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OSCILLATION OF SECOND-ORDER QUASI-LINEAR NEUTRAL FUNCTIONAL DYNAMIC EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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ABSTRACT. In this paper, some sufficient conditions for the oscillation of second-order nonlinear neutral functional dynamic equation

$$\big(r(t)\big([x(t)+p(t)x[\tau(t)]]^{\Delta}\big)^{\gamma}\big)^{\Delta}+\int_a^b q(t,\xi)x^{\gamma}[g(t,\xi)]\Delta\xi=0,\ t\in\mathbb{T}$$

are established. An example is given to illustrate an application of our results.

1. INTRODUCTION

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph. D. Thesis in 1988 [12] in order to unify continuous and discrete analysis. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and difference equations.

The theory of "dynamic equations" unifies the theories of differential and difference equations and it also extends these classical cases to cases "in between". Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [1] and the references cited therein. The books on the subject of time scales, i.e., measure chain, by Bohner and Peterson [4, 5], summarize and organize much of time scale calculus.

Recently, there has been much research activity concerning the oscillation and non-oscillation of solutions of various dynamic equations on time scales, e.g.,

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see [2, 6–11, 13–29] and the references cited therein. Agarwal et al. [2], Saker [18], Tripathy [26] established some oscillation criteria for second-order nonlinear neutral delay dynamic equation

$$\big(r(t)\big([x(t)+p(t)x(t-\tau)]^{\Delta}\big)^{\gamma}\big)^{\Delta}+f\big(t,x(t-\delta)\big)=0, \tag{1.1}$$

where $r(t)>0,\ p(t)\geq 0$, and $f(t,u)\mathrm{sgn}u\geq q(t)|u|^{\alpha}$ such that $r,\ p$ and q are real valued rd-continuous functions defined on $\mathbb{T}$. We note that the results established in [2, 18] also require that

$$0\leq p(t)<1. \tag{1.2}$$

To the best of our knowledge, nothing is known regarding the oscillation of (1.1) under the case when $0\leq p(t)\leq a<\infty$ other than the work in [26]. In [26], the author obtained some new oscillation criteria for (1.1) under the cases $\gamma\geq 1,\ \delta\geq\tau$, and $r^{\Delta}(t)\geq 0$. Chen [6], Şahiner [16], Saker et al. [19, 21], Saker and O'Regan [22], Wu et al. [27], Zhang and Wang [28] considered the second-order nonlinear neutral dynamic equation with variable delays

$$\big(r(t)\big([x(t)+p(t)x[\tau(t)]]^{\Delta}\big)^{\gamma}\big)^{\Delta}+f\big(t,x[\delta(t)]\big)=0, \tag{1.3}$$

where the case (1.2) holds. Han et al. [10] and Saker et al. [24] examined the oscillation of (1.3) when $\gamma=1$. In particular, Han et al. [10] investigated the case where $\gamma=1$ and $p\in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}},[0,p_0])$, where p_0 is a constant. Regarding the oscillation of dynamic equations with distributed deviating arguments, Thandapani and Piramanantham [25] studied the equation

$$\big(r(t)\big([x(t)+p(t)x[\tau(t)]]^{\Delta}\big)^{\gamma}\big)^{\Delta}+\int_a^b q(t,\xi)f\big(x[g(t,\xi)]\big)\Delta\xi=0. \tag{1.4}$$

Chen and Liu [7] examined the oscillation behavior of the third-order equation

$$\Big(B(t)\big(A(t)(x(t)+p(t)x[\tau(t)])^{\Delta}\big)^{\Delta}\Big)^{\Delta}+\int_a^b q(t,\xi)f\big(t,\xi,x[g(t,\xi)]\big)\Delta\xi=0. \tag{1.5}$$

Following [10, 25, 26], we shall consider the oscillatory behavior of equation

$$\big(r(t)\big([x(t)+p(t)x[\tau(t)]]^{\Delta}\big)^{\gamma}\big)^{\Delta}+\int_a^b q(t,\xi)x^{\gamma}[g(t,\xi)]\Delta\xi=0,\ t\in\mathbb{T}. \tag{1.6}$$

Since we are interested in oscillatory behavior of solutions, we will suppose that the time scale $\mathbb{T}$ under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. Throughout the paper we assume that: $\gamma\geq 1$ is a ratio of odd positive integers,

(H_1) $r\in \mathrm{C_{rd}}\big([t_0,\infty)_{\mathbb{T}},(0,\infty)\big)$, $\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t=\infty$, $q(t,\xi)\in \mathrm{C_{rd}}\big([t_0,\infty)_{\mathbb{T}}\times[a,b]_{\mathbb{T}},\mathbb{T}\big)$ and $q(t,\xi)>0$;

(H_2) $\tau\in \mathrm{C^1_{rd}}\big([t_0,\infty)_{\mathbb{T}},\mathbb{T}\big)$, $\tau^{\Delta}(t)\geq\tau_0>0$, here τ_0 is a constant, $\tau\big([t_0,\infty)_{\mathbb{T}}\big)=[\tau(t_0),\infty)_{\mathbb{T}}$;

(H_3) $\tau[g(t,\xi)]=g[\tau(t),\xi]$ for $(t,\xi)\in[t_0,\infty)_{\mathbb{T}}\times[a,b]_{\mathbb{T}}$;

(H_4) $p\in \mathrm{C_{rd}}\big([t_0,\infty)_{\mathbb{T}},[0,a_0]\big)$, where a_0 is a constant;

(H_5) $a,b\in[t_0,\infty)_{\mathbb{T}}$, $g(t,\xi)\in \mathrm{C_{rd}}\big([t_0,\infty)_{\mathbb{T}}\times[a,b]_{\mathbb{T}},\mathbb{T}\big)$, $[a,b]_{\mathbb{T}}=\{t\in\mathbb{T}:a\leq t\leq b\}$, $g(t,a)\leq g(t,\xi)$ for $(t,\xi)\in[t_0,\infty)_{\mathbb{T}}\times[a,b]_{\mathbb{T}}$, and $\lim_{t\to\infty}g(t,a)=\infty$.

We set $z(t) := x(t) + p(t)x[\tau(t)]$. By a solution of (1.6), we mean a nontrivial real-valued function x which has the properties $z \in \mathrm{C}^1_{\mathrm{rd}}([t_x,\infty)_{\mathbb{T}},\mathbb{R})$ and $r(z^\Delta)^\gamma \in \mathrm{C}^1_{\mathrm{rd}}([t_x,\infty)_{\mathbb{T}},\mathbb{R})$, $t_x \in [t_0,\infty)_{\mathbb{T}}$ and satisfying (1.6) for all $t \in [t_x,\infty)_{\mathbb{T}}$. Our attention is restricted to those solutions of (1.6) which exist on some half-line $[t_x,\infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t \in [t_1,\infty)_{\mathbb{T}}\} > 0$ for any $t_1 \in [t_x,\infty)_{\mathbb{T}}$. A solution x of (1.6) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non-oscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Our purpose in this paper is to derive some new criteria for the oscillation of (1.6). This paper is organized as follows: In the next section, we present some conceptions on time scales. In Section 3, we give some lemmas. In Section 4, we will use the Riccati transformation technique to establish some oscillation results for the case where $g(t,a) \geq \tau(t)$. In Section 5, we shall establish some oscillation criteria under the case when $g(t,a) \leq \tau^\sigma(t)$. In Section 6, we will give an example and a remark to illustrate our main results.

2. Some preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$. On any time scale we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}, \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} | s < t\}.$$

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may actually be replaced by any Banach space), the (delta) derivative is defined by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is not right-scattered then the derivative is defined by

$$f^\Delta(t) = \lim_{s\to t^+} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s\to t^+} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$.

f is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$.

The derivative and the shift operator σ are related by the formula

$$f^\sigma(t) = f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

Let f be a real-valued function defined on an interval $[a, b]$. We say that f is increasing, decreasing, nondecreasing, and non-increasing on $[a, b]$ if $t_1,\ t_2 \in [a, b]$ and $t_2 > t_1$ imply $f(t_2) > f(t_1),\ f(t_2) < f(t_1),\ f(t_2) \geq f(t_1)$ and $f(t_2) \leq f(t_1)$, respectively. Let f be a differentiable function on $[a, b]$. Then f is increasing, decreasing, nondecreasing, and non-increasing on $[a, b]$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a, b)$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $g(t)g(\sigma(t)) \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

For $a,\ b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^{\infty} f(s)\Delta s = \lim_{t\to\infty}\int_a^t f(s)\Delta s.$$

3. Some lemmas

Below, we give the following lemmas, which we will use in the proofs of our main results.

Lemma 3.1. [4, Theorem 1.93] *Assume that $\upsilon : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \upsilon(\mathbb{T})$ is a time scale. Let $y : \tilde{\mathbb{T}} \to \mathbb{R}$. If $y^{\tilde{\Delta}}[\upsilon(t)]$ and $\upsilon^{\Delta}(t)$ exist for $t \in \mathbb{T}^k$, then*

$$(y[\upsilon(t)])^{\Delta} = y^{\tilde{\Delta}}[\upsilon(t)]\upsilon^{\Delta}(t).$$

Remark 3.2. In condition (H_2), we assume condition $\tau\big([t_0, \infty)_{\mathbb{T}}\big) = [\tau(t_0), \infty)_{\mathbb{T}}$ which indicates that $\tau(\mathbb{T}) \subseteq \mathbb{T}$ is a time scale, and the derivative in $\mathbb{T}$ is the same as that in $\tau(\mathbb{T})$.

Lemma 3.3. [4, Theorem 1.90] *Assume that $y \in \mathrm{C}^1_{\mathrm{rd}}\big([t_0, \infty)_{\mathbb{T}}, \mathbb{R}\big)$. Then*

$$(y^{\gamma}(t))^{\Delta} = \gamma y^{\Delta}(t)\int_0^1 \big[hy^{\sigma}(t) + (1-h)y(t)\big]^{\gamma-1}\mathrm{d}h. \tag{3.1}$$

4. Oscillation results for the case when $g(t,a) \ge \tau(t)$

In this section, we will establish some oscillation criteria for (1.6) under the case where $g(t,a) \ge \tau(t)$ for $t \in [t_0,\infty)_{\mathbb{T}}$.

Theorem 4.1. *Assume that (H_1)–(H_5) hold, $\tau(t) \le \sigma(t)$, and $g(t,a) \ge \tau(t)$ for $t \in [t_0,\infty)_{\mathbb{T}}$. Suppose further that there exists a positive function $\eta \in \mathrm{C}^1_{\mathrm{rd}}\big([t_0,\infty)_{\mathbb{T}},\mathbb{R}\big)$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \eta(s) \left\{ \frac{\int_a^b Q(s,\xi)\Delta\xi}{2^{\gamma-1}} - \frac{1+\frac{a_0{}^\gamma}{\tau_0}}{(\gamma+1)^{\gamma+1}} \frac{r[\tau(s)]((\eta^\Delta(s))_+)^{\gamma+1}}{\tau_0^\gamma \eta^{\gamma+1}(s)} \right\} \Delta s = \infty, \quad (4.1)$$

where $Q(t,\xi) := \min\{q(t,\xi), q(\tau(t),\xi)\}$, $(\eta^\Delta(t))_+ := \max\{0,\eta^\Delta(t)\}$. Then (1.6) is oscillatory.

Proof. Let x be a non-oscillatory solution of (1.6). Without loss of generality, we assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x[\tau(t)] > 0$, and $x[g(t,\xi)] > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$ and $\xi \in [a,b]_{\mathbb{T}}$. From (1.6) and (H_1), we see that there exists $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that

$$\big(r(t)(z^\Delta(t))^\gamma\big)^\Delta < 0, \ z^\Delta(t) > 0, \ t \in [t_2,\infty)_{\mathbb{T}}. \quad (4.2)$$

Using (1.6) and Lemma 3.1, for all sufficiently large t, we obtain

$$\int_a^b q(\tau(t),\xi) x^\gamma[g(\tau(t),\xi)]\Delta\xi + \frac{\big(r[\tau(t)](z^\Delta[\tau(t)])^\gamma\big)^\Delta}{\tau^\Delta(t)} = 0,$$

and so

$$\big(r(t)(z^\Delta(t))^\gamma\big)^\Delta + \int_a^b q(t,\xi)x^\gamma[g(t,\xi)]\Delta\xi + a_0{}^\gamma \int_a^b q(\tau(t),\xi)x^\gamma[g(\tau(t),\xi)]\Delta\xi + \frac{a_0{}^\gamma}{\tau_0}\big(r[\tau(t)](z^\Delta[\tau(t)])^\gamma\big)^\Delta \le 0.$$

By applying inequality

$$c^\gamma + d^\gamma \ge \frac{1}{2^{\gamma-1}}(c+d)^\gamma \text{ for } c \ge 0, \ d \ge 0, \ \gamma \ge 1,$$

(H_3)–(H_5) and the definitions of Q and z, we conclude that

$$\big(r(t)(z^\Delta(t))^\gamma\big)^\Delta + \frac{a_0{}^\gamma}{\tau_0}\big(r[\tau(t)](z^\Delta[\tau(t)])^\gamma\big)^\Delta + \frac{z^\gamma[g(t,a)]}{2^{\gamma-1}} \int_a^b Q(t,\xi)\Delta\xi \le 0. \quad (4.3)$$

Next, we define a Riccati substitution

$$\omega(t) := \eta(t)\frac{r(t)(z^\Delta(t))^\gamma}{(z[\tau(t)])^\gamma}, \ t \in [t_2,\infty)_{\mathbb{T}}. \quad (4.4)$$

Then $\omega(t) > 0$. From (4.2) and condition $\tau(t) \le \sigma(t)$, we have

$$z^\Delta[\tau(t)] \ge (r^\sigma(t)/r[\tau(t)])^{1/\gamma} z^{\Delta\sigma}(t). \quad (4.5)$$

From (4.4), we obtain

$$\omega^{\Delta}(t) = \left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\sigma}\left[\frac{\eta(t)}{(z[\tau(t)])^{\gamma}}\right]^{\Delta} + \frac{\eta(t)}{(z[\tau(t)])^{\gamma}}\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\Delta}.$$

Thus

$$\begin{aligned}\omega^{\Delta}(t) &= \frac{\eta(t)}{(z[\tau(t)])^{\gamma}}\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\Delta} \\ &\quad + \left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\sigma}\frac{\eta^{\Delta}(t)(z[\tau(t)])^{\gamma} - \eta(t)[(z[\tau(t)])^{\gamma}]^{\Delta}}{(z[\tau(t)])^{\gamma}(z[\tau^{\sigma}(t)])^{\gamma}}. \qquad (4.6)\end{aligned}$$

Using (3.1), (H_2), Lemma 3.1 and Lemma 3.3, we get

$$\left[(z[\tau(t)])^{\gamma}\right]^{\Delta} \geq \gamma[z[\tau(t)]]^{\gamma-1}z^{\Delta}[\tau(t)]\tau^{\Delta}(t). \qquad (4.7)$$

Then, we have by (4.4), (4.5), (4.6) and (4.7) that

$$\begin{aligned}\omega^{\Delta}(t) &\leq \frac{\eta(t)}{(z[\tau(t)])^{\gamma}}\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\Delta} + \frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}\omega^{\sigma}(t) \\ &\quad -\gamma\frac{\eta(t)\tau^{\Delta}(t)}{r^{1/\gamma}[\tau(t)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}(\omega^{\sigma}(t))^{(\gamma+1)/\gamma}. \qquad (4.8)\end{aligned}$$

On the other hand, we define another function u by

$$u(t) := \eta(t)\frac{r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}}{(z[\tau(t)])^{\gamma}},\ t \in [t_2, \infty)_{\mathbb{T}}. \qquad (4.9)$$

Then $u(t) > 0$. From (4.2), we have

$$z^{\Delta}[\tau(t)] \geq \left(r[\tau^{\sigma}(t)]/r[\tau(t)]\right)^{1/\gamma} z^{\Delta}[\tau^{\sigma}(t)]. \qquad (4.10)$$

By virtue of (4.9), we obtain

$$u^{\Delta}(t) = \left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\sigma}\left[\frac{\eta(t)}{(z[\tau(t)])^{\gamma}}\right]^{\Delta} + \frac{\eta(t)}{(z[\tau(t)])^{\gamma}}\left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\Delta}.$$

Thus

$$\begin{aligned}u^{\Delta}(t) &= \frac{\eta(t)}{(z[\tau(t)])^{\gamma}}\left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\Delta} \\ &\quad + \left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\sigma}\frac{\eta^{\Delta}(t)(z[\tau(t)])^{\gamma} - \eta(t)[(z[\tau(t)])^{\gamma}]^{\Delta}}{(z[\tau(t)])^{\gamma}(z[\tau^{\sigma}(t)])^{\gamma}}. \qquad (4.11)\end{aligned}$$

Using (3.1) and Lemma 3.1, we get (4.7). Hence by (4.7), (4.9), (4.10) and (4.11), we find that

$$\begin{aligned}u^{\Delta}(t) &\leq \frac{\eta(t)}{(z[\tau(t)])^{\gamma}}\left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\Delta} \\ &\quad +\frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}u^{\sigma}(t) - \gamma\frac{\eta(t)\tau^{\Delta}(t)}{r^{1/\gamma}[\tau(t)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}(u^{\sigma}(t))^{(\gamma+1)/\gamma}. \qquad (4.12)\end{aligned}$$

Thus, we have by (4.8) and (4.12) that

$$
\begin{aligned}
&\omega^{\Delta}(t)+\frac{a_0{}^{\gamma}}{\tau_0}u^{\Delta}(t)\\
\leq\ &\eta(t)\frac{\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\Delta}+\frac{a_0{}^{\gamma}}{\tau_0}\left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\Delta}}{(z[\tau(t)])^{\gamma}}+\frac{(\eta^{\Delta}(t))_{+}}{\eta^{\sigma}(t)}\omega^{\sigma}(t)\\
&-\gamma\frac{\eta(t)\tau^{\Delta}(t)}{r^{1/\gamma}[\tau(t)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}(\omega^{\sigma}(t))^{(\gamma+1)/\gamma}\\
&+\frac{a_0{}^{\gamma}}{\tau_0}\left\{\frac{(\eta^{\Delta}(t))_{+}}{\eta^{\sigma}(t)}u^{\sigma}(t)-\gamma\frac{\eta(t)\tau^{\Delta}(t)}{r^{1/\gamma}[\tau(t)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}(u^{\sigma}(t))^{(\gamma+1)/\gamma}\right\}.
\end{aligned}
$$

Therefore, by (4.3), conditions $g(t,a)\geq\tau(t)$ and $\tau^{\Delta}(t)\geq\tau_0>0$, we have

$$
\begin{aligned}
&\omega^{\Delta}(t)+\frac{a_0{}^{\gamma}}{\tau_0}u^{\Delta}(t)\\
\leq\ &-\ \frac{\int_a^b Q(t,\xi)\Delta\xi}{2^{\gamma-1}}\eta(t)+\frac{(\eta^{\Delta}(t))_{+}}{\eta^{\sigma}(t)}\omega^{\sigma}(t)\\
&-\ \gamma\frac{\tau_0\eta(t)}{r^{1/\gamma}[\tau(t)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}(\omega^{\sigma}(t))^{(\gamma+1)/\gamma}\\
&+\ \frac{a_0{}^{\gamma}}{\tau_0}\left\{\frac{(\eta^{\Delta}(t))_{+}}{\eta^{\sigma}(t)}u^{\sigma}(t)-\gamma\frac{\tau_0\eta(t)(u^{\sigma}(t))^{(\gamma+1)/\gamma}}{r^{1/\gamma}[\tau(t)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}\right\}. \qquad (4.13)
\end{aligned}
$$

In view of (4.13) and inequality

$$
Bu-Au^{(\gamma+1)/\gamma}\leq\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}},\ A>0, \qquad (4.14)
$$

we get

$$
\begin{aligned}
&\omega^{\Delta}(t)+\frac{a_0{}^{\gamma}}{\tau_0}u^{\Delta}(t)\\
\leq\ &\eta(t)\left\{-\frac{\int_a^b Q(t,\xi)\Delta\xi}{2^{\gamma-1}}+\frac{1}{(\gamma+1)^{\gamma+1}}\left(1+\frac{a_0{}^{\gamma}}{\tau_0}\right)\frac{r[\tau(t)]((\eta^{\Delta}(t))_{+})^{\gamma+1}}{\tau_0^{\gamma}\eta^{\gamma+1}(t)}\right\}.
\end{aligned}
$$

Integrating the above inequality from t_2 to t, we obtain

$$
\begin{aligned}
&\int_{t_2}^{t}\eta(s)\left\{\frac{\int_a^b Q(s,\xi)\Delta\xi}{2^{\gamma-1}}-\frac{1}{(\gamma+1)^{\gamma+1}}\left(1+\frac{a_0{}^{\gamma}}{\tau_0}\right)\frac{r[\tau(s)]((\eta^{\Delta}(s))_{+})^{\gamma+1}}{\tau_0^{\gamma}\eta^{\gamma+1}(s)}\right\}\Delta s\\
\leq\ &\omega(t_2)+\frac{a_0{}^{\gamma}}{\tau_0}u(t_2),
\end{aligned}
$$

which contradicts (4.1). This completes the proof. □

In view of Theorem 4.1, we can obtain different conditions for oscillation of all solutions of (1.6) with different choices of η. For example, if $\eta(t)=1$ and $\eta(t)=t$ for $t\in[t_0,\infty)_{\mathbb{T}}$, we have the following results, respectively.

Corollary 4.2. *Assume that* (H_1)–(H_5) *hold,* $\tau(t) \leq \sigma(t)$*, and* $g(t,a) \geq \tau(t)$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$*. If*

$$\int_{t_0}^{\infty} \int_a^b Q(s,\xi)\Delta\xi\Delta s = \infty, \tag{4.15}$$

where Q *is defined as in Theorem 4.1, then* (1.6) *is oscillatory.*

Corollary 4.3. *Assume that* (H_1)–(H_5) *hold,* $\tau(t) \leq \sigma(t)$*, and* $g(t,a) \geq \tau(t)$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$*. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} s \left\{ \frac{\int_a^b Q(s,\xi)\Delta\xi}{2^{\gamma-1}} - \frac{1 + \frac{a_0{}^{\gamma}}{\tau_0}}{(\gamma+1)^{\gamma+1}} \frac{r[\tau(s)]}{\tau_0^{\gamma} s^{\gamma+1}} \right\} \Delta s = \infty, \tag{4.16}$$

where Q *is defined as in Theorem 4.1, then* (1.6) *is oscillatory.*

Now, we establish the following Philos-type oscillation criterion for the oscillation of (1.6).

Theorem 4.4. *Assume that* (H_1)–(H_5) *hold,* $\tau(t) \leq \sigma(t)$*, and* $g(t,a) \geq \tau(t)$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$*. Suppose also that there exist functions* H, $h \in \mathrm{C_{rd}}(\mathbb{D}, \mathbb{R})$*, where* $\mathbb{D} \equiv \{(t,s) : t \geq s \geq t_0\}$ *such that*

$$H(t,t) = 0,\ t \geq t_0,\ H(t,s) > 0,\ t > s \geq t_0, \tag{4.17}$$

and H *has a non-positive continuous* Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *with respect to the second variable and satisfies*

$$H^{\Delta_s}(t,s) + H(t,s)\frac{\eta(s)}{\eta^{\sigma}(s)} = -\frac{h(t,s)}{\eta^{\sigma}(s)}(H(t,s))^{\gamma/(\gamma+1)}, \tag{4.18}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s) \frac{\eta(s)\int_a^b Q(s,\xi)\Delta\xi}{2^{\gamma-1}} - \frac{1 + \frac{a_0{}^{\gamma}}{\tau_0}}{(\gamma+1)^{\gamma+1}} \frac{r[\tau(s)](h_-(t,s))^{\gamma+1}}{(\tau_0\eta(s))^{\gamma}} \right] \Delta s = \infty, \tag{4.19}$$

where η *is a positive* Δ*-differential function,* Q *is as in Theorem 4.1,* $h_-(t,s) := \max\{0, -h(t,s)\}$*. Then* (1.6) *is oscillatory.*

Proof. By (4.13), the proof is similar to the proof of Philos-type oscillation theorems by [3, 20, 21], so we omit the details. □

5. Oscillation results for the case when $g(t,a) \leq \tau^{\sigma}(t)$

In this section, we will establish some oscillation criteria for (1.6) under the case when $g(t,a) \leq \tau^{\sigma}(t)$ for $t \in [t_0, \infty)_{\mathbb{T}}$.

Theorem 5.1. *Assume that* (H_1)–(H_5) *hold,* $g(t,a) \leq \tau^{\sigma}(t)$, $g(t,a) \leq \sigma(t)$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$, $g(t,a) \in \mathrm{C^1_{rd}}\big([t_0,\infty)_{\mathbb{T}}, \mathbb{T}\big)$, $g^{\Delta}(t,a) > 0$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$*, and*

$g\big([t_0,\infty)_{\mathbb{T}},a)\big)=[g(t_0,a),\infty)_{\mathbb{T}}$. *Suppose further that there exists a positive function* $\eta\in \mathrm{C}^1_{\mathrm{rd}}\big([t_0,\infty)_{\mathbb{T}},\mathbb{R}\big)$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\eta(s)\left\{\frac{\int_a^b Q(s,\xi)\Delta\xi}{2^{\gamma-1}} - \frac{1+\frac{a_0{}^{\gamma}}{\tau_0}}{(\gamma+1)^{\gamma+1}}\frac{r[g(s,a)]((\eta^{\Delta}(s))_+)^{\gamma+1}}{(g^{\Delta}(s,a))^{\gamma}\eta^{\gamma+1}(s)}\right\}\Delta s=\infty, \quad (5.1)$$

where Q *and* $(\eta^{\Delta})_+$ *are defined as in Theorem 4.1. Then* (1.6) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (1.6). Without loss of generality, we assume that there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$, $x[\tau(t)]>0$, and $x[g(t,\xi)]>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$ and $\xi\in[a,b]_{\mathbb{T}}$. Proceeding as in the proof of Theorem 4.1, we have (4.2) and (4.3). We define a Riccati substitution

$$\omega(t):=\eta(t)\frac{r(t)(z^{\Delta}(t))^{\gamma}}{(z[g(t,a)])^{\gamma}},\ t\in[t_2,\infty)_{\mathbb{T}}. \quad (5.2)$$

Then $\omega(t)>0$. From (4.2) and condition $g(t,a)\le\sigma(t)$, we have

$$z^{\Delta}[g(t,a)]\ge\left(r^{\sigma}(t)/r[g(t,a)]\right)^{1/\gamma}z^{\Delta\sigma}(t). \quad (5.3)$$

By virtue of (5.2), we obtain

$$\omega^{\Delta}(t)=\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\sigma}\left[\frac{\eta(t)}{(z[g(t,a)])^{\gamma}}\right]^{\Delta}+\frac{\eta(t)}{(z[g(t,a)])^{\gamma}}\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\Delta}.$$

Thus

$$\begin{aligned}\omega^{\Delta}(t) &= \frac{\eta(t)}{(z[g(t,a)])^{\gamma}}\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\Delta}\\ &\quad+\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\sigma}\frac{\eta^{\Delta}(t)(z[g(t,a)])^{\gamma}-\eta(t)[(z[g(t,a)])^{\gamma}]^{\Delta}}{(z[g(t,a)])^{\gamma}(z[g(\sigma(t),a)])^{\gamma}}. \quad (5.4)\end{aligned}$$

Using (3.1), Lemma 3.1 and Lemma 3.3, we get

$$\left[(z[g(t,a)])^{\gamma}\right]^{\Delta}\ge\gamma[z[g(t,a)]]^{\gamma-1}z^{\Delta}[g(t,a)]g^{\Delta}(t,a). \quad (5.5)$$

Thus, by (5.2), (5.3), (5.4) and (5.5), we see that

$$\begin{aligned}\omega^{\Delta}(t) &\le \frac{\eta(t)}{(z[g(t,a)])^{\gamma}}\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\Delta}+\frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}\omega^{\sigma}(t)\\ &\quad-\gamma\frac{\eta(t)g^{\Delta}(t,a)}{r^{1/\gamma}[g(t,a)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}(\omega^{\sigma}(t))^{(\gamma+1)/\gamma}. \quad (5.6)\end{aligned}$$

On the other hand, we define another function u by

$$u(t):=\eta(t)\frac{r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}}{(z[g(t,a)])^{\gamma}},\ t\in[t_2,\infty)_{\mathbb{T}}. \quad (5.7)$$

Then $u(t)>0$. From (4.2) and condition $g(t,a)\le\tau^{\sigma}(t)$, we have

$$z^{\Delta}[g(t,a)]\ge\left(r[\tau^{\sigma}(t)]/r[g(t,a)]\right)^{1/\gamma}z^{\Delta}[\tau^{\sigma}(t)]. \quad (5.8)$$

From (5.7), we obtain

$$u^{\Delta}(t)=\left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\sigma}\left[\frac{\eta(t)}{(z[g(t,a)])^{\gamma}}\right]^{\Delta}+\frac{\eta(t)}{(z[g(t,a)])^{\gamma}}\left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\Delta}.$$

Thus

$$\begin{aligned} u^{\Delta}(t) &= \frac{\eta(t)}{(z[g(t,a)])^{\gamma}}\left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\Delta} \\ &\quad+\left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\sigma}\frac{\eta^{\Delta}(t)(z[g(t,a)])^{\gamma}-\eta(t)[(z[g(t,a)])^{\gamma}]^{\Delta}}{(z[g(\sigma(t),a)])^{\gamma}(z[g(t,a)])^{\gamma}} \end{aligned} \tag{5.9}$$

Applying (3.1) and Lemma 3.1, we get (5.5). Hence by (5.5), (5.7), (5.8), and (5.9), we find that

$$\begin{aligned} u^{\Delta}(t) &\le \frac{\eta(t)}{(z[g(t,a)])^{\gamma}}\left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\Delta} \\ &\quad+\frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}u^{\sigma}(t)-\gamma\frac{\eta(t)g^{\Delta}(t,a)}{r^{1/\gamma}[g(t,a)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}(u^{\sigma}(t))^{(\gamma+1)/\gamma}. \end{aligned} \tag{5.10}$$

Hence from (5.6) and (5.10), we obtain

$$\begin{aligned} &\omega^{\Delta}(t)+\frac{{a_0}^{\gamma}}{\tau_0}u^{\Delta}(t) \\ \le\ &\eta(t)\frac{\left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\Delta}+\frac{{a_0}^{\gamma}}{\tau_0}\left(r[\tau(t)](z^{\Delta}[\tau(t)])^{\gamma}\right)^{\Delta}}{(z[g(t,a)])^{\gamma}}+\frac{(\eta^{\Delta}(t))_{+}}{\eta^{\sigma}(t)}\omega^{\sigma}(t) \\ &-\gamma\frac{\eta(t)g^{\Delta}(t,a)}{r^{1/\gamma}[g(t,a)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}(\omega^{\sigma}(t))^{(\gamma+1)/\gamma} \\ &+\frac{{a_0}^{\gamma}}{\tau_0}\left\{\frac{(\eta^{\Delta}(t))_{+}}{\eta^{\sigma}(t)}u^{\sigma}(t)-\gamma\frac{\eta(t)g^{\Delta}(t,a)}{r^{1/\gamma}[g(t,a)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}(u^{\sigma}(t))^{(\gamma+1)/\gamma}\right\}. \end{aligned}$$

Therefore, (4.3) yields

$$\begin{aligned} &\omega^{\Delta}(t)+\frac{{a_0}^{\gamma}}{\tau_0}u^{\Delta}(t) \\ \le\ &-\ \frac{\int_a^b Q(t,\xi)\Delta\xi}{2^{\gamma-1}}\eta(t)+\frac{(\eta^{\Delta}(t))_{+}}{\eta^{\sigma}(t)}\omega^{\sigma}(t) \\ &-\ \gamma\frac{\eta(t)g^{\Delta}(t,a)}{r^{1/\gamma}[g(t,a)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}(\omega^{\sigma}(t))^{(\gamma+1)/\gamma} \\ &+\ \frac{{a_0}^{\gamma}}{\tau_0}\left\{\frac{(\eta^{\Delta}(t))_{+}}{\eta^{\sigma}(t)}u^{\sigma}(t)-\gamma\frac{\eta(t)g^{\Delta}(t,a)(u^{\sigma}(t))^{(\gamma+1)/\gamma}}{r^{1/\gamma}[g(t,a)](\eta^{\sigma}(t))^{(\gamma+1)/\gamma}}\right\}. \end{aligned} \tag{5.11}$$

In view of (4.14) and (5.11), we get

$$\begin{aligned} &\omega^{\Delta}(t)+\frac{{a_0}^{\gamma}}{\tau_0}u^{\Delta}(t) \\ \le\ &\eta(t)\left\{-\frac{\int_a^b Q(t,\xi)\Delta\xi}{2^{\gamma-1}}+\frac{1}{(\gamma+1)^{\gamma+1}}\left(1+\frac{{a_0}^{\gamma}}{\tau_0}\right)\frac{r[g(t,a)]((\eta^{\Delta}(t))_{+})^{\gamma+1}}{(g^{\Delta}(t,a))^{\gamma}\eta^{\gamma+1}(t)}\right\}. \end{aligned}$$

Integrating the above inequality from t_2 to t, we obtain

$$\int_{t_2}^{t} \eta(s) \left\{ \frac{\int_a^b Q(s,\xi)\Delta\xi}{2^{\gamma-1}} - \frac{1}{(\gamma+1)^{\gamma+1}} \left(1 + \frac{a_0{}^{\gamma}}{\tau_0}\right) \frac{r[g(s,a)]((\eta^{\Delta}(s))_+)^{\gamma+1}}{(g^{\Delta}(s,a))^{\gamma}\eta^{\gamma+1}(s)} \right\} \Delta s$$
$$\leq \omega(t_2) + \frac{a_0{}^{\gamma}}{\tau_0} u(t_2),$$

which contradicts (5.1). This completes the proof. $\square$

From Theorem 5.1, we can obtain different conditions for oscillation of all solutions of (1.6) with different choices of η. For example, if $\eta(t) = 1$ and $\eta(t) = t$ for $t \in [t_0, \infty)_{\mathbb{T}}$, we have the following results, respectively.

Corollary 5.2. *Assume that* (H_1)–(H_5) *hold,* $g(t,a) \leq \tau^{\sigma}(t)$, $g(t,a) \leq \sigma(t)$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$, $g(t,a) \in \mathrm{C}^1_{\mathrm{rd}}\big([t_0,\infty)_{\mathbb{T}}, \mathbb{T}\big)$, $g^{\Delta}(t,a) > 0$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$*, and* $g\big([t_0,\infty)_{\mathbb{T}}, a)\big) = [g(t_0,a),\infty)_{\mathbb{T}}$. *If* (4.15) *holds, where* Q *is defined as in Theorem 4.1, then* (1.6) *is oscillatory.*

Corollary 5.3. *Assume that* (H_1)–(H_5) *hold,* $g(t,a) \leq \tau^{\sigma}(t)$, $g(t,a) \leq \sigma(t)$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$, $g(t,a) \in \mathrm{C}^1_{\mathrm{rd}}\big([t_0,\infty)_{\mathbb{T}}, \mathbb{T}\big)$, $g^{\Delta}(t,a) > 0$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$*, and* $g\big([t_0,\infty)_{\mathbb{T}}, a)\big) = [g(t_0,a),\infty)_{\mathbb{T}}$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} s \left\{ \frac{\int_a^b Q(s,\xi)\Delta\xi}{2^{\gamma-1}} - \frac{1 + \frac{a_0{}^{\gamma}}{\tau_0}}{(\gamma+1)^{\gamma+1}} \frac{r[g(s,a)]}{(g^{\Delta}(s,a))^{\gamma} s^{\gamma+1}} \right\} \Delta s = \infty, \tag{5.12}$$

where Q *is defined as in Theorem 4.1, then* (1.6) *is oscillatory.*

Now, we derive the following Philos-type oscillation criterion for the oscillation of (1.6).

Theorem 5.4. *Assume that* (H_1)–(H_5) *hold,* $g(t,a) \leq \tau^{\sigma}(t)$, $g(t,a) \leq \sigma(t)$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$, $g(t,a) \in \mathrm{C}^1_{\mathrm{rd}}\big([t_0,\infty)_{\mathbb{T}}, \mathbb{T}\big)$, $g^{\Delta}(t,a) > 0$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$*, and* $g\big([t_0,\infty)_{\mathbb{T}}, a)\big) = [g(t_0,a),\infty)_{\mathbb{T}}$. *Suppose also that there exist functions* H, $h \in \mathrm{C}_{\mathrm{rd}}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,s) : t \geq s \geq t_0\}$ *such that* (4.17) *holds, and* H *has a non-positive continuous* Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *with respect to the second variable and satisfies* (4.18)*, and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} K(t,s)\Delta s = \infty, \tag{5.13}$$

where

$$K(t,s) = \frac{H(t,s)\eta(s)\int_a^b Q(s,\xi)\Delta\xi}{2^{\gamma-1}} - \frac{1 + \frac{a_0{}^{\gamma}}{\tau_0}}{(\gamma+1)^{\gamma+1}} \frac{r[g(s,a)](h_-(t,s))^{\gamma+1}}{(g^{\Delta}(s,a)\eta(s))^{\gamma}},$$

η *is a positive* Δ*-differential function,* Q *is defined as in Theorem 4.1,* $h_-(t,s) := \max\{0, -h(t,s)\}$. *Then* (1.6) *is oscillatory.*

Proof. By (5.11), the proof is similar to the proof of Philos-type oscillation theorems by [3, 20, 21], so we omit the details. $\square$

6. Application

Firstly, we consider the following example.

Example 6.1. Consider the second-order neutral functional differential equation

$$[x(t) + x(t-2\pi)]'' + \int_{-3\pi}^{0} x[t+\xi]\mathrm{d}\xi = 0, \ t \geq 1. \tag{6.1}$$

Let $\alpha = 1$, $a = -3\pi$, $b = 0$, $r(t) = 1$, $p(t) = 1$, $\tau(t) = t - 2\pi$, $q(t,\xi) = 1$, $g(t,\xi) = t + \xi$. Then $Q(t,\xi) = \min\{q(t,\xi), q(\tau(t),\xi)\} = 1$, $g'(t,a) = 1$, $g(t,a) = t - 3\pi \leq t + \xi$ for $\xi \in [-3\pi, 0]$ and $g(t,a) \leq \tau(t) \leq t$. Moreover, letting $\tau_0 = 1$, we see that equation (6.1) is oscillatory due to Corollary 5.2.

Secondly, we give a remark to summarize our main results.

Remark 6.2. In this paper, we have introduced some new theorems for investigation of the oscillation of delayed and advanced equation (1.6). We can use similar method to examine equation (1.6) when $g(t,b) \leq g(t,\xi)$ for $(t,\xi) \in [t_0,\infty)_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$. To the best of our knowledge, there are no known results can be applied to the enclosed example. It would be interesting to study (1.6) when $\tau[g(t,\xi)] \not\equiv g[\tau(t),\xi]$, $\lim_{t\to\infty} p(t) = \infty$, or $p(t) < 0$.

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