ at p have the forms* (3.3) *and* (3.4).

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