J. Nonlinear Sci. Appl. 4 (2011), no. 3, 200–209

THE JOURNAL OF NONLINEAR SCIENCES AND APPLICATIONS
http://www.tjnsa.com

EXISTENTIAL RESULTS FOR NONLINEAR SINGULAR INTERFACE PROBLEMS INVOLVING SECOND ORDER NONLINEAR DYNAMIC EQUATIONS USING PICARD'S ITERATIVE TECHNIQUE

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The authors dedicate this work to the Chancellor of Sri Sathya Sai Institute of Higher Learning, Bhagwan Sri Sathya Sai Baba.

ABSTRACT. In this paper we give existential results for nonlinear interface problems with a singular interface. The solution is proved to exist for an IVP satisfying matching interface conditions. The picards iterative technique is used. We discuss the theory developed to a problem in the field of applied elasticity.

1. INTRODUCTION

Solving boundary value problems with different types of singularities has remained a challenge for mathematicians over the ages. While regular problems, those over finite intervals with well-behaved coefficients pose no difficulties, there are applications wherein either the domain of the problem is not well defined, or the continuity and/or smoothness of the functions, coefficients involved are not guaranteed in some parts of the domain, sometimes in the boundary or parts of

Date: Received: 1 Jan, 2010; Revised: 20 April, 2011.
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2000 *Mathematics Subject Classification.* Primary 58J47; Secondary 58Cxx, 46Txx.
Key words and phrases. regular problems, singular problems, singular interface problems, picards iterative technique.
THIS STUDY IS FUNDED UNDER THE RESEARCH PROJECT NO. ERIP/ER/0803728/M/01/1158, BY DRDO, MINISTRY OF DEFENCE, GOVT. OF INDIA.

the boundary. In all such cases the problem is considered to be a singular problem. The definition of the problem and therefore the description of the solution becomes a highly difficult task.

In the literature we find a class of interface problems, termed as mixed pair of equations, discussed in the papers [3],[8]–[12], [16]–[22] where two different differential equations are defined on two adjacent intervals and the solutions satisfy a matching condition at the point of interface. These problems are called as matching interface problems. If the boundary is well defined then we call the problem to be a regular interface problem. These interface problems with singularities in the domain are always of great interest.

We see that these interface problems for regular case has been discussed in [16]–[22] and the problem of having singularity at the boundary is dealt in [3]. In [3], authors discuss an application of the classical Weyl limit criterion to define the coefficients with well-known Wronskian boundary conditions to tackle the singularity at the boundary for this class of problems. Though this work is specifically for Sturm–Liouville problems, it paves a way to study the problem of singularity at the end boundary points.

The problem of having a singularity at the point of interface is a challenge. Study of these problems using classical analytical tools is tedious. We term these problems as singular interface problems [4]–[7],[13]–[14].

The singularity at the point of interface in the domain of definition of the mixed pair of equations could be of the following three types satisfying certain matching conditions at the singular interface.

To describe the singularities in the domain of definition we take help of the terminology used on Time Scale [2]. The new framework of the dynamic equations on time scale with facilities of the two jump operators with various definitions of continuity and derivatives make one's job simple to study the interface problems with mixed operators along with a singular interface. Recently we have worked on the linear singular interface problems as seen in [4]–[7],[13]–[14]. Here we discuss the corresponding nonlinear problem.

In this paper we present existential results for a IVP associated with the nonlinear singular interface problems. The singular interface problem is described using a pair of dynamic equations on a time scale. The picards iterative technique is used for proving the existential results for the IVP. Also the theory developed will be applied to a problem in the branch of elasticity.

2. Mathematical Preliminaries

Definition 2.1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is *right-scattered,* while $\rho(t) < t$ we say that t is *left-scattered.* Points that are right-scattered and left-scattered at the same time are called *isolated.* Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense,* and if $t >$ inf $\mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense.* Points that are right-dense and left-dense at the same time are called *dense.* Finally, the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t$$

Definition 2.2. $\mathbb{T}^\kappa = \left\{ \begin{array}{l} \mathbb{T} - \{m\} \text{ if sup } \mathbb{T} < \infty \\ \mathbb{T} \text{ if sup } \mathbb{T} = \infty \end{array} \right\}$, where m is the left scattered maximum.

Definition 2.3. Let f be a function defined on $\mathbb{T}$. We say that f is *delta differentiable* at $t \in \mathbb{T}^\kappa$ provided there exists an α such that for all $\epsilon > 0$ there is a neighborhood $\mathcal{N}$ around t with

$$|f(\sigma(t) - f(s) - \alpha(\sigma(t) - s)| \leq \epsilon\,|\sigma(t) - s| \text{ for all } s \in \mathcal{N}$$

Definition 2.4.

$$f^\Delta(t) = \begin{cases} lim_{s\to t, s\in\mathbb{T}} \frac{f(t)-f(s)}{t-s} \text{ if } \mu(t) = 0 \\ \frac{f(\sigma(t))-f(t)}{\mu(t)} \text{ if } \mu(t) > 0 \end{cases}$$

Note 2.5. For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative $f^{\Delta\Delta}$ provided f^Δ is differentiable on $\mathbb{T}^{\kappa 2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta\Delta} = (f^\Delta)^\Delta : \mathbb{T}^{\kappa 2} \to \mathbb{R}$. Similarly we define the higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa n} \to \mathbb{R}$.

Definition 2.6. (Comparison Test for real series)
Let $\sum A_n$ and $\sum B_n$ be two real series. Let $N \in \mathbb{N}$ such that for all $n > N, A_n < B_n$. Then if $\sum B_n$ converges, $\sum A_n$ converges as well.

3. Definition of the IVP

Let $\mathbb{T}_1 = [0, \rho(a)]_\mathbb{T}$, $\mathbb{T}_2 = [\sigma(a), l]_\mathbb{T}$, where $0 < \rho(a) < \sigma(a) < l < +\infty$. Also let (f_1, f_2) be nonlinear function tuple in $\mathcal{C}(\mathbb{T}_1 \times \mathbb{T}_1 \times {\mathbb{T}_1}^\kappa) \times \mathcal{C}(\mathbb{T}_2 \times \mathbb{T}_2 \times {\mathbb{T}_2}^\kappa)$. In this chapter we consider the following IVP associated with singular interface problem(IVP-SIP).

$$y_1^{\Delta\Delta}(t) = f_1(t, y_1^\sigma, {y_1}^{\Delta\sigma}),\ t \in \mathbb{T}_1^{\kappa^2} \tag{3.1}$$
$$y_2^{\Delta\Delta}(t) = f_2(t, y_2^\sigma, {y_2}^{\Delta\sigma}),\ t \in \mathbb{T}_2^{\kappa^2} \tag{3.2}$$

with the initial conditions

$$y_1(0) = 0 \tag{3.3}$$
$$y_1^\Delta(0) = 0 \tag{3.4}$$

followed by the matching interface conditions

$$\rho_1 y_1(a) = \rho_2 y_2(\sigma(a)) \tag{3.5}$$
$$\rho_3 y_1^\Delta(a) = \rho_4 y_2^\Delta(\sigma(a)),\ \rho_i > 0,\ i = 1, 2, 3, 4. \tag{3.6}$$

4. Existential Results for the IVP-SIP

In this section we prove the existence of solution for the IVP-SIP using Picard's Iterative Technique.

Theorem 4.1. *If (f_1, f_2) is bounded, then there exists a bounded solution for the IVP-SIP.*

Proof. As shown in [15] it can be clearly seen that the IVP-SIP is equivalent to the integral equation

$$\begin{aligned} T(y_1, y_2) &= \Bigg(\int_0^{t_1} \int_0^{m} f_1(s, y_1^\sigma, y_1^{\Delta\sigma}) \Delta s \Delta m, \int_{\sigma(a)}^{t_2} \int_{\sigma(a)}^{m'} f_2(s, y_2^\sigma, y_2^{\Delta\sigma}) \Delta s \Delta m' \\ &+ \int_{\sigma(a)}^{t_2} \frac{\rho_3}{\rho_4} \Big(\int_0^{\rho(a)} f_1(s, y_1^\sigma, y_1^{\Delta\sigma}) \Delta s \Big) \Delta m' \\ &+ \frac{\rho_1}{\rho_2} \Big(\int_0^{\rho(a)} \int_0^{m'} f_1(s, y_1^\sigma, y_1^{\Delta\sigma}) \Delta s \Delta m' \Big) \Bigg) \end{aligned}$$

where $t_1, m \in \mathbb{T}_1$ and $t_2, m' \in \mathbb{T}_2$.

We now show that there exists (y_1, y_2) satisfying the above integral equation using the picard's iterative technique.

Case I Let $u_0 = 0$. We let

$$|f_1(t, p_{11}^\sigma, q_{11}^\sigma) - f_1(t, p_{12}^\sigma, q_{12}^\sigma)| \leq K_{11}^\sigma(t)|p_{11}^\sigma - p_{12}^\sigma| + K_{12}^\sigma(t)|q_{11}^\sigma - q_{12}^\sigma|$$

where

$$\int_0^t \int_0^m (K_{11}^\sigma(s) + K_{12}^\sigma(s)) \Delta s \Delta m = B < 1, \text{ for some } K_{11}, K_{12} \in \mathcal{C}(\mathbb{T}_1).$$

For $t \in \mathbb{T}_1$, we define

$$\|u\| = max_{t\in\mathbb{T}_1}\{|u(t)|, |u^\Delta(t)|\}$$

Now we let

$$u_{n+1} = \int_0^t \int_0^m f_1(s, u_n^\sigma, {u_n}^{\Delta\sigma}) \Delta s \Delta m, \ \forall t \in \mathbb{T}_1$$

and

$$r_p(t) = |u_p(t) - u_{p-1}(t)|.$$

So

$$|r_1(t)| = |u_1(t) - u_0(t)| = |u_1(t) - 0| = |u_1(t)|$$

$$
\begin{aligned}
|u_1(t)| &\le \int_0^t \int_0^m |f_1(s, u_0^\sigma, {u_0}^{\Delta\sigma})\Delta s\Delta m - f_1(s, p_{11}^\sigma, {p_{11}}^{\Delta\sigma})\Delta s\Delta m| \\
&+ \int_0^t \int_0^m |f_1(s, p_{11}^\sigma, {p_{11}}^{\Delta\sigma})|\Delta s\Delta m \\
&\le \int_0^t \int_0^m (K_{11}^\sigma(s)|u_0^\sigma - p_{11}^\sigma| + K_{12}^\sigma(s)|u_0^{\Delta\sigma} - p_{11}^{\Delta\sigma}|)\Delta s\Delta m \\
&+ \int_0^t \int_0^m |f_1(s, p_{11}^\sigma, {p_{11}}^{\Delta\sigma})|\Delta s\Delta m \\
&\le \|u_0 - p_{11}\| \int_0^t \int_0^m (K_{11}^\sigma(s) + K_{12}^\sigma(s))\Delta s\Delta m \\
&+ \int_0^t \int_0^m |f_1(s, p_{11}^\sigma, {p_{11}}^{\Delta\sigma})|\Delta s\Delta m \\
&= \|u_0 - p_{11}\|B + \int_0^t \int_0^m |f_1(s, p_{11}^\sigma, {p_{11}}^{\Delta\sigma})|\Delta s\Delta m \\
&= A
\end{aligned}
$$

Therefore $|u_1(t) - u_0| \le A$ where A is finite. Let us assume that

$$r_{p-1}(t) \le AB^{p-2} \text{ for } 2 < p < n.$$

Now

$$
\begin{aligned}
r_p(t) &= |u_p(t) - u_{p-1}(t)| \\
&= |\int_0^t \int_0^m f_1(s, u_{p-1}^\sigma, {u_{p-1}}^{\Delta\sigma})\Delta s\Delta m \\
&- \int_0^t \int_0^m f_2(s, u_{p-2}^\sigma, {u_{p-2}}^{\Delta\sigma})\Delta s\Delta m| \\
&\le \int_0^t \int_0^m [K_{11}^\sigma(s)|u_{p-1}^\sigma - u_{p-2}^\sigma| + K_{12}^\sigma(s)|{u_{p-1}}^{\Delta\sigma} - {u_{p-2}}^{\Delta\sigma}|]\Delta s\Delta m \\
&\le \|u_{p-1} - u_{p-2}\| \int_0^t \int_0^m (K_{11}^\sigma(s) + K_{12}^\sigma(s))\Delta s\Delta m \\
&\le r_{p-1}(t) \int_0^t \int_0^m (K_{11}^\sigma(s) + K_{12}^\sigma(s))\Delta s\Delta m \\
&\le AB^{p-2}B = AB^{p-1}
\end{aligned}
$$

Thus we have shown that

$$r_p(t) \le AB^{p-1}.$$

Therefore the infinite series,

$$\sum_{p=1}^{\infty} r_p(t) \le \sum_{p=1}^{\infty} AB^{p-1}$$

converges uniformly for all $t \in \mathbb{T}_1$ by the comparison test since $B < 1$. Let

$$lim_{p\to\infty} u_p(t) = u(t).$$

Now we will show that the sequence of functions $u_p(t)$ converges uniformly to $u(t)$. We see that

$$\begin{aligned}
|u(t) - u_p(t)| &= |u_0 + \sum_{i=1}^{\infty}[u_i(t) - u_{i-1}(t)] - u_0 - \sum_{i=1}^{p}[u_i(t) - u_{i-1}(t)]| \\
&\le \sum_{i=p+1}^{\infty} |[u_i(t) - u_{i-1}(t)]| \\
&= \sum_{i=p+1}^{\infty} r_i(t) \\
&\le \sum_{i=p+1}^{\infty} AB^i \le A\frac{B^{p+1}}{1-B}
\end{aligned}$$

Therefore as $p \to \infty$, we have $u_p(t) \to u(t)$(as $B < 1$). Now we will show that $u(t)$ is a continuous function for $t \in \mathbb{T}_1$. Let $\epsilon > 0$ be given.

$$\begin{aligned}
|u(t+h) - u(t)| &= |u(t+h) - u_p(t+h) + u_p(t+h) - u_p(t) + u_p(t) - u(t)| \\
&\le |u(t+h) - u_p(t+h)| \\
&+ |u_p(t+h) - u_p(t)| + |u_p(t) - u(t)| \\
&\le 2\frac{AB^{p+1}}{1-B} + |u_p(t+h) - u_p(t)|
\end{aligned}$$

For sufficiently large m and arbitrarily small h, we have

$$|u(t+h) - u(t)| < \epsilon, \; t \in \mathbb{T}_1.$$

The fact that $u(t)$ is bounded follows from the fact that $f_1(s, y, y^{'})$ is bounded on $\mathbb{T}_1$.

Case II Let $v_0 = \frac{\rho_1}{\rho_2}\Big(\int_0^{\rho(a)} \int_0^{m^{'}} f_1(s, u_0, {u_0}^{'})\Delta s \Delta m^{'}\Big)$. We let

$$|f_2(t, p_{21}^{\sigma}, q_{21}^{\sigma}) - f_2(t, p_{22}^{\sigma}, q_{22}^{\sigma})| \le K_{21}^{\sigma}(t)|p_{21}^{\sigma} - p_{22}^{\sigma}| + K_{22}^{\sigma}(t)|q_{21}^{\sigma} - q_{22}^{\sigma}|$$

where

$$\int_0^t \int_0^m (K_{21}^{\sigma}(s) + K_{22}^{\sigma}(s))\Delta s \Delta m = B^{'} < 1, \text{ for some } K_{21}, K_{22} \in \mathcal{C}(\mathbb{T}_2).$$

For $t \in \mathbb{T}_2$, we define

$$\|v\| = max_{t\in\mathbb{T}_2}\{|v(t)|, |v^{'}(t)|\}$$

Now we let

$$\begin{aligned} v_{n+1} &= \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'} f_2(s, v_n^{\sigma}, {v_n}^{\Delta\sigma})\Delta s\Delta m' \\ &+ \int_{\sigma(a)}^{t}\frac{\rho_3}{\rho_4}\bigg(\int_{0}^{\rho(a)} f_1(s, u_n^{\sigma}, {u_n}^{\Delta\sigma})\Delta s\bigg)\Delta m' \\ &+ \frac{\rho_1}{\rho_2}\bigg(\int_{0}^{\rho(a)}\int_{0}^{m'} f_1(s, u_n^{\sigma}, {u_n}^{\Delta\sigma})\Delta s\Delta m'\bigg)\ \forall t \in \mathbb{T}_2 \end{aligned}$$

and

$$g_p(t) = |v_p(t) - v_{p-1}(t)|$$

So

$$|g_1(t)| = |v_1(t) - v_0(t)|$$

We see that

$$\begin{aligned} &\int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'} f_2(s, v_0^{\sigma}, {v_0}^{\Delta\sigma})\Delta s\Delta m' - \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'} f_2(s, p_{21}^{\sigma}, {p_{21}}^{\Delta\sigma})\Delta s\Delta m' \\ &\leq \|v_0 - p_{21}\|\int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'}(K_{21}^{\sigma}(s) + K_{22}^{\sigma}(s))\Delta s\Delta m \\ &= \|v_0 - p_{21}\|B' \end{aligned}$$

Now $|g_1(t)|$

$$\begin{aligned} &= |v_1(t) - v_0(t)| \\ &= |\int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'} f_2(s, v_0^{\sigma}, {v_0}^{\Delta\sigma})\Delta s\Delta m' + \int_{\sigma(a)}^{t}\frac{\rho_3}{\rho_4}\bigg(\int_{0}^{\rho(a)} f_1(s, u_0^{\sigma}, {u_0}^{\Delta\sigma})\Delta s\bigg)\Delta m'| \\ &\leq |\int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'} f_2(s, v_0^{\sigma}, {v_0}^{\Delta\sigma})\Delta s\Delta m' - \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'} f_2(s, p_{21}^{\sigma}, {p_{21}}^{\Delta\sigma})\Delta s\Delta m'| \\ &+ |\int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'} f_2(s, p_{21}^{\sigma}, {p_{21}}^{\Delta\sigma})\Delta s\Delta m'| \\ &+ |\int_{\sigma(a)}^{t}\frac{\rho_3}{\rho_4}\bigg(\int_{0}^{\rho(a)} f_1(s, u_0^{\sigma}, {u_0}^{\Delta\sigma})\Delta s\bigg)\Delta m'| \\ &\leq \|v_0 - p_{21}\|B' + \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'} f_2(s, p_{21}^{\sigma}, {p_{21}}^{\Delta\sigma})\Delta s\Delta m'| \\ &+ |\int_{\sigma(a)}^{t}\frac{\rho_3}{\rho_4}\bigg(\int_{0}^{\rho(a)} f_1(s, u_0^{\sigma}, {u_0}^{\Delta\sigma})\Delta s\bigg)\Delta m'| \\ &= A' \text{ which is finite.} \end{aligned}$$

Therefore $|g_1(t)| \leq A'$ where A' is finite. Let us assume that

$$g_{p-1}(t) \leq A'B'^{p-2} \text{ for } 2 < p < n.$$

Now

$$\begin{aligned}
g_p(t) &= |v_p(t) - v_{p-1}(t)| \\
&= |\int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'} f_2(s, v_{p-1}^{\sigma}, {v_{p-1}}^{\Delta\sigma})\Delta s \Delta m' \\
&- \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'} f_2(s, v_{p-2}^{\sigma}, {v_{p-2}}^{\Delta\sigma})\Delta s \Delta m'| \\
&\leq A' B'^{p-1} \text{ (similar to Case I).}
\end{aligned}$$

In similar lines to Case I it can be shown that the sequence of functions $v_p(t)$ converges uniformly to $v(t)$ where

$$lim_{p\to\infty} v_p(t) = v(t).$$

Also $v(t)$ is a continuous function for $t \in \mathbb{T}_2$. The fact that $v(t)$ is bounded follows from the fact that $f_1(s, y^{\sigma}, y^{\Delta\sigma})$ is bounded on $\mathbb{T}_1$ and $f_2(s, z^{\sigma}, z^{\Delta\sigma})$ is bounded on $\mathbb{T}_2$.

We are done through the proof if we can show that $(u(t), v(t))$ is a fixed point of the operator T. We see that

$lim_{n\to\infty} u_{n+1}(t)$

$$\begin{aligned}
&= lim_{n\to\infty} \int_0^t \int_0^m f_1(s, u_n^{\sigma}, {u_n}^{\Delta\sigma})\Delta s \Delta m \\
&= \int_0^t \int_0^m f_1(s, lim_{n\to\infty} u_n^{\sigma}, lim_{n\to\infty} {u_n}^{\Delta\sigma})\Delta s \Delta m \text{ (since } f_1 \text{ is continuous)} \\
&= \int_0^t \int_0^m f_1(s, u^{\sigma}, u^{\Delta\sigma})\Delta s \Delta m
\end{aligned}$$

Hence we have

$$u(t) = \int_0^t \int_0^m f_1(s, u^{\sigma}, u^{\Delta\sigma})\Delta s \Delta m$$

Similarly, using the fact that f_2 is continuous it can be shown that

$$\begin{aligned}
v(t) &= \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m'} f_2(s, v^{\sigma}, v^{\Delta\sigma})\Delta s \Delta m' \\
&+ \int_{\sigma(a)}^{t} \frac{\rho_3}{\rho_4}\Big(\int_0^{\rho(a)} f_1(s, u^{\sigma}, u^{\Delta\sigma})\Delta s\Big)\Delta m' \\
&+ \frac{\rho_1}{\rho_2}\Big(\int_0^{\rho(a)}\int_0^{m'} f_1(s, u^{\sigma}, u^{\Delta\sigma})\Delta s \Delta m'\Big)
\end{aligned}$$

So $(u(t), v(t))$ is a fixed point of the operator T. Hence there exists a solution for the IVP-SIP. □ □

Remark 4.2. The above theorem can be proved for IVPs for the Interface II and Interface III with suitable changes in the notations.

5. Applications

The results presented here are general in nature and holds true for both the interface problems for the regular case and singular interface problems. Evidently we need to consider $\rho(a) = a = \sigma(a)$ for the regular case where we have the delta derivative becoming the ordinary derivative. These results can be applied for a pair of ordinary nonlinear differential equations with matching interface conditions which is the subject matter of discussion in [3],[8]–[12], [16]–[22].

Here we discuss the theory developed for a regular interface problems in the field of applied elasticity.

[1] In the branch of applied elasticity [23], we encounter the problem of buckling of columns of variable cross sections given by

$$\frac{d^2u_1}{dx^2} + K_1^2 u_1 = 0,\ 0 \le x \le l_1$$
$$\frac{d^2u_2}{dx^2} + K_2^2 u_2 = 0,\ l_1 \le x \le l$$

where $K_i^2 = \frac{P}{EI_i}, E$ is the modulus of elasticity, P is the load applied, I_i are the moments of inertia, $i = 1, 2$, and u_1, u_2 are the displacements of the cross sections for the thinner and thicker portions of the column respectively. The physical conditions on the system are given by

$$\begin{aligned} u_1(0) &= u_1^{'}(0) = 0 \\ u_1(l_1) &= u_2(l_1) \\ u_1^{'}(l_1) &= u_2^{'}(l_1) \end{aligned}$$

where $x = l_1$ denotes the point of interface.

Here we see that $a = l_1 = \sigma(a)$ is the regular interface. Hence from Theorem(4.1) we see that a solution exists for the buckling of columns of variable cross sections.

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