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L_p-APPROXIMATION BY A LINEAR COMBINATION OF SUMMATION-INTEGRAL TYPE OPERATORS

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This paper is dedicated to Professor R.K.S. Rathore

ABSTRACT. The present paper is a study of some direct results in L_p-approximation by a linear combination of summation-integral type operators. We obtain an error estimate in terms of the higher order modulus of smoothness using some properties of the Steklov mean.

1. INTRODUCTION

Motivated by the integral modification of Bernstein polynomials by Durrmeyer [2] and subsequent work by Derriennic [3] on Bernstein Durrmeyer operators, Gupta

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and Mohapatra [7] considered hybrid type operators by combining the weights of Szász and Baskakov operators in order to approximate Lebesgue integrable functions on the interval $[0,\infty)$ as follows:

$$M_n(f,x)=\sum_{d=0}^{\infty}p_{n,d}(x,c)\int_0^{\infty}b_{n,d}(t,c)f(t)\,dt, \tag{1.1}$$

where $p_{n,d}(x,c)=(-1)^d\dfrac{x^d}{d!}\phi_{n,c}^{(d)}(x)$, $b_{n,d}(t,c)=(-1)^{d+1}\dfrac{t^d}{d!}\phi_{n,c}^{(d+1)}(t)$ and $\{\phi_{n,c}\}_{n\in\mathbb{N}}$ be a sequence of functions defined on an interval $[0,b]$, $b>0$ having the following properties for every $n\in\mathbb{N},k\in\mathbb{N}^0$ (the set of non-negative integers):

(i) $\phi_{n,c}\in C^{\infty}([a,b])$; (ii) $\phi_{n,c}(0)=1$;

(iii) $\phi_{n,c}$ is completely monotone i.e $(-1)^k\phi_{n,c}^{(k)}\geq 0$;

(iv) there exists an integer c such that $\phi_{n,c}^{(k+1)}=-n\phi_{n+c,c}^{(k)}, n>\max\{0,-c\}$.

For $f\in L_p[0,\infty)$, the operators $M_n(f;x)$ can be expressed as

$$M_n(f;x)=\int_0^{\infty}W_n(t,x,c)f(t)\,dt,$$

where

$$W_n(t,x,c)=\sum_{d=0}^{\infty}p_{n,d}(x,c)b_{n,d}(t,c)$$

is the kernel of the operators.

Gupta [4] established that operators with different weights give better results than the corresponding symmetric operators. Here, we observe that for the case $c>0$ and $\phi_{n,c}(x)=(1+cx)^{-n/c}$, the operators M_n reduce to Baskakov-Durrmeyer operators and when $c=0$ and $\phi_{n,c}(x)=e^{-nx}$, these become Szász-Durrmeyer operators. Some approximation properties of these operators were studied in [5]. The rate of convergence by the operators M_n for the particular value $c=1$ was studied in [6].

It turns out that the order of approximation by these operators is at best $O(n^{-1})$, however smooth the function may be. In order to speed up the rate of convergence by the operators M_n, Agrawal and Gairola [1] considered the linear combination $M_n(f,k,.)$ of the operators M_n, as

$$M_n(f,k,x) = \sum_{j=0}^{k} C(j,k) M_{d_j n}(f,x),$$

where

$$C(j,k) = \prod_{i=0, i\neq j}^{k} \frac{d_j}{d_j - d_i}, k \neq 0 \text{ and } C(0,0) = 1, \tag{1.2}$$

$d_0, d_1, ...d_k$ being $(k+1)$ arbitrary but fixed distinct positive integers.

Let $m \in \mathbb{N}$ (the set of positive integers) and $0 < a < b < \infty$. For $f \in L_p[a,b]$, $1 \leq p \leq \infty$, the $m-$th order integral modulus of smoothness of f is defined as

$$\omega_m(f, \delta, p, [a,b]) = \sup_{0<h\leq\delta} \|\Delta_h^m f(t)\|_{L_p[a,b-mh]},$$

where $\Delta_h^m f(t)$ is the $m-$th order forward difference of the function f with step length h and $0 < \delta \leq (b-a)/m$.

In what follows, we suppose that $0 < a < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < b < \infty$ and $I_j = [a_j, b_j]; j = 1,2,3$. Let $AC[a,b]$ and $BV[a,b]$ denote the classes of absolutely continuous functions and functions of bounded variations respectively on the interval $[a,b]$. Further, C is a constant not always the same at each occurrence.

For $1 \leq p \leq \infty$, let

$$L_p^{(2k+2)}(I_1) = \left\{ f \in L_p[0,\infty) : f^{(2k+1)} \in AC(I_1) \, \text{and} \, f^{(2k+2)} \in L_p(I_1) \right\}.$$

2. Preliminaries

In this section we give some results which are useful in establishing our main theorems.

Lemma 2.1. [7] *For $m \in \mathbb{N} \cup \{0\}$, if we define the $m-th$ order moment for the operators M_n by*

$$\mu_{n,m}(x,c) = \sum_{d=0}^{\infty} p_{n,d}(x,c) \int_0^{\infty} b_{n,d}(t,c)(t-x)^m \, dt$$

then

$$\mu_{n,0}(x,c) = 1 \;\; \mu_{n,1}(x,c) = \frac{1+cx}{n-c}$$

and

$$\mu_{n,2}(x,c) = \frac{2cx^2(n+c) + 2x(n+2c) + 2}{(n-c)(n-2c)}$$

Also the following recurrence relation holds

$$[n - c(m+1)]\mu_{n,m+1}(x,c) = x(1+cx)[\mu_{n,m}^{(1)}(x,c) + 2m\mu_{n,m-1}(x,c)] + [(1+2cx)(m+1) - cx]\mu_{n,m}(x,c).$$

Further we have,

(i) $\mu_{n,m}(x)$ is a polynomial in x of degree m, $m \neq 1$;

(ii) for every $x \in [0,\infty)$, $\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$

Corollary 2.2. *For each $r > 0$ and for every $x \in [0,\infty)$, we have*

$$M_n(|t-x|^r, x) = O\left(n^{-r/2}\right), \text{ as } n \to \infty.$$

Proof. Let $I =: M_n(|t-x|^r, x)$ and s be an even integer $> r$. Then, using Hölder's inequality and Lemma 2.1, we obtaim

$$I = \int_0^{\infty} (W_n(t,x,c))^{\frac{r}{s} + (1-\frac{r}{s})} |t-x|^r \, dt$$

$$= \left(\int_0^\infty W_n(t,x,c)|t-x|^s\,dt\right)^{r/s}\left(\int_0^\infty (W_n(t,x,c)\,dt\right)^{1-\frac{r}{s}}$$
$$\leq C(n^{-s/2})^{r/s} = Cn^{-r/2}.$$

□

The dual operator $\hat{M}_n$ corresponding to the operator M_n is defined as

$$\hat{M}_n(f;t) = \int_0^\infty W_n(t,x,c)f(x)\,dx.$$

Then the corresponding $m-$th order moment is given by

$$\hat{\mu}_{n,m}(t) = \int_0^\infty W_n(t,x,c)(x-t)^m\,dx.$$

Lemma 2.3. *For the function $\hat{\mu}_{n,m}(t), n/c > m+2$ there holds the recurrence relation*

$$[n-c(m+2)]\hat{\mu}_{n,m+1}(x,c)$$
$$= x(1+cx)[\hat{\mu}^{(1)}_{n,m}(x,c) + 2m\hat{\mu}_{n,m-1}(x,c)] + [(1+2cx)(m+1)+cx]\hat{\mu}_{n,m}(x,c). \tag{2.1}$$

Further we have,

(i) $\hat{\mu}_{n,m}(x)$ is a polynomial in x of degree m, $m \neq 1$;

(ii) for every $x \in [0,\infty)$, $\hat{\mu}_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$

Proof. We make use of the expressions $x(1+cx)p'_{n,d}(x,c) = (d-nx)p_{n,d}(x,c)$ and $t(1+ct)b'_{n,d}(t,c) = (d-(n+c)t)b_{n,d}(t,c)$. Thus, we get

$$t(1+ct)[\hat{\mu}^{(1)}_{n,m}(t) + m\hat{\mu}_{n,m-1}(t)] + ct\hat{\mu}^{(1)}_{n,m}(t)$$

$$= \sum_{d=0}^{\infty}(d-nt)b_{n,d}(t,c)\int_0^{\infty} p_{n,d}(x,c)(x-t)^m\,dx.$$

This gives

$$t(1+ct)[\hat{\mu}^{(1)}_{n,m}(t)+m\hat{\mu}_{n,m-1}(t)]+ct\hat{\mu}^{(1)}_{n,m}(t)-\hat{\mu}_{n,m+1}(t)$$
$$= \sum_{d=0}^{\infty} b_{n,d}(t,c)\int_0^{\infty} t(1+ct)p'_{n,d}(x,c)(x-t)^m\,dx$$
$$= \sum_{d=0}^{\infty} b_{n,d}(t,c)\int_0^{\infty} \{c(x-t)^2+t(1+ct)+(1+2cx)(x-t)\}p'_{n,d}(x,c)(x-t)^m\,dx$$
$$=: T_1+T_2+T_3. \quad (2.2)$$

Now, for $n/c > m+2$ integration by parts yields $T_1 = -c(m+2)\hat{\mu}_{n,m+1}(t)$, $T_2 = -mt(1+ct)\hat{\mu}_{n,m-1}(t)$ and $T_3 = -(m+1)(1+2ct)\hat{\mu}_{n,m}(t)$. Using these expressions for $T_1 - T_3$ in (2.2) and rearranging the terms we obtain (2.1). □

Lemma 2.4. [8] *For $r \in \mathbb{N}$ and n sufficiently large, there holds*

$$M_n((t-x)^r,k,x) = n^{-(k+1)}\{Q(r,k,x)+o(1)\},$$

where $Q(r,k,x)$ is certain polynomials in x of degree r.

Let $f \in L_p[0,\infty)$, $1 \leqslant p < \infty$. Then for sufficiently small $\eta > 0$ the Steklov mean $f_{\eta,m}$ of $m-$th order corresponding to f is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}\Big(f(t)+(-1)^{m-1}\Delta^m_{\sum_{i=1}^m t_i}f(t)\Big)\prod_{i=1}^m dt_i,\ ,\ t\in I_1,$$

where Δ_h^m is $m-$the order forward difference operator with step length h.

Lemma 2.5. *For the function $f_{\eta,m}$, we have* *(a)* $f_{\eta,m}$ *has derivatives up to order* m *over* I_1, $f_{\eta,m}^{(m-1)} \in AC(I_1)$ *and* $f_{\eta,m}^{(m)}$ *exists a.e. and belongs to* $L_p(I_1)$ *;*

(b) $\|f_{\eta,m}^{(r)}\|_{L_p(I_2)} \leqslant C_r\, \eta^{-r}\omega_r(f,\eta,p,I_1), r = 1,2,...,m;$

(c) $\|f - f_{\eta,m}\|_{L_p(I_2)} \leqslant C_{m+1}\ \omega_m(f,\eta,p,I_1);$

(d) $\|f_{\eta,m}\|_{L_p(I_2)} \leqslant C_{m+2}\|f\|_{L_p(I_1)};$

(e) $\|f_{\eta,m}^{(m)}\|_{L_p(I_2)} \leqslant C_{m+3}\,\eta^{-m}\|f\|_{L_p(I_1)},$

where $C_i^{\prime}s$ *are certain constants that depend on* i *but are independent of* f *and* η.

Following [[10], Theorem 18.17] or [[12], pp.163-165], the proof of the above lemma easily follows hence the details are omitted.

Let $f \in L_p[a,b], 1 \leqslant p < \infty$. Then the Hardy-Littlewood majorant $h_f(x)$ of the function f is defined as

$$h_f(x) = \sup_{\xi \neq x} \frac{1}{\xi - x} \int_x^{\xi} f(t)\, dt,\ (a \leq \xi \leq b).$$

Lemma 2.6. [13] *If* $1 < p < \infty$ *and* $f \in L_p[a,b]$, *then* $h_f \in L_p[a,b]$ *and*

$$\left\|h_f\right\|_{L_p[a,b]} \leqslant 2^{1/p}\frac{p}{p-1}\|f\|_{L_p[a,b]}.$$

The next lemma gives a bound for the intermediate derivatives of f in terms of the highest order derivative and the function in L_p-norm.

Lemma 2.7. [9] *Let* $1 \leqslant p < \infty, f \in L_p[a,b]$. *Suppose* $f^{(k)} \in AC[a,b]$ *and* $f^{(k+1)} \in L_p[a,b]$. *Then*

$$\left\|f^{(j)}\right\|_{L_p[a,b]} \leqslant K_j\left(\left\|f^{(k+1)}\right\|_{L_p[a,b]} + \|f\|_{L_p[a,b]}\right),\ j = 1,2,...,k,$$

where K_j *are certain constants independent of* f.

Lemma 2.8. *Let $f \in BV(I_1)$. The following inequality holds:*

$$\left\| M_n \left(\phi(t) \int_x^t (t-w)^{2k+1} df(w); x \right) \right\|_{L_1(I_2)} \leq C n^{-(k+1)} \|f\|_{BV(I_1)},$$

where $\phi(t)$ is the characteristic function of I_1.

Proof. For each n there exists a nonnegative integer $r = r(n)$ such that $rn^{-1/2} \leq \max\{b_1 - a_2, b_2 - a_1\} \leq (r+1)n^{-1/2}$. Then,

$$\begin{aligned} K &= \left\| M_n \left(\int_x^t (t-w)^{2k+1} df(w) \phi(t); x \right) \right\|_{L_1(I_2)} \\ &\leq \sum_{l=0}^{r} \int_{a_2}^{b_2} \Bigg\{ \int_{x+ln^{-1/2}}^{x+(l+1)n^{-1/2}} \phi(t) W_n(t,x,c) |t-x|^{2k+1} \left[\int_x^{x+(l+1)n^{-1/2}} \phi(w) |df(w)| \right] dt \\ &+ \int_{x-(l+1)n^{-1/2}}^{x-ln^{-1/2}} \phi(t) W_n(t,x,c) |t-x|^{2k+1} \left[\int_{x-(l+1)n^{-1/2}}^{x} \phi(w) |df(w)| \right] dt \Bigg\} dx. \end{aligned}$$

Let $\phi_{x,d,e}$ denote the characteristic function of the interval $[x - dn^{-1/2}, x + en^{-1/2}]$, where d and e are nonnegative integers. Then we have

$$\begin{aligned} K &\leq \sum_{l=1}^{r} n^2 \Bigg(l^{-4} \int_{a_2}^{b_2} \Bigg\{ \int_{x+ln^{-1/2}}^{x+(l+1)n^{-1/2}} \phi(t) W_n(t,x,c) |t-x|^{2k+5} \left[\int_{a_1}^{b_1} \phi_{x,0,l+1}(w) |df(w)| \right] dt \\ &+ \int_{x-(l+1)n^{-1/2}}^{x-ln^{-1/2}} \phi(t) W_n(t,x,c) |t-x|^{2k+5} \left[\int_{a_1}^{b_1} \phi_{x,l+1,0}(w) |df(w)| \right] dt \Bigg\} dx \Bigg) \\ &+ \int_{a_2}^{b_2} \int_{a_2-n^{-1/2}}^{b_2+n^{-1/2}} \phi(t) W_n(t,x,c) |t-x|^{2k+1} \left[\int_{a_1}^{b_1} \phi_{x,1,1}(w) |df(w)| \right] dt\, dx. \end{aligned}$$

Using the moment estimates given by Corollary 2.2 to obtain a bound for $\int_0^\infty W_n(t,x,c)|t-x|^{2k+5} dt$ and then applying Fubini's theorem, we get

$$\begin{aligned} K &\le Cn^{-(2k+1)/2}\Big\{\sum_{l=1}^{r} l^{-4}\Big[\int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2}\phi_{x,0,l+1}(w)dx\Big)|df(w)| \\ &+ \int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2}\phi_{x,l+1,0}(w)dx\Big)|df(w)|\Big] + \int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2}\phi_{x,1,1}(w)dx\Big)|df(w)|\Big\} \\ &\le Cn^{-(2k+1)/2}\Big\{\sum_{l=1}^{r} l^{-4}\Big[\int_{a_1}^{b_1}\Big(\int_{w-(l+1)n^{-1/2}}^{w} dx\Big)|df(w)| \\ &+ \int_{a_1}^{b_1}\Big(\int_{w}^{w+(l+1)n^{-1/2}} dx\Big)|df(w)|\Big] + \int_{a_1}^{b_1}\Big(\int_{w-n^{-1/2}}^{w+n^{-1/2}} dx\Big)|df(w)|\Big\} \\ &\le Cn^{-(k+1)}\left(4\left(\sum_{l=1}^{r} l^{-3}\right)+2\right)\|f\|_{BV(I_1)} \\ &\le C'n^{-(k+1)}\|f\|_{BV(I_1)}. \end{aligned}$$

□

3. Main Result

In order to prove our main result, we first discuss the approximation in the smooth subspace $L_p^{(2k+2)}(I_1)$ of $L_p[0,\infty)$.

Theorem 3.1. *If $p>1$, $f \in L_p^{(2k+2)}(I_1)$, then for sufficiently large n*

$$\|M_n(f,k,\cdot)-f(\cdot)\|_{L_p(I_2)} \leqslant C_1 n^{-(k+1)}\left[\left\|f^{(2k+2)}\right\|_{L_p(I_1)} + \|f\|_{L_p[0,\infty)}\right]. \quad (3.1)$$

Moreover, if $f \in L_1[0,\infty)$, f has derivatives up to the order $(2k+1)$ on I_1 with $f^{(2k)} \in AC(I_1)$ and $f^{(2k+1)} \in BV(I_1)$, then for sufficiently large n there holds

$$\|M_n(f,k,\cdot)-f(\cdot)\|_{L_1(I_2)} \leqslant C_2 n^{-(k+1)}\left[\left\|f^{(2k+1)}\right\|_{BV(I_1)} + \left\|f^{(2k+1)}\right\|_{L_1(I_2)} + \|f\|_{L_1[0,\infty)}\right], \quad (3.2)$$

where C_1 and C_2 are certain constants independent of f and n.

Proof. Let $p > 1$, then for all $t \in [0, \infty)$ and $x \in I_2$, we can write

$$f(t) - f(x) = \sum_{j=1}^{2k+1} \frac{f^{(j)}(x)}{j!}(t-x)^j + \frac{1}{(2k+1)!}\int_x^t \varphi(t)(t-v)^{2k+1} f^{(2k+2)}(v)\,dv + F(t,x)(1-\varphi(t)), \quad (3.3)$$

where $\varphi(t)$ is the characteristic function of the interval I_1 and

$$F(t,x) = f(t) - \sum_{j=0}^{2k+1} \frac{f^{(j)}(x)}{j!}(t-x)^j, \; \forall\, t \in [0,\infty) \,\text{and}\, x \in I_2.$$

Therefore operating by $M_n(.,k,x)$ on both sides of (3.3), we obtain three terms, say E_1, E_2 and E_3 corresponding to the three terms in the right hand side of (3.3).

$$\begin{aligned} M_n(f,k,x) - f(x) &= \sum_{j=1}^{2k+1} \frac{f^{(j)}(x)}{j!} M_n((t-x)^j, k, x) \\ &+ \frac{1}{(2k+1)!} M_n\left(\varphi(t)\int_x^t (t-v)^{2k+1} f^{(2k+2)}(v)\,dv, k, x\right) \\ &+ M_n(F(t,x)(1-\varphi(t)), k, x) \\ &=: E_1 + E_2 + E_3. \end{aligned}$$

In view of Lemma 2.4 and 2.2, we get

$$\|E_1\|_{L_p(I_2)} \leqslant C\, n^{-(k+1)} \left(\left\|f^{(2k+2)}\right\|_{L_p(I_2)} + \|f\|_{L_p(I_2)} \right).$$

To estimate E_2, let $h_{f^{(2k+2)}}$ be the Hardy-Littlewood majorant of $f^{(2k+2)}$ on I_1. Then, using Hölder's inequality and Corollary 2.2, we get

$$|J_1| =: \left| M_n\left(\varphi(t)\int_x^t (t-v)^{2k+1} f^{(2k+2)}(v)\,dv; x\right)\right|$$

$$
\begin{aligned}
&\leq M_n\left(\varphi(t)|t-x|^{2k+1}\left|\int_x^t \left|f^{(2k+2)}(v)\right|\,dv\right|;x\right)\\
&\leq M_n\left(\varphi(t)|t-x|^{2k+2}|h_{|f^{(2k+2)}|}(t)|;x\right)\\
&\leq \left\{M_n\left(\varphi(t)|t-x|^{(2k+2)q};x\right)\right\}^{1/q}\left\{M_n\left(\varphi(t)|h_{|f^{(2k+2)}|}(t)|^p;x\right)\right\}^{1/p}\\
&\leq Cn^{-(k+1)}\left\{\int_{a_1}^{b_1} W_n(t,x,c)|h_{|f^{(2k+2)}|}(t)|^p\,dt\right\}^{1/p}.
\end{aligned}
$$

Hence, by Fubini's theorem and Lemma 2.6, we have

$$
\begin{aligned}
\|J_1\|^p_{L_p(I_2)} &\leq Cn^{-(k+1)p}\int_{a_2}^{b_2}\int_{a_1}^{b_1} W_n(t,x,c)|h_{|f^{(2k+2)}|}(t)|^p\,dt\,dx\\
&\leq Cn^{-(k+1)p}\int_{a_1}^{b_1}\left[\int_{a_2}^{b_2} W_n(t,x,c)\,dx\right]|h_{|f^{(2k+2)}|}(t)|^p\,dt\\
&\leq Cn^{-(k+1)p}\frac{n}{n-c}\int_{a_1}^{b_1}|h_{|f^{(2k+2)}|}(t)|^p\,dt \text{ (in view of } \hat{\mu}_{n,0}(t)=\frac{n}{n-c})\\
&\leq Cn^{-(k+1)p}\left\|h_{|f^{(2k+2)}|}\right\|^p_{L_p(I_1)}, \text{ since } n \text{ is sufficiently large}\\
&\leq Cn^{-(k+1)p}\left\|f^{(2k+2)}\right\|^p_{L_p(I_1)}.
\end{aligned}
$$

Consequently,

$$
\|J_1\|_{L_p(I_2)} \leq Cn^{-(k+1)}\left\|f^{(2k+2)}\right\|_{L_p(I_1)}.
$$

Thus, we have

$$
\|E_2\|_{L_p(I_2)} \leq Cn^{-(k+1)}\left\|f^{(2k+2)}\right\|_{L_p(I_1)}.
$$

In order to estimate E_3, it is sufficient to consider $I := \left|M_n\Big(F(t,x)(1-\varphi(t));x\Big)\right|$.

For $t \in [0,\infty) \setminus [a_1,b_1], x \in I_2$ we can find a $\delta > 0$ such that $|t-x| \geq \delta$. Thus

$$\begin{aligned} I &= \Big| M_n\Big(F(t,x)(1-\varphi(t));x\Big)\Big| \\ &= \left| M_n\Big(\Big(f(t) - \sum_{j=0}^{2k+1} \frac{f^{(j)}(x)}{j!}(t-x)^j\Big)\big(1-\varphi(t)\big);x\Big)\right| \\ &\leqslant \delta^{-(2k+2)} M_n\Big(|f(t)|(t-x)^{2k+2};x\Big) + \delta^{-(2k+2)} \sum_{j=0}^{2k+1} \frac{\big|f^{(j)}(x)\big|}{j!} M_n\Big(|t-x|^{2k+j+2};x\Big) \\ &=: J_2 + J_3, \text{ say.} \end{aligned}$$

On an application of Hölder's inequality and Corollary 2.2, we get

$$\begin{aligned} |J_2| &\leq \delta^{-(2k+2)} \Big\{M_n\Big(|f(t)|^p;x\Big)\Big\}^{1/p} \Big\{M_n\Big(|t-x|^{(2k+2)q};x\Big)\Big\}^{1/q} \\ &\leq Cn^{-(k+1)} \left(\int_{a_1}^{b_1} W_n(t,x,c)\big|f(t)\big|^p\,dt\right)^{1/p}. \end{aligned}$$

Now, applying Fubini's theorem we get

$$\begin{aligned} \big\|J_2\big\|^p_{L_p(I_2)} &\leq C\ n^{-(k+1)p} \int_{a_2}^{b_2}\int_{a_1}^{b_1} W_n(t,x,c)\big|f(t)\big|^p\,dt\,dx \\ &\leq C\ n^{-(k+1)p} \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} W_n(t,x,c)\,dx\right) \big|f(t)\big|^p\,dt \\ &\leq C\ n^{-(k+1)p} \int_{a_1}^{b_1} \big|f(t)\big|^p\,dt \text{ (in view of } \hat{\mu}_{n,0}(t) = \frac{n}{n-c}). \end{aligned}$$

So,

$$\big\|J_2\big\|_{L_p(I_2)} \leq C\ n^{-(k+1)} \|f\|_{L_p[0,\infty)}.$$

Now, in view of Corollary 2.2 and Lemma 2.7, we have the inequality

$$\big\|J_3\big\|_{L_p(I_2)} \leqslant C\ \ n^{-(k+1)} \left(\|f\|_{L_p(I_2)} + \big\|f^{(2k+2)}\big\|_{L_p(I_2)}\right).$$

Therefore,

$$\|E_3\|_{L_p(I_2)} \leqslant C \ \ n^{-(k+1)} \left(\|f\|_{L_p[0,\infty)} + \left\|f^{(2k+2)}\right\|_{L_p(I_2)} \right).$$

Combining the estimates for $E_1 - E_3$, (3.1) follows.

Now, let $p = 1$. Since $f^{(2k+1)} \in BV(I_1)$, it follows from Theorem 17.17 of [10] that $f^{(2k+1)}$ is continuous a.e. on I_1. This alongwith Theorem 14.1 of [11] implies that for almost all values of $x \in I_2$ and for all $t \in [0, \infty)$,

$$\begin{aligned} f(t) - f(x) &= \sum_{j=1}^{2k+1} \frac{f^{(j)}(x)}{j!}(t-x)^j + \frac{1}{(2k+1)!}\int_x^t \varphi(t)(t-v)^{2k+1}\, df^{(2k+1)}(v) \\ &+ F(t,x)(1-\varphi(t)), \end{aligned} \tag{3.4}$$

where $\varphi(t)$ and $F(t, x)$ are defined as above. Therefore operating by $M_n(., k, x)$ on both sides of (3.4),we obtain three terms E_4, E_5 and E_6, say corresponding to the three terms on the right hand side of (3.4).

Now proceeding as in the case of the estimate of E_1, we have

$$\left\|E_4\right\|_{L_1(I_2)} \leqslant C\ n^{-(k+1)} \left(\|f\|_{L_1(I_2)} + \left\|f^{(2k+1)}\right\|_{L_1(I_2)} \right). \tag{3.5}$$

To estimate E_5, since $f^{(2k+1)} \in BV(I_1)$, using Corollary 2.2 and Lemma 2.8, we obtain

$$\left\|E_5\right\|_{L_1(I_2)} \leqslant C\ n^{-(k+1)} \left\|f^{(2k+1)}\right\|_{BV(I_1)}.$$

To estimate E_6, it is sufficient to consider $M_n\Big(F(t, x)(1 - \varphi(t)); x\Big)$. Since $t \in [0, \infty) \setminus [a_1, b_1]$ and $x \in I_2$, we can choose a $\delta > 0$ such that $|t - x| \geq \delta$ which implies that

$$\left\|E_6\right\|_{L_1(I_2)} \leqslant C \int_{a_2}^{b_2}\int_0^\infty W_n(t,x,c)|f(t)(1-\varphi(t))|\, dt\, dx$$

$$+ \sum_{i=0}^{2k+1} \frac{1}{i!} \int_{a_2}^{b_2} \int_{0}^{\infty} W_n(t,x,c)|f^{(i)}(x)|\,|t-x|^i(1-\varphi(t))\,dt\,dx =: S_1 + S_2, \text{ say.}$$

For sufficiently large t, we can find positive constant M and C' such that

$$\frac{(t-x)^{2k+2}}{t^{2k+2}+1} > C' \ \ \forall t \geq M, \ x \in I_2.$$

By Fubini's theorem,

$$S_1 = \left(\int_0^M \int_{a_2}^{b_2} + \int_M^{\infty} \int_{a_2}^{b_2} \right) W_n(t,x,c)|f(t)|(1-\varphi(t))|\,dx\,dt =: S_3 + S_4$$

Now, using Lemma 2.3 we have

$$\begin{aligned} S_3 &\leq \delta^{-(2k+2)} \int_0^M \int_{a_2}^{b_2} W_n(t,x,c)|f(t)|(t-x)^{2k+2}\,dx\,dt \\ &\leq Cn^{-(k+1)} \int_0^M |f(t)|dt \end{aligned}$$

and

$$\begin{aligned} S_4 &\leq \frac{1}{C'} \int_M^{\infty} \int_{a_2}^{b_2} W_n(t,x,c) \frac{(t-x)^{2k+2}}{t^{2k+2}+1} |f(t)|\,dx\,dt \\ &\leq Cn^{-(k+1)} \int_M^{\infty} |f(t)|dt, \end{aligned}$$

Combining the estimates of S_3 and S_4, we get

$$S_1 \leq Cn^{-(k+1)} \|f\|_{L_1[0,\infty)}$$

Further, using Lemma 2.1 and Lemma 2.7, we obtain

$$S_2 \leqslant C\ n^{-(k+1)} \left(\|f\|_{L_1(I_2)} + \left\|f^{(2k+1)}\right\|_{L_1(I_2)} \right).$$

Hence,

$$\|M_n(F(t,x))(1-\varphi(t));x\|_{L_1(I_2)} \leq C\, n^{-(k+1)} \Bigg(\|f\|_{L_1[0,\infty)} + \|f^{(2k+1)}\|_{L_1(I_2)} \Bigg).$$

Consequently,

$$\|E_6\|_{L_1(I_2)} \leq C\, n^{-(k+1)} \Bigg(\|f\|_{L_1[0,\infty)} + \|f^{(2k+1)}\|_{L_1(I_2)} \Bigg).$$

From these estimates of E_4, E_5, E_6 we get (3.2). □

Finally, we establish the following direct theorem:

Theorem 3.2. *Let $f \in L_p[0,\infty), p \geq 1$. Then, for all n sufficiently large there holds*

$$\|M_n(f,k,.) - f\|_{L_p(I_2)} \leqslant C_k \ \left(\omega_{2k+2}\Big(f, \frac{1}{\sqrt{n}}, p, I_1\Big) + n^{-(k+1)} \|f\|_{L_p[0,\infty)} \right),$$

where C_k is a constant independent of f and n.

Proof. Let $f_{\eta,2k+2}$ be the Steklov mean of order $(2k+2)$ corresponding to the function f over I_1, where $\eta > 0$ is sufficiently small. Then we have

$$\begin{aligned} \|M_n(f,k,.) - f\|_{L_p(I_2)} &\leqslant \|M_n(f - f_{\eta,2k+2}, k, .)\|_{L_p(I_2)} + \|M_{n,k}(f_{\eta,2k+2}, k, .) - f_{\eta,2k+2}\|_{L_p(I_2)} \\ &+ \|f_{\eta,2k+2} - f\|_{L_p(I_2)} =: J_1 + J_2 + J_3, \text{ say.} \end{aligned}$$

Letting $\phi(t)$ to be the characteristic function of I_3, we get

$$M_n((f - f_{\eta,2k+2})(t);x) \quad = \quad M_n(\phi(t)(f - f_{\eta,2k+2})(t);x)$$

$$+ \quad M_n((1-\phi(t))(f-f_{\eta,2k+2})(t);x)$$
$$=: \quad \Sigma_1+\Sigma_2, \text{ say.}$$

Clearly, the following inequality holds for $p=1$, for $p>1$, it follows from Hölder's inequality

$$\int_{a_2}^{b_2} |\Sigma_1|^p dx \leq \int_{a_2}^{b_2}\int_{a_3}^{b_3} W_n(t,x,c)|(f-f_{\eta,2k+2})(t)|^p dt\, dx.$$

Using Fubini's theorem and Lemma 2.3 we get

$$\|\Sigma_1\|_{L_p(I_2)} \leq 2\|f-f_{\eta,2k+2}\|_{L_p(I_3)}.$$

Proceeding in similar manner, for all $p\geq 1$

$$\|\Sigma_2\|_{L_p(I_2)} \leq Cn^{-(k+1)}\|f-f_{\eta,2k+2}\|_{L_p[0,\infty)}.$$

Consequently, by the property (c) of Steklov mean, we get

$$J_1 \leqslant C\left(\omega_{2k+2}\big(f,\eta,p,I_1\big)+n^{-(k+1)}\|f\|_{L_p[0,\infty)}\right).$$

Since $\|f^{(2k+1)}_{\eta,2k+2}\|_{BV(I_3)} = \|f^{(2k+1)}_{\eta,2k+2}\|_{L_1(I_3)}$, using Theorem 3.1 and properties (b) and (d) of Steklov mean, we obtain

$$\begin{aligned} J_2 &\leqslant C\, n^{-(k+1)}\left(\|f_{\eta,2k+2}\|_{L_p[0,\infty)}+\|f^{(2k+2)}_{\eta,2k}\|_{L_p(I_3)}\right) \\ &\leqslant C\, n^{-k}\left(\|f\|_{L_p(I_1)}+\eta^{-2k}\omega_{2k}\big(f,\eta,p,I_1\big)\right). \end{aligned}$$

Finally, by the property (c) of Steklov mean, we get

$$J_3 \leqslant C\,\omega_{2k+2}\big(f,\eta,p,I_1\big).$$

Choosing $\eta=1/\sqrt{n}$ and combining the estimates J_1-J_3, the required result follows. $\square$

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