J. Nonlinear Sci. Appl. 4 (2011), no. 2, 170–179

THE JOURNAL OF NONLINEAR SCIENCE AND APPLICATIONS
http://www.tjnsa.com

# ON A CERTAIN CLASS OF HARMONIC MULTIVALENT FUNCTIONS

SAURABH PORWAL[1], POONAM DIXIT[2,*] AND VINOD KUMAR[3]

*Dedicated to Themistocles M. Rassias on the occasion of his sixtieth birthday*
*Communicated by Choonkil Park*

ABSTRACT. The purpose of the present paper is to study some results involving coefficient conditions, extreme points, distortion bounds, convolution conditions and convex combination for a new class of harmonic multivalent functions in the open unit disc. Relevant connections of the results presented here with various known results are briefly indicated.

## 1. INTRODUCTION

A continuous complex-valued function $f = u+iv$ defined in a simply connected domain $D$ is said to be harmonic in $D$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain we can write $f = h+\overline{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$.

A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$, $z \in D$. See Clunie and Shiel-Small [7], (see also Ahuja [1] and Duren [11].)

Let $H$ denotes the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ \ g(z) = \sum_{k=1}^{\infty} b_k z^k,\ \ |b_1| < 1. \tag{1.1}$$

*Date*: Received: August 27, 2010; Revised: November 29, 2010.
*Corresponding author

2000 *Mathematics Subject Classification.* Primary 30C45, 30C50, 30C55.
*Key words and phrases.* Harmonic; Univalent; Multivalent functions; Fractional calculus.

Recently, Ahuja and Jahangiri [3] defined the class

$$H_p\left(k\right),\ \left(p,k\in N=\{1,2,3....\}\right)$$

consisting of all p-valent harmonic functions $f=h+\overline{g}$ which are sense-preserving in $U$ and $h$ and $g$ are of the form

$$h(z)=z^p+\sum_{k=2}^{\infty}a_{k+p-1}z^{k+p-1},g(z)=\sum_{k=1}^{\infty}b_{k+p-1}z^{k+p-1},\ \ |b_p|<1. \tag{1.2}$$

Note that $H$ and $H_p\left(k\right)$ reduce to the class $S$ and $S_p(k)$ of analytic univalent and multivalent functions, respectively, if the co-analytic part of its members are zero. For these classes $f\left(z\right)$ may be expressed as

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \tag{1.3}$$

and

$$f(z)=z^p+\sum_{k=2}^{\infty}a_{k+p-1}z^{k+p-1}. \tag{1.4}$$

The following definitions of fractional integrals and fractional derivatives are due to Owa [17] and Srivastava and Owa [23].

**Definition 1.1.** The fractional integral of order $\lambda$ is defined for a function $f\left(z\right)$ by

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma\lambda}\int_0^z\frac{f(\varsigma)}{\left(z-\varsigma\right)^{1-\lambda}}d\varsigma, \tag{1.5}$$

where $\lambda>0,\ f\left(z\right)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $\left(z-\varsigma\right)^{\lambda-1}$ is removed by requiring $\log\left(z-\varsigma\right)$ to be real when $\left(z-\varsigma\right)>0$.

**Definition 1.2.** The fractional derivative of order $\lambda$ is defined for a function $f\left(z\right)$ by

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma\left(1-\lambda\right)}\frac{d}{dz}\int_0^z\frac{f(\varsigma)}{\left(z-\varsigma\right)^{\lambda}}d\varsigma, \tag{1.6}$$

where $0\le\lambda<1,\ f\left(z\right)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $\left(z-\varsigma\right)^{-\lambda}$ is removed as in Definition 1.1 above.

**Definition 1.3.** Under the hypothesis of Definition 1.2 the fractional derivative of order $n+\lambda$ is defined for a function $f\left(z\right)$ by

$$D_z^{n+\lambda}f(z)=\frac{d^n}{dz^n}D_z^{\lambda}f(z), \tag{1.7}$$

where $0\le\lambda<1$ and $n\in N_0=\{0,1,2.....\}$.

Motivated with the definition of Salagean operator, we introduced an interesting operator $(D_z^{\lambda,p})^n f(z)$ for function $f(z)$ of the form (1.4)

$$(D_z^{\lambda,p})^0 f(z) = \frac{\Gamma(p-\lambda+1)}{\Gamma(p+1)} z^\lambda D_z^\lambda f(z)$$

$$(D_z^{\lambda,p})^1 f(z) = \frac{z}{p}\left(\frac{\Gamma(p-\lambda+1)}{\Gamma(p+1)} z^\lambda D_z^\lambda f(z)\right)'$$

$$(D_z^{\lambda,p})^n f(z) = (D_z^{\lambda,p})((D_z^{\lambda,p})^{n-1} f(z)).$$

Thus we have

$$(D_z^{\lambda,p})^n f(z) = z^p + \sum_{k=2}^{\infty}\left(\frac{(k+p-1)\Gamma(p-\lambda+1)\Gamma(k+p)}{p\Gamma(p+1)\Gamma(k+p-\lambda)}\right)^n a_{k+p-1} z^{k+p-1}$$

$$= z^p + \sum_{k=2}^{\infty} (\varphi(k,p,\lambda))^n a_{k+p-1} z^{k+p-1} \tag{1.8}$$

where and throughout this paper

$$\varphi(k,p,\lambda) = \frac{k\Gamma(p-\lambda+1)\Gamma(k+p)}{\Gamma(p+1)\Gamma(k+p-\lambda)} \ (k,p \in N). \tag{1.9}$$

Now, we define $(D_z^{\lambda,p})^n f(z)$ for function of the form (1.2) as follows

$$(D_z^{\lambda,p})^n f(z) = (D_z^{\lambda,p})^n h(z) + (-1)^n \overline{(D_z^{\lambda,p})^n g(z)}. \tag{1.10}$$

We note that the study of the above operator $(D_z^{\lambda,p})^n$ is of special interest because it includes a variety of well-known operator. For example

1. If we put $n = 0$, $p = 1$ then it reduces to Owa-Srivastava Operator.
2. If we put $\lambda = 0$, then it reduces to well-known and widely used Salagean Operator [19].

Now for

$$m \in N, n \in N_0, m > n,\ 0 \le \gamma < 1,\ \beta \ge 0,\ 0 \le \lambda < 1,\ 0 \le t \le 1,\ \alpha \in R$$

and $z \in U$, suppose that $H_p(m,n;\beta;\gamma;t;\lambda)$ denote the family of harmonic functions $f$ of the form (1.2) such that

$$Re\left\{(1+\beta e^{i\alpha})\frac{(D_z^{\lambda,p})^m f(z)}{(D_z^{\lambda,p})^n f_t(z)} - \beta e^{i\alpha}\right\} \ge \gamma, \tag{1.11}$$

where $(D_z^{\lambda,p})^m f(z)$ is defined by (1.10) and $f_t(z) = (1-t)z + t(h(z) + \overline{g(z)})$.

Further, let the subclass $\overline{H_p}(m,n;\beta;\gamma;t;\lambda)$ consisting of harmonic functions $f_m = h + \overline{g_m}$ in $H_p(m,n;\beta;\gamma;t;\lambda)$ so that $h$ and $g_m$ are of the form

$$h(z) = z^p - \sum_{k=2}^{\infty} |a_{k+p-1}| z^{k+p-1}, g_m(z) = (-1)^{m-1}\sum_{k=1}^{\infty} |b_{k+p-1}| z^{k+p-1},\ \ |b_p| < 1. \tag{1.12}$$

By specializing the parameters in subclass $H_p(m,n;\beta;\gamma;t;\lambda)$, we obtain the following known subclasses studied earlier by various authors.

(1) $H_p(m,n;0;\gamma;1;0) \equiv H_p(m,n;\gamma)$ and $\overline{H_p}(m,n;0;\gamma;1;0) \equiv \overline{H_p}(m,n;\gamma)$ studied by Sker and Eker [20].

(2) $H_p(n+1,n;1;\gamma;1;0) \equiv H_p(n,\gamma)$ and $\overline{H_p}(n+1,n;1;\gamma;1;0) \equiv \overline{H_p}(n,\gamma)$ studied by Dixit et al. [9].
(3) $H_p(1,0;0;\gamma;1;0) \equiv H_p(\gamma)$ and $\overline{H_p}(1,0;0;\gamma;1;0) \equiv \overline{H_p}(\gamma)$ studied by Ahuja and Jahangiri [3].
(4) $H_1(n+q,n,p,\gamma,1,0) \equiv R_H(n,\gamma,p,q)$ studied by Dixit et al. [8].
(5) $H_1(n+1,n,1,\gamma,1,0) \equiv RS_H(n,\gamma)$ and $\overline{H_1}(n+1,n,1,\gamma,1,0) \equiv \overline{RS_H}(n,\gamma)$ studied by Yalcin et al. [25].
(6) $H_1(1,0,1,\gamma,1,0) \equiv G_H(\gamma)$ and $\overline{H_1}(1,0,1,\gamma,1,0) \equiv \overline{G_H}(\gamma)$ studied by Rosy et al. [18].
(7) $H_1(2,1,\beta,\gamma,1,0) \equiv HCV(\beta,\gamma)$ and $\overline{H_1}(2,1,\beta,\gamma,1,0) \equiv \overline{HCV}(\beta,\gamma)$ studied by Kim et al. [15].
(8) $H_1(1,0,\beta,\gamma,t,0) \equiv G_H(\beta,\gamma,t)$ and $\overline{H_1}(1,0,\beta,\gamma,t,0) \equiv \overline{G_H}(\beta,\gamma,t)$ studied by Ahuja et al. [2].
(9) $H_1(m,n;0;\gamma;1;0) \equiv S_H(m,n;\gamma)$ and $\overline{H_1}(m,n;0;\gamma;1;0) \equiv \overline{S_H}(m,n;\gamma)$ studied by Yalcin [24].
(10) $H_1(n+1,n;0;\gamma;1;0) \equiv H(n,\gamma)$ and $\overline{H_1}(n+1,n;0;\gamma;1;0) \equiv \overline{H}(n,\gamma)$ studied by Jahangiri et al. [13].
(11) $H_1(1,0;0;\gamma;t;\lambda) \equiv S^*_H(\gamma,t,\lambda)$ and $\overline{H_1}(1,0;0;\gamma;t;\lambda) \equiv \overline{S^*_H}(\gamma,t,\lambda)$ studied by Kumar et al. [16].
(12) $H_1(1,0;0;\gamma;1;\lambda) \equiv S^*_H(\gamma,\lambda)$ and $\overline{H_1}(1,0;0;\gamma;1;\lambda) \equiv TS^*_H(\gamma,\lambda)$ studied by Dixit and Porwal [10].
(13) $H_1(2,1;0;\gamma;1;0) \equiv HK(\gamma)$ and $H_1(1,0;0;\gamma;1;0) \equiv S^*_H(\gamma)$ studied by Jahangiri [12].
(14) $H_1(2,1;0;0;1;0) \equiv HK(0)$ and $H_1(2,1;0;0;1;0) \equiv S^*_H(0)$ studied by Avci and Zlotkiewicz [6], Silverman [21] and Silverman and Silvia [22].
(15) $H_1(1,0,0,\gamma,0,0) \equiv N_H(\gamma)$ studied by Ahuja and Jahangiri [4].
(16) $S_p(1,0,k,0,1,0) \equiv k-ST$ studied by Aouf [5], see also Kanas and Srivastava [14].

In the present paper, results involving the coefficient condition, extreme points, distortion bounds, convolution and convex combinations for the above classes $H_p(m,n,\beta,\gamma,t\lambda)$ and $\overline{H_p}(m,n,\beta,\gamma,t,\lambda)$ of harmonic multivalent functions have been investigated.

## 2. Main results

In our first theorem, we introduce a sufficient condition for functions in $H_p(m,n,\beta,\gamma,t,\lambda)$.

**Theorem 2.1.** *Let $f=h+\overline{g}$ be such that $h$ and $g$ are given by* (1.2). *Furthermore, let*

$$\sum_{k=1}^{\infty}\{\psi(m,n,\beta,\gamma,t,\lambda)|a_{k+p-1}|+\Theta(m,n,\beta,\gamma,t,\lambda)|b_{k+p-1}|\}\leq 2, \tag{2.1}$$

*where*

$$\psi(m,n,\beta,\gamma,t,\lambda)=\frac{(\varphi(k,p,\lambda))^m(1+\beta)-t(\beta+\gamma)(\varphi(k,p,\lambda))^n}{1-\gamma},$$

*and*

$$\Theta(m,n,\beta,\gamma,t,\lambda)=\frac{(\varphi(k,p,\lambda))^m(1+\beta)-(-1)^{m-n}t(\beta+\gamma)(\varphi(k,p,\lambda))^n}{1-\gamma},$$

$a_p=1,\ m\in N, n\in N_0,\ m>n,\ 0\leq\gamma<1,\ \beta\geq 0,\ 0\leq\lambda<1$ *and* $0\leq t\leq 1$, *then* $f$ *is sense-preserving in* $U$ *and* $f\in H_p(m,n,\beta,\gamma,t,\lambda)$.

*Proof.* Using the fact that $Re\ \omega\geq\gamma$ if and only if $|1-\gamma+\omega|\geq|1+\gamma-\omega|$, it suffices to show that

$$\begin{aligned}&|(1-\gamma)(D_z^{\lambda,p})^n f_t(z)+(1+\beta e^{i\alpha})(D_z^{\lambda,p})^m f(z)-\beta e^{i\alpha}(D_z^{\lambda,p})^n f_t(z)|\\ &-|(1+\gamma)(D_z^{\lambda,p})^n f_t(z)-(1+\beta e^{i\alpha})(D_z^{\lambda,p})^m f(z)+\beta e^{i\alpha}(D_z^{\lambda,p})^n f_t(z)|.\end{aligned}\qquad(2.2)$$

Substituting for $(D_z^{\lambda,p})^m f(z)$ and $(D_z^{\lambda,p})^n f_t(z)$ in L.H.S. of (2.2), we have

$$\Bigg|(2-\gamma)z^p+\sum_{k=2}^{\infty}[(1-\gamma-\beta e^{i\alpha})(\varphi(k,p,\lambda))^n t+(1+\beta e^{i\alpha})(\varphi(k,p,\lambda))^m]\ a_{k+p-1}z^{k+p-1}$$

$$+(-1)^n\sum_{k=1}^{\infty}[(1-\gamma-\beta e^{i\alpha})(\varphi(k,p,\lambda))^n t+(-1)^{m-n}(1+\beta e^{i\alpha})(\varphi(k,p,\lambda))^m]\bar{b}_{k+p-1}\bar{z}^{k+p-1}\Bigg|$$

$$-\Bigg|\gamma z^p+\sum_{k=2}^{\infty}[(1+\gamma+\beta e^{i\alpha})(\varphi(k,p,\lambda))^n t-(1+\beta e^{i\alpha})(\varphi(k,p,\lambda))^m]\ a_{k+p-1}z^{k+p-1}$$

$$+(-1)^n\sum_{k=1}^{\infty}[(1+\gamma+\beta e^{i\alpha})(\varphi(k,p,\lambda))^n t-(-1)^{m-n}(1+\beta e^{i\alpha})(\varphi(k,p,\lambda))^m]\bar{b}_{k+p-1}\bar{z}^{k+p-1}\Bigg|$$

$$\geq 2(1-\gamma)|z|^p-2\sum_{k=2}^{\infty}[(\varphi(k,p,\lambda))^m(1+\beta)-(\gamma+\beta)t(\varphi(k,p,\lambda))^n]|a_{k+p-1}||z|^{k+p-1}$$

$$-\sum_{k=1}^{\infty}|[(1-\gamma)(\varphi(k,p,\lambda))^n t+(-1)^{m-n}(1+\beta e^{i\alpha})(\varphi(k,p,\lambda))^m-\beta e^{i\alpha}(\varphi(k,p,\lambda))^n t]||b_{k+p-1}||z|^{k+p-1}$$

$$-\sum_{k=1}^{\infty}|[(-1)^{m-n}(1+\beta e^{i\alpha})(\varphi(k,p,\lambda))^m-(1+\gamma+\beta e^{i\alpha})(\varphi(k,p,\lambda))^n t]||b_{k+p-1}||z|^{k+p-1}$$

$$=\begin{cases}2(1-\gamma)|z|^p-2\displaystyle\sum_{k=2}^{\infty}[(1+\beta)(\varphi(k,p,\lambda))^m-(\gamma+\beta)t(\varphi(k,p,\lambda))^n]|a_{k+p-1}||z|^{k+p-1}\\ \quad-2\displaystyle\sum_{k=2}^{\infty}[(1+\beta)(\varphi(k,p,\lambda))^m+(\gamma+\beta)t(\varphi(k,p,\lambda))^n]|b_{k+p-1}||z|^{k+p-1},\ \text{if } m-n \text{ is odd}\\ 2(1-\gamma)|z|^p-2\displaystyle\sum_{k=2}^{\infty}[(1+\beta)(\varphi(k,p,\lambda))^m-(\gamma+\beta)t(\varphi(k,p,\lambda))^n]|a_{k+p-1}||z|^{k+p-1}\\ \quad-2\displaystyle\sum_{k=1}^{\infty}[(1+\beta)(\varphi(k,p,\lambda))^m-(\gamma+\beta)t(\varphi(k,p,\lambda))^n]|b_{k+p-1}||z|^{k+p-1},\ \text{if } m-n \text{ is odd}\end{cases}$$

$$= 2(1-\gamma)|z|^p\Bigg\{1-\sum_{k=2}^{\infty}\frac{(\varphi(k,p,\lambda))^m(1+\beta)-(\gamma+\beta)t(\varphi(k,p,\lambda))^n}{1-\gamma}|a_{k+p-1}||z|^{k-1}$$
$$-\sum_{k=1}^{\infty}\frac{(\varphi(k,p,\lambda))^m(1+\beta)-(-1)^{m-n}(\gamma+\beta)t(\varphi(k,p,\lambda))^n}{1-\gamma}|b_{k+p-1}||z|^{k-1}\Bigg\}$$
$$> 2(1-\gamma)\Bigg\{1-\sum_{k=2}^{\infty}\frac{(\varphi(k,p,\lambda))^m(1+\beta)-(\gamma+\beta)t(\varphi(k,p,\lambda))^n}{1-\gamma}|a_{k+p-1}|$$
$$-\sum_{k=1}^{\infty}\frac{(\varphi(k,p,\lambda))^m(1+\beta)-(-1)^{m-n}(\gamma+\beta)t(\varphi(k,p,\lambda))^n}{1-\gamma}|b_{k+p-1}|\Bigg\}.$$

The last expression is non negative by (2.1), and so the proof is complete. ☐

In the following theorem, it is proved that the condition (2.1) is also necessary for functions $f_m = h + \overline{g_m}$, where h and $g_m$ are of the form (1.12).

**Theorem 2.2.** *Let* $f_m = h+\overline{g_m}$ *be given by* (1.12). *Then* $f_m \in \overline{H}_p(m,n,\beta,\gamma,t,\lambda)$, *if and only if*

$$\sum_{k=1}^{\infty}\left\{\psi\left(m,n,\beta,\gamma,t,\lambda\right)|a_{k+p-1}|+\Theta\left(m,n,\beta,\gamma,t,\lambda\right)|b_{k+p-1}|\right\}\leq 2. \tag{2.3}$$

*Proof.* Since $\overline{H_p}\left(m,n;\beta;\gamma;t;\lambda\right) \subset H_p\left(m,n;\beta;\gamma;t;\lambda\right)$, we only need to prove the "only if" part of the theorem.

To this end, for functions $f_m$ of the form (1.12), we notice that condition

$$Re\left\{\left(1+\beta e^{i\alpha}\right)\frac{\left(D_z^{\lambda,p}\right)^m f\left(z\right)}{\left(D_z^{\lambda,p}\right)^n f_t\left(z\right)}-\beta e^{i\alpha}\right\}\geq\gamma$$

is equivalent to

$$Re\left\{\frac{\begin{array}{c}(1-\gamma)z^p-\sum_{k=2}^{\infty}[(\varphi(k,p,\lambda))^m(1+\beta e^{i\alpha})-(\beta e^{i\alpha}+\gamma)(\varphi(k,p,\lambda))^n t]\\ |a_{k+p-1}|z^{k+p-1}+(-1)^{2m-1}\\ \sum_{k=1}^{\infty}[(\varphi(k,p,\lambda))^m(1+\beta e^{i\alpha})-(-1)^{m-n}(\beta e^{i\alpha}+\gamma)(\varphi(k,p,\lambda))^n t]\\ |b_{k+p-1}|\bar{z}^{k+p-1}\end{array}}{\begin{array}{c}z-\sum_{k=2}^{\infty}\left(\varphi\left(k,p,\lambda\right)\right)^n t\left|a_{k+p-1}\right|z^{k+p-1}+(-1)^{m+n-1}\\ \sum_{k=1}^{\infty}\left(\varphi\left(k,p,\lambda\right)\right)^n t\left|b_{k+p-1}\right|\bar{z}^{k+p-1}\end{array}}\right\}\geq 0. \tag{2.4}$$

The above required condition (2.4) must hold for all values of z in $U$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$Re\left\{\frac{\begin{array}{c}(1-\gamma)-\sum_{k=2}^{\infty}\left[(\varphi(k,p,\lambda))^{m}-\gamma t(\varphi(k,p,\lambda))^{n}\right]|a_{k+p-1}|r^{k-1}\\ -\sum_{k=1}^{\infty}\left[(\varphi(k,p,\lambda))^{m}-(-1)^{m-n}\gamma t(\varphi(k,p,\lambda))^{n}\right]|b_{k+p-1}|r^{k-1}\end{array}}{1-\sum_{k=2}^{\infty}(\varphi(k,p,\lambda))^{n}t|a_{k+p-1}|r^{k-1}-(-1)^{m-n}\sum_{k=1}^{\infty}(\varphi(k,p,\lambda))^{n}t|b_{k+p-1}|r^{k-1}}\right.$$

$$\left.-e^{i\alpha}\frac{\begin{array}{c}\sum_{k=2}^{\infty}\beta\left((\varphi(k,p,\lambda))^{m}-t(\varphi(k,p,\lambda))^{n}\right)|a_{k+p-1}|r^{k-1}\\ -\sum_{k=1}^{\infty}\beta\left((\varphi(k,p,\lambda))^{m}-(-1)^{m-n}t(\varphi(k,p,\lambda))^{n}\right)|b_{k+p-1}|r^{k-1}\end{array}}{\begin{array}{c}1-\sum_{k=2}^{\infty}(\varphi(k,p,\lambda))^{n}t|a_{k+p-1}|r^{k-1}-(-1)^{m-n}\\ \sum_{k=1}^{\infty}(\varphi(k,p,\lambda))^{n}t|b_{k+p-1}|r^{k-1}\end{array}}\right\}\geq 0.$$

Since $Re\left(-e^{i\alpha}\right) \geq -\left|e^{i\alpha}\right| = -1$, the above inequality reduces to

$$\frac{\begin{array}{c}(1-\gamma)-\sum_{k=2}^{\infty}\left[(\varphi(k,p,\lambda))^{m}(1+\beta)-(\beta+\gamma)(\varphi(k,p,\lambda))^{n}t\right]|a_{k+p-1}|r^{k-1}\\ -\sum_{k=1}^{\infty}\left[(\varphi(k,p,\lambda))^{m}(1+\beta)-(-1)^{m-n}(\beta+\gamma)(\varphi(k,p,\lambda))^{n}t\right]|b_{k+p-1}|r^{k-1}\end{array}}{1-\sum_{k=2}^{\infty}(\varphi(k,p,\lambda))^{n}t|a_{k+p-1}|r^{k-1}-(-1)^{m-n}\sum_{k=1}^{\infty}(\varphi(k,p,\lambda))^{n}t|b_{k+p-1}|r^{k-1}}\geq 0. \tag{2.5}$$

If the condition (2.3) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Hence there exists a $z_0 = r_0$ in (0,1) for which the quotient in (2.5) is negative. This contradicts the required condition for $f_m \in \overline{H_p}(m, n; \beta; \gamma; t; \lambda)$ and so the proof is complete. □

We employ the techniques of Dixit et al. [9] in the proofs of Theorems 2.3, 2.4, 2.6 and 2.7.

**Theorem 2.3.** *Let $f_m$ be given by (1.12). Then $f_m \in \overline{H_p}(m, n; \beta; \gamma; t; \lambda)$, if and only if*

$$f_m(z) = \sum_{k=1}^{\infty} \left( x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{m_{k+p-1}}(z) \right), \text{ where } h_p(z) = z^p,$$

$$h_{k+p-1}(z) = z^p - \frac{1}{\psi(m,n,\beta,\gamma,t,\lambda)} z^{k+p-1}, (k = 2,3,4....), g_{m_{k+p-1}}(z) = z^p + (-1)^{m-1}$$

$$\frac{1}{\Theta(m,n,\beta,\gamma,t,\lambda)} \bar{z}^{k+p-1}, (k = 1,2,3....), x_{k+p-1} \geq 0,$$

$$y_{k+p-1} \geq 0, \sum_{k=1}^{\infty} (x_{k+p-1} + y_{k+p-1}) = 1.$$

*In particular, the extreme points of $\overline{H_p}(m, n; \beta; \gamma; t; \lambda)$ are $\{h_{k+p-1}\}$ and $\{g_{m_{k+p-1}}\}$.*

**Theorem 2.4.** *Let $f_m \in \overline{H_p}(m, n; \beta; \gamma; t; \lambda)$. Then for $|z| = r < 1$ we have*

$$|f_m(z)| \leq (1 + |b_p|) r^p + \frac{1}{\left(\frac{(p+1)^2}{p(p-\lambda+1)}\right)^n}$$

$$\left( \frac{1-\gamma}{\left(\frac{(p+1)^2}{p(p-\lambda+1)}\right)^{m-n} (1+\beta) - t(\gamma+\beta)} - \frac{(1+\beta) - (-1)^{m-n} t(\gamma+\beta)}{\left(\frac{(p+1)^2}{p(p-\lambda+1)}\right)^{m-n} (1+\beta) - t(\gamma+\beta)} |b_p| \right) r^{p+1}$$

*and*

$$|f_m(z)| \geq (1 - |b_p|) r^p - \frac{1}{\left(\frac{(p+1)^2}{p(p-\lambda+1)}\right)^n}$$

$$\left( \frac{1-\gamma}{\left(\frac{(p+1)^2}{p(p-\lambda+1)}\right)^{m-n} (1+\beta) - t(\gamma+\beta)} - \frac{(1+\beta) - (-1)^{m-n} t(\gamma+\beta)}{\left(\frac{(p+1)^2}{p(p-\lambda+1)}\right)^{m-n} (1+\beta) - t(\gamma+\beta)} |b_p| \right) r^{p+1}.$$

*The following covering result follows from the left hand inequality in Theorem 2.4.*

**Corollary 2.5.** *Let $f_m$ of the form (1.12) be so that $f_m \in \overline{H_p}(m, n; \beta; \gamma; t; \lambda)$. Then*

$$\left\{ \omega : |\omega| < \frac{(\varphi(p,\lambda))^m (1+\beta) - (\varphi(p,\lambda))^n (\gamma+\beta) t - 1 + \gamma}{(\varphi(p,\lambda))^m (1+\beta) - (\varphi(p,\lambda))^n (\gamma+\beta) t} \right.$$
$$\left. - \frac{((\varphi(p,\lambda))^m - 1)(1+\beta) - t(\gamma+\beta)\left((\varphi(p,\lambda))^n - (-1)^{m-n}\right)}{(\varphi(p,\lambda))^m (1+\beta) - (\varphi(p,\lambda))^n (\gamma+\beta) t} |b_p| \subset f_m(U) \right\},$$

*where $\phi(p,\lambda) = \left\{ \frac{(p+1)^2}{p(p-\lambda+1)} \right\}$.*

*For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form*

$$f_m(z) = z^p - \sum_{k=2}^{\infty} |a_{k+p-1}| z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} |b_{k+p-1}| \bar{z}^{k+p-1}$$

*and*

$$F_m(z) = z^p - \sum_{k=2}^{\infty} |A_{k+p-1}| z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} |B_{k+p-1}| \bar{z}^{k+p-1}$$

*we define the convolution of two harmonic functions* $f_m$ *and* $F_m$ *as*

$$\begin{aligned}(f_m * F_m)(z) &= f_m(z) * F_m(z) \\ &= z^p - \sum_{k=2}^{\infty} |a_{k+p-1}A_{k+p-1}| z^{k+p-1} + (-1)^{m-1} \\ &\quad \sum_{k=1}^{\infty} |b_{k+p-1}B_{k+p-1}| \bar{z}^{k+p-1}.\end{aligned} \tag{2.6}$$

*Using this definition, we show that the class* $\overline{H_p}(m,n;\beta;\gamma;t;\lambda)$ *is closed under convolution.*

**Theorem 2.6.** *For* $0 \le \gamma_1 \le \gamma_2 < 1$ *let* $f_m \in \overline{H_p}(m,n;\beta;\gamma_1;t;\lambda)$ *and* $F_m \in \overline{H_p}(m,n;\beta;\gamma_2;t;\lambda)$. *Then* $f_m * F_m \in \overline{H_p}(m,n;\beta;\gamma_2;t;\lambda) \subseteq \overline{H_p}(m,n;\beta;\gamma_1;t;\lambda)$.

*Next, we show that* $\overline{H}_p(m,n,\beta,\gamma,t,\lambda)$ *is closed under convex combinations of its members.*

**Theorem 2.7.** *The class* $\overline{H_p}(m,n;\beta;\gamma;t;\lambda)$ *is closed under convex combination.*

## Acknowledgement

The authors are thankful to referee for his valuable comments and suggestions.

## References

[1] O.P. Ahuja, Planar harmonic univalent and related mappings, J. Inequal. Pure Appl. Math., 6 (4) (2005), Art.122, 1-18.

[2] O.P. Ahuja, R. Aghalary and S.B. Joshi, Harmonic univalent functions associated with $k$-Uniformly starlike functions, J. Math. Res. Sci., 9 (1) (2005), 9-17.

[3] O.P. Ahuja and J.M. Jahangiri, Multivalent harmonic starlike functions, Ann. Univ. Marie Curie-Sklodowska Sect. A, LV1 (2001), 1-13.

[4] O.P. Ahuja and J.M. Jahangiri, Noshiro-type harmonic univalent functions, Sci. Math. Japon., 6 (2) (2002), 253-259.

[5] M.K. Aouf, A generalization of multivalent functions with negative coefficients II, Bull. Korean. Math. Soc., 25 (2) (1998), 221-232.

[6] Y. Avci and E. Zlotkiewicz, On harmonic univalent mappings, Ann. Univ. Mariae Curie-Sklodowska Sect. A 44 (1990), 1-7.

[7] J. Clunie and T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser.AI Math 9 (1984), 3-25.

[8] K. K. Dixit, A. L. Pathak, S. Porwal and R. Agarwal, A new subclass of harmonic univalent functions defined by Salagean operator, Int. J. Contemp. Math. Sci., 4 (8) (2009), 371-383.

[9] K.K. Dixit, A.L. Pathak and R. Tripathi, A new subclass of Goodman-Rønning type harmonic multivalent functions, Proc. Int. Sym. GFTA Malaysia, (2008), 79-94.

[10] K.K. Dixit and Saurabh Porwal, An application of fractional calculus to harmonic univalent functions, Bull. Cal. Math. Soc., 102(4), (2010), 343-352.

[11] P. Duren, Harmonic Mappings in the Plane, Cambridge Tracts in Mathematics, Vol.156, Cambridge University Press, Cambridge,2004, ISBN 0-521-64121-7.

[12] J.M. Jahangiri, Harmonic functions starlike in the unit disc, J. Math. Anal. Appl. 235, (1999) 470-477.
[13] J.M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya, Salagean-type harmonic univalent functions, Southwest J. Pure Appl. Math., 2 (2002), 77-82.
[14] S. Kanas and H.M. Srivastava, Linear operators associated with $k$-Uniformly convex functions, Integral Transform Spec. Funct., 9 (2) (2000), 121-132.
[15] Y.C. Kim, J.M. Jahangiri and J.H. Choi, Certain convex harmonic functions, Int. J. Math. Math. Sci., 29 (8) (2002), 459-465.
[16] V. Kumar, Saurabh Porwal and Poonam Dixit, A New Subclass of Harmonic Univalent Functions Defined by Fractional Calculus, Ind. J. Math., 52(3), (2010), (Accepted).
[17] S. Owa, On the distortion theorem I, Kyungpook Math. J., 18 (1978), 53-59.
[18] T. Rosy, B.A. Stephen, K.G. Subramanian and J.M. Jahangiri, Goodman-Ronning-type harmonic univalent functions, Kyungpook Math. J., 41 (1) (2001), 45-54.
[19] G.S. Salagean, Subclasses of univalent functions, Complex Analysis-Fifth Romanian Finish Seminar, Bucharest, 1 (1983), 362-372.
[20] Bilal Seker and Sevtap Sümer Eker, On Salagean-type harmonic multivalent functions, General Mathematics, 15(2-3) (2007), 52-63.
[21] H. Silverman, Harmonic univalent functions with negative coefficients, J. Math. Anal. Appl., 220 (1998), 283-289.
[22] H. Silverman and E.M. Silvia, Subclasses of Harmonic univalent functions, New Zealand J. Math., 28 (1999), 275-284.
[23] H.M. Srivastava and S. Owa, An application of the fractional derivative, Math. Japon., 29 (1984), 383-389.
[24] Sibel Yalcin, A new class of Salagean-type harmonic univalent functions, Appl. Math. Lett., 18 (2005), 191-198.
[25] S. Yalcin, M. Öztürk and M. Yamankaradeniz, On the subclass of Salagean-type harmonic univalent functions, J. Inequl. Pure Appl. Math., 8 (2)(2007), Art. 54, 1-17.

[1]Department of Mathematics, Janta College, Bakewar, Etawah (U.P.) India – 206124

[2,3]Department of Mathematics, Christ Church College, Kanpur (U.P.) India – 208001

*E-mail address*: saurabhjcb@rediffmail.com