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TERNARY JORDAN HOMOMORPHISMS IN C^*-TERNARY ALGEBRAS

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Dedicated to Themistocles M. Rassias on the occasion of his sixtieth birthday

ABSTRACT. In this note, we prove the Hyers-Ulam-Rassias stability of Jordan homomorphisms in C^*-ternary algebras for the following generalized Cauchy-Jensen additive mapping:

$$rf\left(\frac{s\sum_{j=1}^{p} x_j + t\sum_{j=1}^{d} x_j}{r}\right) = s\sum_{j=1}^{p} f(x_j) + t\sum_{j=1}^{d} f(x_j)$$

and generalize some results concerning this functional equation.

1. INTRODUCTION

Ternary algebraic structures appear more or less naturally in various domain of theoretical and mathematical physics, for example the quark model inspired a particular brand of ternary algebraic system. One of such attempt has been proposed by Y. Nambu in 1973, and known under the name of "Nambu mechanics" since then [43] (see also [1, 45] and [46]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \longmapsto [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, u, v]] = [x, [u, z, y], v] = [[x, y, z], u, v]$, and satisfies $\|[x, y, z]\| \leq \|x\|.\|y\|.\|z\|$ and $\|[x, x, x]\| = \|x\|^3$. If a C^*-ternary algebra $(A, [., ., .])$ has an identity, i.e., an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is

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routine to verify that A, endowed with $xoy := [x, e, y]$ and $x^* := [e, x, e]$, is a unital C^*-algebra. Conversely, if (A, o) is a unital C^*- algebra, then $[x, y, z] := xoy^*oz$ makes A into a C^*-ternary algebra.
A $\mathbb{C}$-linear mapping $H : A \to B$ between C^*-ternary algebras is called a ternary Jordan homomorphism if

$$H([x, x, x]) = [H(x), H(x), H(x)]$$

for all $x \in A$.

The stability of functional equations started with the following question concerning stability of group homomorphisms proposed by S.M. Ulam [44] during a talk before a Mathematical Colloquium at the University of Wisconsin, Madison, in 1940:

Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(., .)$. Given $\varepsilon > 0$, can a $\delta > 0$ be found so if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$, for all $x, y \in G_1$, then a homomorphism $H : G_1 \longrightarrow G_2$ exists with $d(h(x), H(x)) < \varepsilon$, for all $x \in G_1$.
In 1941, Hyers [18] provide the first (partial) answer to Ulam's problem as follows:

If E and E' are Banach spaces and $f : E \longrightarrow E'$ is a mapping for which there is $\varepsilon > 0$ such that $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in E$, then there is a unique additive mapping $L : E \longrightarrow E'$ such that $\|f(x) - L(x)\| \leq \varepsilon$ for all $x \in E$.
Hyers?theorem was generalized by Aoki [3] for additive mappings and by Rassias [35] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [35] has provided a lot of influence in the development of what we now call generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. On the other hand, J.M. Rassias (see [32]–[34]) solved the Ulam problem by involving a product of different powers of norms. In 1994, a generalization of the Rassias' theorem was obtained by Găvruta [17] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. For more details about the results concerning such problems the reader is referred to [2]–[31] and [36]–[41].

In this paper, we have analyzed some detail of C^*-ternary algebra. A detailed study of how we can have the Hyers-Ulam-Rassias stability of Jordan homomorphism in C^* ternary algebra associated with the following generalized Cauchy-Jensen additive mapping

$$rf(\frac{s\sum_{j=1}^{p} x_j + t\sum_{j=1}^{d} x_j}{r}) = s\sum_{j=1}^{p} f(x_j) + t\sum_{j=1}^{d} f(x_j)$$

is given.

2. Stability of Jordan homomorphisms

Let A, B be C^*-ternary algebras. For a given mapping $f : A \longrightarrow B$, we define

$$C_\mu f(x_1, ..., x_p, y_1, ..., y_d) := rf(\frac{s\sum_{j=1}^{p} \mu x_j + t\sum_{j=1}^{d} \mu x_j}{r}) - s\sum_{j=1}^{p} \mu f(x_j) - t\sum_{j=1}^{d} \mu f(x_j)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x_1, ..., x_p, y_1, ..., y_d \in A$.
One can easily show that a mapping $f : A \longrightarrow A$ satisfies

$$C_\mu f(x_1, ..., x_p, y_1, ..., y_d) = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, ..., x_p, y_1, ..., y_d \in A$ if and only if

$$f(\mu x + \lambda y) = \mu f(x) + \lambda f(y)$$

for all $\mu, \lambda \in \mathbb{T}^1$ and all $x, y \in A$.
We will use the following lemmas in this paper:

Lemma 2.1. [29] *Let $f : A \longrightarrow A$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.2. [26] *Let $\{x_n\}_n, \{y_n\}_n$ and $\{z_n\}_n$ be convergent sequences in A. Then the sequence $\{[x_n, y_n, z_n]\}_n$ is convergent in A.*

Theorem 2.3. *Let r,θ be non-negative real numbers such that $r \in (-\infty, 1) \cup (3, +\infty)$, and let $f : A \longrightarrow A$ be a mapping such that*

$$\|C_\mu f(x_1, ..., x_p, y_1, ..., y_d)\|_A \leq \theta(\sum_{j=1}^{p} \|x_j\|_A^r + \sum_{j=1}^{d} \|y_j\|_A^r) \tag{2.1}$$

and

$$\|f([x, x, x]) - [f(x), f(x), f(x)]\|_A \leq 3\theta \|x\|_A^r \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, x_1, ..., x_p, y_1, ..., y_d \in A$. Then there exists a unique ternary Jordan homomorphism $h : A \longrightarrow A$ such that

$$\|f(x) - h(x)\|_A \leq \frac{2^r(p + d)\theta}{|2(p + 2d)^r - (p + 2d)2^r|} \|x\|_A^r \tag{2.3}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $x_1 = ... = x_p = y_1, ..., y_d = x$ and $s = 1, t = 2$ in (2.1), we get

$$\|f((p + 2d)x) - (p + 2d)f(x)\| \leq (p + d)\theta \|x\|_A^r \tag{2.4}$$

for all $x \in A$. So

$$\|f(x) - (p + 2d)f(\frac{x}{p + 2d})\| \leq \frac{(p + d)\theta}{2^r(p + 2d)^r} \|x\|_A^r$$

for all $x \in A$. Hence

$$\|(p + 2d)^l f(\frac{x}{(p + 2d)^l} - (p + 2d)^m f(\frac{x}{(p + 2d)^m}\| \leq \sum_{j=l}^{m-1} \|(p + 2d)^j f(\frac{x}{(p + 2d)^j})$$
$$- (p + 2d)^{j+1} f(\frac{x}{(p + 2d)^{j+1}})\| \leq \frac{\theta}{2^r} \sum_{j=l}^{m-1} \frac{(p + 2d)^j}{(p + 2d)^{rj}} \|x\|_A^r \tag{2.5}$$

for all non-negative integers m and l with $m > 1$ and all $x \in A$. It follows from (2.5) that the sequence $\{(p + 2d)^n f(\frac{x}{(p+2d)^n})\}$ is a Cauchy sequence for all $x \in A$.

Since A is complete, the sequence $\{(p+2d)^n f(\frac{x}{(p+2d)^n})\}$ converges. So one can define the mapping $h : A \longrightarrow A$ by

$$h(x) := \lim_{n\longrightarrow\infty}(p+2d)^n f(\frac{x}{(p+2d)^n})$$

for all $x \in A$.

Moreover letting $l = 0$ and passing the limit $m \longrightarrow \infty$ in (2.5), we get (2.3). It follows from (2.1) that

$$\begin{aligned}\|rh(\frac{(p+2d)x}{r}) - (p+2d)h(x)\| &\leq \lim_{n\longrightarrow\infty}(p+2d)^n\|rf(\frac{x}{(p+2d)^{n-1}})\\ &- (p+2d)f(\frac{x}{(p+2d)^n})\| \leq \lim_{n\longrightarrow\infty}\frac{(p+2d)^n}{(p+2d)^{nr}}(3\theta\|x\|_A^r)\\ &= 0\end{aligned}$$

for all $x \in A$. So

$$rh(\frac{s\sum_{j=1}^p x_j + t\sum_{j=1}^d x_j}{r}) = s\sum_{j=1}^p h(x_j) + t\sum_{j=1}^d h(x_j)$$

for all $x \in A$. By Lemma 2.1, the mapping $h : A \longrightarrow A$ is Cauchy additive. By the same reasoning as in the proof of Theorem 2.1 of [29], the mapping $h : A \longrightarrow A$ is $\mathbb{C}-$linear.

It follows from (2.2) that

$$\begin{aligned}\|h([x,x,x]) - [h(x),h(x),h(x)]\| &\leq \lim_{n\longrightarrow\infty}(p+2d)^{3n}\|f(\frac{[x,x,x]}{(p+2d)^{3n}})\\ &- [f(\frac{x}{(p+2d)^n}), f(\frac{x}{(p+2d)^n}), f(\frac{x}{(p+2d)^n})]\|\\ &\leq \lim_{n\longrightarrow\infty}\frac{(p+2d)^{3n}}{(p+2d)^{nr}}(3\theta\|x\|_A^r) = 0\end{aligned}$$

for all $x \in A$. So

$$h([x,x,x]) = [h(x),h(x),h(x)]$$

for all $x \in A$.

Now, let $T : A \longrightarrow A$ be another Cauchy-Jensen additive mapping satisfying (2.3). Then we have

$$\begin{aligned}\|h(x) - T(x)\| &= (p+2d)^n\|h(\frac{x}{(p+2d)^n}) - T(\frac{x}{(p+2d)^n})\|\\ &\leq (p+2d)^n(\|h(\frac{x}{(p+2d)^n}) - f(\frac{x}{(p+2d)^n})\| + \|T(\frac{x}{(p+2d)^n}) - f(x(p+2d)^n)\|)\\ &\leq \frac{6(p+2d)^n\theta}{((2)^r-(2))(p+2d)^{nr}}\|x\|_A^r\end{aligned}$$

which tends to zero as $n \longrightarrow \infty$ for all $x \in A$. So we can conclude that $h(x) = T(x)$ for all $x \in A$. This proves the uniqueness property of h. Thus the mapping

$h : A \longrightarrow A$ is unique C^*-ternary algebra Jordan homomorphism satisfying (2.3). $\square$

Theorem 2.4. *Let r, s and θ be non–negative real numbers such that $0 < r < 1, 0 < s < 3$ (respectively, $r > 1, s > 3$) and let $d \geq 2$. Suppose that $f : A \longrightarrow A$ is a mapping with $f(0) = 0$, satisfying (2.1) and*

$$\|f([x, x, x]) - [f(x), f(x), f(x)]\|_A \leq 3\theta\|x\|_A^s \qquad (2.4)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Then there exists a unique C^*-ternary algebra Jordan homomorphism $h : A \longrightarrow A$ such that

$$\|f(x) - h(x)\|_A \leq \frac{d\theta}{2|d - d^r|}\|x\|_A^r \qquad (2.5)$$

for all $x \in A$.

Proof. **Case I.** $0 < r < 1$ and $0 < s < 3$.
Letting $\mu = 1, x_1 = ... = x_p = 0$ and $y_1 = ... = y_d = x$ and $t = 1$ in (2.1), we get

$$\|f(dx) - df(x)\|_A \leq \frac{d\theta}{2}\|x\|_A^r \qquad (2.6)$$

for all $x \in A$. If we replace x by d^n in (2.6) and divide both sides of (2.6) to d^{n+1}, we get

$$\|\frac{1}{d^{n+1}}f(d^{n+1}x) - \frac{1}{d^n}f(d^n x)\|_A \leq \frac{\theta}{2}d^{(r-1)n}\|x\|_A^r$$

for all $x \in A$ and all non–negative integers n. Therefore,

$$\|\frac{1}{d^{n+1}}f(d^{n+1}x) - \frac{1}{d^m}f(d^m x)\|_A \leq \frac{\theta}{2}\sum_{i=m}^{n} d^{(r-1)i}\|x\|_A^r \qquad (2.7)$$

for all $x \in A$ and all non–negative integers $n \geq m$. From this it follows that the sequence $\{\frac{1}{d^n}f(d^n x)\}$ is Cauchy for all $x \in A$. Since A is complete, the sequence $\{\frac{1}{d^n}f(d^n x)\}$ converges. Thus one can define the mapping $h : A \longrightarrow A$ by

$$h(x) := \lim_{n\longrightarrow\infty} \frac{1}{d^n}f(d^n x)$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \longrightarrow \infty$ in (2.7) we get (2.5). It follows from (2.1) that

$$\|rh(\frac{s\sum_{j=1}^{p}\mu x_j + t\sum_{j=1}^{d}\mu y_j}{r}) - s\sum_{j=1}^{p}\mu h(x_j) - t\sum_{j=1}^{d}\mu h(y_j)\|_A$$

$$= \lim_{n\longrightarrow\infty}\frac{1}{d^n}\|rf(d^n\frac{s\sum_{j=1}^{p}\mu x_j + t\sum_{j=1}^{d}\mu y_j}{r}) - s\sum_{j=1}^{p}\mu f(d^n x_j) - t\sum_{j=1}^{d}\mu f(d^n y_j)\|_A$$

$$\leq \lim_{n\longrightarrow\infty}\frac{d^{nr}}{d^n}\theta(\sum_{j=1}^{p}\|x_j\|_A^r + \sum_{j=1}^{d}\|y_j\|_A^r) = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, ..., x_p, y_1, ..., y_d \in A$. Hence

$$rh(\frac{s\sum_{j=1}^{p}\mu x_j + t\sum_{j=1}^{d}\mu y_j}{r}) = s\sum_{j=1}^{p}\mu h(x_j) + t\sum_{j=1}^{d}\mu h(y_j)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, ..., x_p, y_1, ..., y_d \in A$. So $h(\lambda x + \mu y) = \lambda h(x) + \mu h(y)$ for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y \in A$. Therefore by Lemma 2.1, the mapping $h : A \longrightarrow A$ is $\mathbb{C}-$linear.

It follows from Lemma 2.2 and (2.4) that

$$\begin{aligned}\|h([x,x,x]) - [h(x),h(x),h(x)]\|_A &= \lim_{n\longrightarrow\infty}\frac{1}{d^{3n}}\|f([d^n x, d^n x, d^n x])\\ &- [f(d^n x), f(d^n x), f(d^n x)]\|_A \leq \theta \lim_{n\longrightarrow\infty}\frac{d^{ns}}{d^{3n}}(\|x\|_A^s + \|x\|_A^s + \|x\|_A^s)\\ &= 0\end{aligned}$$

for all $x \in A$. Thus

$$h([x,x,x]) = [h(x),h(x),h(x)]$$

for all $x \in A$.

We can proved that the mapping $h : A \longrightarrow A$ is a unique C^*-ternary algebra Jordan homomorphism satisfying (2.5), as desired (see [26]).

Case II.$r > 1, s > 3$.

We can define the mapping $h : A \longrightarrow A$ by

$$h(x) := \lim_{n\longrightarrow\infty} d^n f(d^{-n}x)$$

for all $x \in A$. The rest of the proof is similar to the proof of case I. □

Theorem 2.5. *Let r, θ be non–negative real numbers such that $r \in (-\infty, \frac{1}{p+d}) \cup (1, +\infty)$, and let $f : A \longrightarrow A$ be a mapping such that*

$$\|C_\mu f(x_1, ..., x_p, y_1, ..., y_d)\|_A \leq \theta \prod_{j=1}^{p}\|x_j\|_A^r \cdot \prod_{j=1}^{d}\|y_j\|_A^r \tag{2.8}$$

and

$$\|f([x,x,x]) - [f(x),f(x),f(x)]\|_A \leq \theta\|x\|_A^{3r} \tag{2.9}$$

for all $\mu \in \mathbb{T}^1$ and all $x, x_1, ..., x_p, y_1, ..., y_d \in A$. Then there exists a unique ternary Jordan homomorphism $h : A \longrightarrow A$ such that

$$\|f(x) - h(x)\|_A \leq \frac{2^{(p+d)r}\theta}{|2(p+2d)^{(p+d)r} - 2^{(p+d)r}(p+2d)|}\|x\|_A^{(p+d)r}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $x_1 = ... = x_p = y_1, ..., y_d = x$ and $s = 1, t = 2$ in (2.8), we get

$$\|f((p+2d)x) - (p+2d)f(x)\| \leq (p+d)\theta\|x\|_A^{3r} \tag{2.10}$$

for all $x \in A$. So

$$\|f(x) - (p+2d)f(\frac{x}{p+2d})\| \leq \frac{\theta}{(p+2d)^{(p+d)r}}\|x\|_A^{(p+d)r}$$

for all $x \in A$. Hence,

$$\begin{aligned}
&\|(p+2d)^l f(\frac{x}{(p+2d)^l} - (p+2d)^m f(\frac{x}{(p+2d)^m}\| \\
&\qquad \le \sum_{j=l}^{m-1} \|(p+2d)^j f(\frac{x}{(p+2d)^j}) - (p+2d)^{j+1} f(\frac{x}{(p+2d)^{j+1}})\| \\
&\qquad \le \frac{\theta}{(p+2d)^{(p+d)r}} \sum_{j=l}^{m-1} \frac{(p+2d)^j}{(p+2d)^{(p+d)rj}} \|x\|_A^{(p+d)r} \qquad (2.11)
\end{aligned}$$

for all non–negative integers m and l with $m > 1$ and all $x \in A$. It follows from (2.11) that the sequence $\{(p+2d)^n f(\frac{x}{(p+2d)^n})\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{(p+2d)^n f(\frac{x}{(p+2d)^n})\}$ converges. So one can define the mapping $h : A \longrightarrow A$ by

$$h(x) := \lim_{n\longrightarrow\infty} (p+2d)^n f(\frac{x}{(p+2d)^n})$$

for all $x \in A$.
Moreover letting $l = 0$ and passing the limit $m \longrightarrow \infty$ in (2.11), we get (2.9). The rest of the proof is similar to the proof of Theorem 2.3. □

Theorem 2.6. *Let $r, s, p, r_1, ..., r_p, s_1, ..., s_d$ and θ be non–negative real numbers such that $r+s+p \neq 3$ and $r_k > 0 (s_k > 0)$ for some $1 \le k \le p (1 \le k \le d)$. Let $f : A \longrightarrow A$ be a mapping satisfying*

$$\|C_\mu f(x_1, ..., x_p, y_1, ..., y_d)\|_A \le \theta \prod_{j=1}^{p} \|x_j\|_A^{r_j} \cdot \prod_{j=1}^{d} \|y_j\|_A^{s_j} \qquad (2.12)$$

and

$$\|f([x,x,x]) - [f(x), f(x), f(x)]\|_A \le \theta \|x\|_A^{r+s+p} \qquad (2.13)$$

for all $\mu \in \mathbb{T}^1$ and all $x, x_1, ..., x_p, y_1, ..., y_d \in A$. Then the mapping $f : A \longrightarrow A$ is a ternary Jordan homomorphism (we put $\| \cdot \|_A^0 = 1$).

Proof. We can show that $f(\mu x + \lambda y) = \mu f(x) + \lambda f(y)$ for all $\lambda, \mu \in \mathbb{T}^1$ and $x, y \in A$ (see [26]). Therefor, by Lemma 2.1 the mapping $f : A \longrightarrow A$ is $\mathbb{C}$-linear. Let $r + s + p > 3$. Then it follows from (2.13) that

$$\begin{aligned}
&\|f([x,x,x]) - [f(x), f(x), f(x)]\|_A \\
&\lim_{n\longrightarrow\infty} 8^n \|f([\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}]) - [f(\frac{x}{2^n}), f(\frac{x}{2^n}), f(\frac{x}{2^n})]\|_A \\
&\le \theta \|x\|_A^r \|x\|_A^s \|x\|_A^p \lim_{n\longrightarrow\infty} (\frac{8}{2^{r+s+p}})^n = 0
\end{aligned}$$

for all $x \in A$. Therefore,

$$f([x,x,x]) = [f(x), f(x), f(x)] \qquad (2.14)$$

for all $x \in A$. Similarly, for $r + s + p < 3$, we get (2.14). □

3. Superstability of ternary Jordan homomorphisms

Throughout this section, assume that A is a unital C^*-algebra with unite element e, and with norm $\|\cdot\|_A$.
We investigate superstability of ternary Jordan homomorphisms in C^*-ternary algebras associated with the functional equation $C_\mu f(x_1,...,x_p,y_1,...,y_d)=0$.

Theorem 3.1. *Let $r>1, s>3$ and θ be non-negative real numbers, and let $f: A\longrightarrow A$ be a mapping satisfying* (2.1) *and* (2.2)*. If there exists a real number $\lambda>1 (0<\lambda<1)$ and an element $x_0\in A$ such that $\lim_{n\longrightarrow\infty}\frac{1}{\lambda^n}f(\lambda^n x_0)=e(\lim_{n\longrightarrow\infty}\lambda^n f(\frac{x_0}{\lambda^n})=e))$, then the mapping $f: A\longrightarrow A$ is a ternary Jordan homomorphism.*

Proof. By using of Section 2, there exists a unique ternary Jordan homomorphism $h: A\longrightarrow A$ such that

$$h(x)=\lim_{n\longrightarrow\infty}\frac{1}{\lambda^n}f(\lambda^n x),(h(x)=\lim_{n\longrightarrow\infty}\lambda^n f(\frac{x}{\lambda^n}) \tag{3.2}$$

for all $x\in A, \lambda>1(0<\lambda<1)$. Therefore , by the assumption we get that $h(x_0)=e$. Let $\lambda>1$ and $\lim_{n\longrightarrow\infty}\frac{1}{\lambda^n}f(\lambda^n x_0)=e$. It follows from (2.4) that

$$\begin{aligned}\|[h(x),h(x),h(x)]-[h(x),h(x),f(x)]\|_A&=\|h[x,x,x]-[h(x),h(x),f(x)]\|_A\\&=\lim_{n\longrightarrow\infty}\frac{1}{\lambda^{2n}}\|f([\lambda^n x,\lambda^n x,x])-[f(\lambda^n x),f(\lambda^n x),f(x)\|_A\\&\leq\theta\lim_{n\longrightarrow\infty}\frac{1}{\lambda^{2n}}(\lambda^{ns}\|x\|_A^s+\lambda^{ns}\|x\|_A^s+\|x\|_A^s)=0\end{aligned}$$

for all$x\in A$. So $[h(x),h(x),h(x)]=[h(x),h(x),f(x)]$ for all $x\in A$. Letting $x=x_0$ in the last equality, we get $f(x)=h(x)$ for all $x\in A$. Similarly, one can show that $h(x)=f(x)$ for all $x\in A$ when $0<\lambda<1$ and $\lim_{n\longrightarrow\infty}\lambda^n f(\frac{x_0}{\lambda^n})=e$. Therefore, the mapping $f: A\longrightarrow A$ is a ternary Jordan homomorphism. □

Theorem 3.2. *Let $r<1, s<2$ and θ be non-negative real numbers, and let $f: A\longrightarrow A$ be a mapping satisfying (2.1) and (2.2). If there exists a real number $\lambda>1(0<\lambda<1)$ and an element $x_0\in A$ such that $\lim_{n\longrightarrow\infty}\frac{1}{\lambda^n}f(\lambda^n x_0)=e(\lim_{n\longrightarrow\infty}\lambda^n f(\frac{x_0}{\lambda^n})=e)$, then the mapping $f: A\longrightarrow A$ is a ternary Jordan homomorphism.*

Proof. The proof is similar to the proof of Theorem 3.1. □

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