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EXISTENCE, UNIQUENESS AND STABILITY RESULTS OF IMPULSIVE STOCHASTIC SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAYS

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Abstract. This article presents the results on existence, uniqueness and stability of mild solution for impulsive stochastic semilinear functional differential equations with non-Lipschitz condition and Lipschitz condition. The results are obtained by using the method of successive approximation and Bihari's inequality.

1. Introduction

Impulsive differential equations are suitable for mathematical model to simulate the evolution of large classes of real processes. These processes are subjected to short temporary perturbations. The duration of these perturbations is negligible compared to the duration of whole process. These perturbations occurs in the form of impulses (see [8, 14]). There is much notice in the field of impulsive differential equations [1, 7] and the references therein.

The study of impulsive stochastic differential equations (ISDEs) is a new area of research. There are few publications in the theory of ISDEs. Jun Yang et al.[15], studied the stability analysis of ISDEs with delays. Zhiguo Yang et al.[16], studied the exponential p- stability of ISDEs with delays. In [12, 13], Sakthivel and Luo studied the existence and asymptotic stability in p-th moment of mild solutions to ISDEs with and without infinite delays through fixed point theory. In [2], the author studied the impulsive stochastic partial neutral functional differential equations under non-Lipschitz condition and Lipschitz condition. Motivated by

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[3, 10, 11], we will generalize the existence and uniqueness of the solution to impulsive stochastic partial functional differential equations (ISFDEs) under non-Lipschitz condition and Lipschitz condition. Moreover, we study the stability through the continuous dependence on the initial values by means of Corollary of Bihari's inequality. Further, we refer [3, 5, 6, 9].

The paper is organized as follows. In section 2, we recall briefly the notations, definitions, lemmas and preliminaries which are used throughout this paper. In section 3, we study the existence and uniqueness of ISFDEs by relaxing the linear growth conditions. In section 4, we study stability through the continuous dependence on the initial values. Finally in section 5, an example is given to illustrate our results.

2. Preliminaries

Let X, Y be real separable Hilbert spaces and $L(Y, X)$ be the space of bounded linear operators mapping Y into X. For convenience, we shall use the same notation $\|.\|$ to denote the norms in X, Y and $L(Y, X)$ without any confusion. Let (Ω, B, P) be a complete probability space with an increasing right continuous family $\{B_t\}_{t\geq 0}$ of complete sub σ-algebra of B. Let $\{w(t) : t \geq 0\}$ denote a Y-valued Wiener process defined on the probability space (Ω, B, P) with covariance operator Q, that is

$$E < w(t), x >_Y < w(s), y >_Y = (t \wedge s) < Qx, y >_Y, \quad \text{for all } x, y \in Y,$$

where Q is a positive, self-adjoint, trace class operator on Y. In particular, we denote $w(t)$, $t \geq 0$, a Y- valued Q- Wiener process with respect to $\{B_t\}_{t\geq 0}$.

In order to define stochastic integrals with respect to the Q- Wiener process $w(t)$, we introduce the subspace $Y_0 = Q^{1/2}(Y)$ of Y which, endowed with the inner product
$< u, v >_{Y_0} = < Q^{-1/2}u, Q^{-1/2}v >_Y$ is a Hilbert space. We assume that there exists a complete orthonormal system $\{e_i\}_{i\geq 1}$ in Y, a bounded sequence of nonnegative real numbers λ_i such that $Qe_i = \lambda_i e_i, i = 1, 2, \ldots,$ and a sequence $\{\beta_i\}_{i\geq 1}$ of independent Brownian motions such that

$$< w(t), e > = \sum_{n=1}^{\infty} \sqrt{\lambda_i} < e_i, e > \beta_i(t), \ e \in Y,$$

and $B_t = B_t^w$, where B_t^w is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Y_0, X)$ denote the space of all Hilbert- Schmidt operators from Y_0 into X. It turns out to be a separable Hilbert space equipped with the norm $\|\mu\|^2_{L_2^0} = tr((\mu Q^{1/2})(\mu Q^{1/2})^*)$ for any $\mu \in L_2^0$. Clearly for any bounded operator $\mu \in L(Y, X)$ this norm reduces to $\|\mu\|^2_{L_2^0} = tr(\mu Q \mu^*)$.

We now make the system (2.1) precise: Let A be the infinitesimal generator of a strongly continuous semigroup $\{S(t), \ t \geq 0\}$ defined on X. Let the functions $f : \Re^+ \times \hat{D} \to X$; $a : \Re^+ \times \hat{D} \to L(Y, X)$, where $\Re^+ = [0, \infty)$, are Borel measurable. Here $\hat{D} = D((-\infty, 0], X)$ denotes the family of all right piecewise continuous functions with left-hand limit φ from $(-\infty, 0]$ to X. The phase space $D((-\infty, 0], X)$

is assumed to be equipped with the norm $\|\varphi\|_t = \sup_{-\infty<\theta\le 0} |\varphi(\theta)|$. We also assume $D^b_{B_0}((-\infty,0],X)$ to denote the family of all almost surely bounded, B_0-measurable, $\hat{D}$- valued random variables. Let $L_2 = L_2(\Omega, B, X)$ denote the Hilbert space of all B- measurable square integrable random variables with values in X. Further, let $\mathcal{B}_\mathcal{T}$ be a Banach space $\mathcal{B}_\mathcal{T}((-\infty,T],L_2)$, the family of all B_t-adapted process $\varphi(t,w)$ with almost surely continuous in t for fixed $w \in \Omega$ with norm defined for any $\varphi \in \mathcal{B}_\mathcal{T}$

$$\|\varphi\|_{\mathcal{B}_\mathcal{T}} = (\sup_{0\le t\le T} E\|\varphi\|_t^2)^{1/2}.$$

In this article, we will examine impulsive stochastic semilinear functional differential equations of the form

$$\begin{aligned} dx(t) &= \Big[Ax(t) + f(t,x_t)\Big]dt + a(t,x_t)dw(t),\ t \neq t_k,\ k = 1,2,\dots m, \\ \Delta x(t_k) &= x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \\ x(t) &= \varphi \in D^b_{B_0}((-\infty,0],X), \end{aligned} \tag{2.1}$$

where $0 \le t \le T$. The fixed moments of time t_k satisfies $0 < t_1 < \dots < t_m < T$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. And $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump in the state x at time t_k with I_k determining the size of the jump. The notation A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $\{S(t),\ t \ge 0\}$ with $D(A) \subset X$.

Lemma 2.1.[4] *Let $T > 0$ and $u_0 \ge 0$, $u(t), v(t)$ be the continuous functions on $[0,T]$. Let $K : \Re^+ \to \Re^+$ be a concave continuous and nondecreasing function such that $K(r) > 0$ for $r > 0$. If*

$$u(t) \le u_0 + \int_0^t v(s)K(u(s))ds \ \textit{for all}\ 0 \le t \le T,$$

then

$$u(t) \le G^{-1}\Big(G(u_0) + \int_0^t v(s)ds\Big) \quad \textit{for all}\ t \in [0,T] \ \textit{such that}$$

$$G(u_0) + \int_0^t v(s)ds \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{K(s)}$ for $r \ge 0$ and G^{-1} is the inverse function of G. In particular, moreover if, $u_0 = 0$ and $\int_{0^+} \frac{ds}{K(s)} = \infty$, then $u(t) = 0$ for all $t \in [0,T]$.

In order to obtain the stability of solutions, we use the following extended Bihari's inequality

Lemma 2.2.[10] *Let the assumptions of Lemma 2.1 hold. If*

$$u(t) \le u_0 + \int_t^T v(s)K(u(s))ds \ \textit{for all}\ 0 \le t \le T,$$

then

$$u(t) \leq G^{-1}\Big(G(u_0) + \int_t^T v(s)ds\Big) \quad \textit{for all } t \in [0,T] \textit{ such that}$$

$$G(u_0) + \int_t^T v(s)ds \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{K(s)}$ *for* $r \geq 0$ *and* G^{-1} *is the inverse function of* G.

Corollary 2.3.[10] *Let the assumptions of Lemma 2.1 hold and* $v(t) \geq 0$ *for* $t \in [0,T]$. *If for all* $\epsilon > 0$, *there exists* $t_1 \geq 0$ *such that for* $0 \leq u_0 < \epsilon$, $\int_{t_1}^T v(s)ds \leq \int_{u_0}^{\epsilon} \frac{ds}{K(s)}$ *holds, then for every* $t \in [t_1, T]$, *the estimate* $u(t) \leq \epsilon$ *holds.*

Lemma 2.4.[5] *For any* $r \geq 1$ *and for arbitrary* L_2^0*- valued predictable process* $\Phi(\cdot)$

$$\sup_{s\in[0,t]} E\| \int_0^s \Phi(u)dw(u)\|_X^{2r} = (r(2r-1))^r \Big(\int_0^t (E\|\Phi(s)\|_{L_2^0}^{2r})ds\Big)^r.$$

Definition 2.5. A semigroup $\{S(t),\ t \geq 0\}$ is said to be uniformly bounded if $\|S(t)\| \leq M$ for all $t \geq 0$, where $M \geq 1$ is some constant.

Definition 2.6. A stochastic process $\{x(t) \in \mathcal{B}_T,\ t \in (-\infty, T]\}, (0 < T < \infty)$ is called a mild solution of the equation (2.1) if
(i) $x(t) \in X$ is B_t- adapted;
(ii) $x(t)$ satisfies the integral equation

$$x(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0], \\ S(t)\varphi(0) + \int_0^t S(t-s)f(s,x_s)ds + \int_0^t S(t-s)a(s,x_s)dw(s) \\ + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), \ a.s \ t \in [0,T]. \end{cases} \tag{2.2}$$

3. Existence and uniqueness

In this section, we discuss the existence and uniqueness of mild solution of the system (2.1). We use the following hypotheses to prove our results.

Hypotheses:

(H_1) : A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, whose domain $D(A)$ is dense in X.

(H_2) : For each $x, y \in \hat{D}$ and for all $t \in [0,T]$, such that,

$$\|f(t,x_t) - f(t,y_t)\|^2 \vee \|a(t,x_t) - a(t,y_t)\|^2 \leq K(\|x-y\|_t^2),$$

where $K(\cdot)$ is a concave non-decreasing function from $\Re^+$ to $\Re^+$, $K(0) = 0$, $K(u) > 0$, for $u > 0$ and $\int_{0+} \frac{du}{K(u)} = \infty$.

(H_3) : The function $I_k \in C(X,X)$ and there exists some constant h_k such that

$$\|I_k(x(t_k)) - I_k(y(t_k))\|^2 \leq h_k \|x-y\|_t^2, \quad \text{for each } x, y \in \hat{D},\ k = 1, 2 \ldots, m.$$

(H_4) : For all $t \in [0,T]$, it follows that $f(t,0), a(t,0), I_k(0) \in L^2$, for $k = 1,2\ldots,m$ such that

$$\|f(t,0)\|^2 \vee \|a(t,0)\|^2 \vee \|I_k(0)\|^2 \leq \kappa_0,$$

where $\kappa_0 > 0$ is a constant.

Let us now introduce the successive approximations to equation (2.2) as follows

$$x^0(t) = \begin{cases} \varphi(t) \ for \ t \in (-\infty, 0], \\ S(t)\varphi(0) \ for \ \ t \in [0,T]. \end{cases} \tag{3.1}$$

and, for $n = 1, 2, \ldots,$

$$x^n(t) = \begin{cases} \varphi(t) \ for \ t \in (-\infty, 0], \\ S(t)\varphi(0) + \int_0^t S(t-s)f(s, x_s^{n-1})ds + \int_0^t S(t-s)a(s, x_s^{n-1})dw(s) \\ + \sum_{0<t_k<t} S(t-t_k)I_k(x^{n-1}(t_k)) \ for \ a.s \ \ t \in [0,T], \end{cases} \tag{3.2}$$

with an arbitrary non-negative initial approximation $x^0 \in \mathcal{B}_{\mathcal{T}}$.

Theorem 3.1. *Assume that $(H_1) - (H_4)$ hold. Then the system (2.1) has unique mild solution $x(t)$ in $\mathcal{B}_{\mathcal{T}}$, provided there is $M \geq 1$ such that $\|S(t)\| \leq M$ and*

$$M^2 m \sum_{k=1}^{m} h_k < \frac{1}{3}.$$

Proof : Let $x^0 \in \mathcal{B}_{\mathcal{T}}$ be a fixed initial approximation to (3.2). First observe that by (H_1) - (H_4), $\|S(t)\| \leq M$ for some $M \geq 1$ and all $t \in [0,T]$. Then for any $n \geq 1$, we have,

$$\begin{aligned} \|x^n(t)\|^2 \leq\ & 4M^2 \|\varphi(0)\|^2 \\ & +8TM^2 \int_0^t \big[\|f(s, x_s^{n-1}) - f(s,0)\|^2 + \|f(s,0)\|^2\big] ds \\ & +8M^2 \int_0^t \big[\|a(s, x_s^{n-1}) - a(s,0)\|^2 + \|a(s,0)\|^2\big] ds \\ & +8M^2 m \sum_{k=1}^{m} \big[\|I_k(x^{n-1}(t_k)) - I_k(0)\|^2 + \|I_k(0)\|^2\big]. \end{aligned}$$

Thus,

$$\begin{aligned} E\|x^n\|_t^2 \leq\ & Q_1 + 8M^2(T+1)E\int_0^t K(\|x^{n-1}\|_s^2)ds \\ & +8M^2 m \sum_{k=1}^{m} h_k \Big\{E\|x^{n-1}\|_t^2\Big\}, \end{aligned}$$

where, $Q_1 = 4M^2\big(E\|\varphi(0)\|^2 + 2\big(T(T+1) + m\sum_{k=1}^{m} h_k\big)\kappa_0\big)$.

Given that $K(\cdot)$ is concave and $K(0) = 0$, we can find a pair of positive constants a and b such that

$$K(u) \leq a + bu, \text{ for all } \ u \geq 0.$$

Then we have,

$$\begin{aligned} E\|x^n\|_t^2 &\leq Q_2 + 8M^2(T+1)b\int_0^t E\|x^{n-1}\|_s^2 ds \\ &\quad +8M^2 m\sum_{k=1}^m h_k\{E\left\|x^{n-1}\right\|_t^2\},\ n=1,2,\ldots \end{aligned} \tag{3.3}$$

where, $Q_2 = Q_1 + 8M^2(T+1)Ta$.
Since

$$E\left\|x^0\right\|_t^2 \leq M^2 E\|\varphi(0)\|^2 = Q_3 < \infty. \tag{3.4}$$

Thus,

$$E\|x^n\|_t^2 \leq Q_4 < \infty, \text{ for all } n = 0,1,2,\ldots \text{ and } t \in [0,T]. \tag{3.5}$$

This proves the boundedness of $\{x^n(t), n \in \mathbb{N}\}$.
Let us next show that $\{x^n(t)\}$ is Cauchy in $\mathcal{B}_T$. For this, for $n, m \geq 1$, we have

$$\begin{aligned} \left\|x^{n+1}(t) - x^{m+1}(t)\right\|^2 &\leq 3M^2(T+1)\int_0^t K(\|x^n(s) - x^m(s)\|^2)ds \\ &\quad +3M^2 m\sum_{k=1}^m h_k\|x^n(t) - x^m(t)\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} \sup_{0\leq s\leq t} E\left\|x^{n+1} - x^{m+1}\right\|_s^2 &\leq Q_5\int_0^t K\Big(\sup_{0\leq r\leq s} E\|x^n - x^m\|_r^2\Big)ds \\ &\quad +Q_6\sup_{0\leq s\leq t} E\|x^n - x^m\|_s^2, \end{aligned} \tag{3.6}$$

where $Q_5 = 3M^2(T+1)$ and $Q_6 = 3M^2 m\sum_{k=1}^m h_k$.

Integrating both sides of equation (3.6) and applying Jensen's inequality gives that

$$\begin{aligned}\int_0^t \sup_{0\le l\le s} E\left\|x^{n+1}-x^{m+1}\right\|_l^2 ds &\le Q_5\int_0^t\int_0^s K\Big(\sup_{0\le r\le l} E\|x^n-x^m\|_r^2\Big)dlds\\ &\quad +Q_6\int_0^t \sup_{0\le l\le s} E\|x^n-x^m\|_l^2 ds,\\ &\le Q_5\int_0^t s\int_0^s K\Big(\sup_{0\le r\le l} E\|x^n-x^m\|_r^2\Big)\frac{1}{s}dlds\\ &\quad +Q_6\int_0^t \sup_{0\le l\le s} E\|x^n-x^m\|_l^2 ds,\\ &\le Q_5 t\int_0^t K\Big(\int_0^s \sup_{0\le r\le l} E\|x^n-x^m\|_r^2\frac{1}{s}dl\Big)ds\\ &\quad +Q_6\int_0^t \sup_{0\le l\le s} E\|x^n-x^m\|_l^2 ds.\end{aligned}$$

Then,

$$\Psi_{n+1,m+1}(t) \;\le\; Q_5\int_0^t K\Big(\Psi_{n,m}(s)\Big)ds + Q_6\Psi_{n,m}(t), \tag{3.7}$$

where

$$\Psi_{n,m}(t) \;=\; \frac{\displaystyle\int_0^t \sup_{0\le l\le s} E\|x^n-x^m\|_l^2 ds}{t}.$$

From (3.5), it is easy to see that

$$\sup_{n,m}\Psi_{n,m}(t) \;<\; \infty.$$

So letting $\Psi(t) = \limsup_{n,m\to\infty}\Psi_{n,m}(t)$ and taking into account the Fatou's lemma, we yield that

$$\Psi(t) \;=\; \hat{Q}\int_0^t K\Big(\Psi(s)\Big)ds, \text{ where } \hat{Q}=\frac{Q_5}{1-Q_6}.$$

Now, applying the Lemma 2.1, immediately reveals $\Psi(t)=0$ for any $t\in[0,T]$. This further means $\{x^n(t), n\in\mathbb{N}\}$ is a Cauchy sequence in $\mathcal{B}_{\mathcal{T}}$. So there is an $x\in\mathcal{B}_{\mathcal{T}}$ such that

$$\lim_{n\to\infty}\int_0^T \sup_{0\le s\le t} E\|x^n-x\|_s^2 dt = 0.$$

In addition, by (3.5), it is easy to follow that $E\|x\|_t^2\le Q_4$. Thus we claim that $x(t)$ is a mild solutions to (2.1). On the other hand, by (H_2) and letting $n\to\infty$, we can also claim that for $t \in [0,T]$

$$\begin{aligned} E\| \int_0^t S(t-s)\Big[f(s,x_s^{n-1}) - f(s,x_s)\Big]ds\|^2 &\to 0, \\ E\| \int_0^t S(t-s)\Big[a(s,x_s^{n-1}) - a(s,x_s)\Big]dw(s)\|^2 &\to 0 \\ \text{and} \quad E\| \sum_{0<t_k<t} S(t-t_k)\Big[I_k(x^{n-1}(t_k)) - I_k(x(t_k))\Big]\|^2 &\to 0. \end{aligned}$$

Hence, taking limits on both sides of (3.2),

$$\begin{aligned} x(t) &= S(t)\varphi(0) + \int_0^t S(t-s)f(s,x_s)ds + \int_0^t S(t-s)a(s,x_s)dw(s) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)). \end{aligned}$$

This certainly demonstrates by the Definition 2.6 that $x(t)$ is a mild solution to (2.1) on the interval $[0,T]$.

Now, we prove the uniqueness of the solutions of (2.2). Let $x_1, x_2 \in \mathcal{B}_{\mathcal{T}}$ be two solutions of (2.1) on some interval $(-\infty, T]$. Then, for $t \in (-\infty, 0]$, the uniqueness is obvious and for $0 \le t \le T$, we have

$$\begin{aligned} E\|x_1 - x_2\|_t^2 &\le Q_6 E\|x_1 - x_2\|_t^2 \\ &\quad + Q_5 \int_0^t K(E\|x_1 - x_2\|_s^2)ds. \end{aligned}$$

Thus,

$$E\|x_1 - x_2\|_t^2 \le \frac{Q_5}{1-Q_6}\int_0^t K(E\|x_1 - x_2\|_s^2)ds.$$

Thus, Bihari's inequality yields that

$$\sup_{t\in[0,T]} E\|x_1 - x_2\|_t^2 = 0, \ 0 \le t \le T.$$

Thus, $x_1(t) = x_2(t)$, for all $0 \le t \le T$. Therefore, for all $-\infty < t \le T$, $x_1(t) = x_2(t)$. This achieve the proof. □

4. Stability

In this section, we study the stability through the continuous dependence on initial values.

Definition 4.1. A mild solution $x(t)$ of the system (2.1) with initial value ϕ is said to be stable in the mean square if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$E\|x - \hat{x}\|_t^2 \le \epsilon \text{ whenever } E\|\phi - \hat{\phi}\|^2 < \delta, \text{ for all } t \in [0,T]. \tag{4.1}$$

where $\hat{x}(t)$ is another mild solution of the system (2.1) with initial value $\hat{\phi}$.

Theorem 4.2. *Let $x(t)$ and $y(t)$ be mild solutions of the system* (2.1) *with initial values φ_1 and φ_2 respectively. If the assumptions of Theorem* 3.1 *are satisfied, then the mild solution of the system* (2.1) *is stable in the mean square.*

Proof: By the assumptions, $x(t)$ and $y(t)$ are two mild solutions of equations (2.1) with initial values φ_1 and φ_2 respectively, then for $0 \le t \le T$

$$\begin{aligned} x(t) - y(t) = & \; S(t)\big[\varphi_1(0) - \varphi_2(0)\big] + \int_0^t S(t-s)\big[f(s,x_s) - f(s,y_s)\big]ds \\ & + \int_0^t S(t-s)\big[a(s,x_s) - a(s,y_s)\big]dw(s) + \sum_{0<t_k<t} S(t-t_k)\big[I_k(x(t_k)) - I_k(y(t_k))\big]. \end{aligned}$$

So, estimating as before, we get

$$\begin{aligned} E\|x-y\|_t^2 \;\le\; & 4M^2 E\|\varphi_1 - \varphi_2\|^2 + 4M^2(T+1)\int_0^t K(E\|x-y\|_s^2)ds \\ & + 4M^2 m \sum_{k=1}^m h_k E\|x-y\|_t^2. \end{aligned}$$

Thus,

$$\begin{aligned} E\|x-y\|_t^2 \;\le\; & \frac{4M^2}{1 - 4M^2 m \sum_{k=1}^m h_k} E\|\varphi_1 - \varphi_2\|^2 \\ & + \frac{4M^2(T+1)}{1 - 4M^2 m \sum_{k=1}^m h_k} \int_0^t K(E\|x-y\|_s^2)ds. \end{aligned}$$

Let $K_1(u) = \frac{4M^2(T+1)}{1-4M^2 m \sum_{k=1}^m h_k} K(u)$, where K is a concave increasing function from $\Re^+$ to $\Re^+$ such that $K(0) = 0$, $K(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{du}{K(u)} = +\infty$. So, $K_1(u)$ is obviously, a concave function from $\Re^+$ to $\Re^+$ such that $K_1(0) = 0$, $K_1(u) \ge K(u)$, for $0 \le u \le 1$ and $\int_{0^+} \frac{du}{K_1(u)} = +\infty$. Now for any $\epsilon > 0$, $\epsilon_1 = \frac{1}{2}\,\epsilon$, we have $\lim\limits_{s \to 0} \displaystyle\int_s^{\epsilon_1} \frac{du}{K_1(u)} = \infty$. So, there is a positive constant $\delta < \epsilon_1$, such that $\int_\delta^{\epsilon_1} \frac{du}{K_1(u)} \ge T$.

Let

$$\begin{aligned} u_0 &= \frac{4M^2}{1 - 4M^2 m \sum_{k=1}^m h_k} E\|\varphi_1 - \varphi_2\|^2, \\ u(t) &= E\|x-y\|_t^2, \; v(t) = 1, \end{aligned}$$

when $u_0 \le \delta \le \epsilon_1$. From Corollary 2.3 we have

$$\int_{u_0}^{\epsilon_1} \frac{du}{K_1(u)} \ge \int_\delta^{\epsilon_1} \frac{du}{K_1(u)} \ge T = \int_0^T v(s)ds.$$

So, for any $t \in [0,T]$, the estimate $u(t) \le \epsilon_1$ holds. This completes the proof. $\square$

Remark 4.3.

If $m = 0$ in (2.1), then the system behave as stochastic partial functional differential equations with infinite delays of the form

$$\begin{cases} dx(t) = \Big[Ax(t) + f(t,x_t)\Big]dt + a(t,x_t)dw(t), \; 0 \le t \le T, \\ x(t) = \varphi \in D^b_{B_0}((-\infty,0],X). \end{cases} \tag{4.2}$$

By applying Theorem 3.1 under the hypotheses $(H_1)-(H_2), (H_4)$ the system (4.2) guarantees the existence and uniqueness of the mild solution.

Remark 4.4.
If the system (4.2) satisfies the Remark 4.1, then by Theorem 4.1, the mild solution of the system (4.2) is stable in the mean square.

5. An example

We conclude this work with an example of the form

$$\begin{aligned} du(t,x) &= \Big[\frac{\partial^2 u(t,x)}{\partial x^2} + H(t,u(tsint,x))\Big]dt + \sigma\, G(t,u(tsint,x))d\beta(t), \\ &\qquad t \neq t_k,\ 0 \le t \le T,\ 0 \le x \le \pi, \end{aligned} \tag{5.1}$$

together with the initial conditions

$$u(t_k^+) - u(t_k^-) = (1+b_k)u(x(t_k)),\ t = t_k,\ k = 1,2,\ldots m, \tag{5.2}$$

$$u(t,0) = u(t,\pi) = 0, \tag{5.3}$$

$$u(t,x) = \Phi(t,x),\ 0 \le x \le \pi,\ -\infty < t \le 0. \tag{5.4}$$

Let $X = L^2([0,\pi])$ and $Y = R^1$, the real number σ is magnitude of continuous noise, $\beta(t)$ is a standard one dimension Brownian motion, $\Phi \in D^b_{B_0}((-\infty,0],X)$, $b_k \ge 0$ for $k = 1,2,\ldots,m$ and $\sum_{k=1}^m b_k < \infty$.

Define an operator A on X by $Au = \frac{\partial^2 u}{\partial x^2}$ with the domain

$$D(A) = \Big\{u \in X \,\Big|\, u \text{ and } \frac{\partial u}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 u}{\partial x^2} \in X,\ u(0) = u(\pi) = 0\Big\}.$$

It is well known that A generates a strongly continuous semigroup $S(t)$ which is compact, analytic and selfadjoint. Moreover, the operator A can be expressed as

$$Au = \sum_{n=1}^{\infty} n^2 < u, u_n > u_n,\ u \in D(A),$$

where $u_n(\zeta) = (\frac{2}{\pi})^{\frac{1}{2}} \sin(n\zeta)$, $n = 1,2,\ldots$, is the orthonormal set of eigenvectors of A, and

$$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} < u, u_n > u_n,\ u \in X.$$

Then the problem (5.1)−(5.4) can be modeled as the abstract impulsive stochastic semilinear functional differential equation of the form (2.1), as follows

$$\begin{aligned} f(t,x_t) &= H(t,u(tsint,x)),\ a(t,x_t) = \sigma\, G(t,u(tsint,x)) \\ \text{and } I_k(x(t_k)) &= (1+b_k)u(x(t_k)) \text{ for } k = 1,2,\ldots m. \end{aligned}$$

The below results are consequence of Theorem 3.1 and Theorem 4.1 respectively.

Proposition 5.1. *If the hypotheses $(H_1)-(H_5)$ hold, then there exists a unique mild solution u of the system $(5.1)-(5.4)$.*

Proposition 5.2. *If all the hypotheses of Proposition* 5.1 *hold, then the mild solution* u *of the system* $(5.1)-(5.4)$ *is stable in the mean square.*

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