J. Nonlinear Sci. Appl. 4 (2011), no. 4, 262–269

THE JOURNAL OF NONLINEAR SCIENCES AND APPLICATIONS
http://www.tjnsa.com

TRIPLE SOLUTIONS FOR NONLINEAR SINGULAR m-POINT BOUNDARY VALUE PROBLEM

FULI WANG

ABSTRACT. In this paper, we study the existence of three solutions to the following nonlinear m-point boundary value problem

$$\begin{cases} u''(t) + \beta^2 u(t) = h(t) f(t, u(t)), & 0 < t < 1, \\ u'(0) = 0, \ u(1) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases}$$

where $0 < \beta < \frac{\pi}{2}$, $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$. $h(t)$ is allowed to be singular at $t = 0$ and $t = 1$. The arguments are based only upon the Leggett-Williams fixed point theorem. We also prove nonexist results.

1. INTRODUCTION

Motivated by the work of Bitsadze and Samarskii on nonlocal linear elliptic boundary value problem [1, 2], Il'in and Moiseev studied a multipoint boundary value problems for linear second-order ordinary differential equations [3]. Since then, great efforts have been devoted to the multipoint boundary value problems for more general nonlinear ordinary differential equations due to its theoretical challenge and its great potential applications; see for example [4]–[13] and the references therein.

In 2007, Han [13] studied the existence of positive solutions for three-point boundary value problem

$$\begin{cases} u''(t) + \beta^2 u(t) = h(t) f(t, u(t)), & 0 < t < 1, \\ u'(0) = 0, \ u(\eta) = u(1), \end{cases} \tag{1.1a}$$

Date: Received: August 14, 2010; Revised: November 15, 2010.

*Corresponding author

2000 *Mathematics Subject Classification.* 34B15; 34B25.

Key words and phrases. m-point boundary value problem; Positive solutions; Fixed point theorem.

where $0 < \beta < \frac{\pi}{2}$, $0 < \eta < 1$, under the assumptions:
(C_1) $h : (0, 1) \to [0, +\infty)$ is continuous, $h(t) \not\equiv 0$, and

$$\int_0^1 h(t)dt < +\infty.$$

(C_2) $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

He established the following result for (1.1a).

Theorem 1.1[13]. *Assume (C_1) and (C_2) hold. Then in each of the following cases:*
(i)

$$\liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda_1; \quad \limsup_{u\to +\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda_1;$$

(ii)

$$\liminf_{u\to +\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} > \lambda_1; \quad \limsup_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \lambda_1.$$

Then BVP (1.1a) has at least one positive solution, where λ_1 is the first eigenvalue of the corresponding linear positive operator.

In this paper, we are concerned with the second-order m-point boundary value problem

$$\begin{cases} u''(t) + \beta^2 u(t) = h(t)f(t, u(t)), & 0 < t < 1, \\ u'(0) = 0, \ u(1) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases} \tag{1.1}$$

where $0 < \beta < \frac{\pi}{2}$, $m \geq 3$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-1} < 1$ and $\alpha_i \geq 0$, for all $i = 1, 2, \ldots, m-2$ such that $\sum\limits_{i=1}^{m-2} \alpha_i < 1$.

The goal of this paper is to obtain the existence of multiple solutions for the singular second-order m-point boundary value problem (1.1) under the sufficient conditions by applying the well-known Leggett-Williams fixed point theorem. The emphasis here is that, in our main result, we don't care whether the following limits exist or not:

$$\liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \ \limsup_{u\to +\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \ \liminf_{u\to +\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \ \limsup_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{u}.$$

Therefore, our conclusion improves and extends the main results contained in [13].

The paper is divided into four sections. In Section 2, we provide some preliminaries and various lemmas, which play key roles in this paper. In Section 3, we obtain the existence of multiple positive solutions of the m-point boundary value problem (1.1). In Section 4, we give the nonexistence of positive solution.

2. Preliminaries and lemmas

In Banach space $C[0, 1]$ in which the norm is defined by $\|u\| = \max\limits_{0\leq t\leq 1} |u(t)|$ for any $u \in C[0, 1]$. We set $K = \{u \in C[0, 1] | u(t) \geq 0, \ t \in \ [0, 1]\}$ be a cone in $C[0, 1]$.

The function u is said to be a positive solution of BVP (1.1) if $u \in C[0,1] \cap C^2(0,1)$ satisfies (1.1) and $u(t) > 0$ for $t \in (0,1)$.

Let $G(t,s)$ be the Green's function of the problem (1.1) with $h(t)f(t,u) \equiv 0$ (see [14]), that is,

$$\begin{aligned} G(t,s) &= \begin{cases} \dfrac{1}{\beta}\sin\beta(t-s), & 0 \le s \le t \le 1, \\ 0, & 0 \le t \le s \le 1 \end{cases} \\ &+\frac{K_m}{\beta}\cos\beta t \begin{cases} \sin\beta(1-s) - \sum\limits_{i=1}^{m-2}\alpha_i \sin\beta(\eta_i - s), & 0 \le s \le \eta_1, \\ \sin\beta(1-s) - \sum\limits_{i=2}^{m-2}\alpha_i \sin\beta(\eta_i - s), & \eta_1 \le s \le \eta_2, \\ \sin\beta(1-s) - \sum\limits_{i=3}^{m-2}\alpha_i \sin\beta(\eta_i - s), & \eta_2 \le s \le \eta_3, \\ \cdots \\ \sin\beta(1-s) - \sum\limits_{i=k}^{m-2}\alpha_i \sin\beta(\eta_i - s), & \eta_{k-1} \le s \le \eta_k, \\ \cdots \\ \sin\beta(1-s), & \eta_{m-2} \le s \le 1. \end{cases} \end{aligned}$$

It is known [14] that there exist a constant $\sigma \in (0,1)$ and a continuous funcition $\Phi : [0,1] \to [0,+\infty)$ such that

$$\sigma\Phi(s) \le G(t,s) \le \Phi(s), \quad \text{for all} \quad t,s \in [0,1]. \tag{2.1}$$

We make the following assumptions:

(H_1) $h : (0,1) \to [0,+\infty)$ is continuous, $h(t) \not\equiv 0$, and

$$\int_0^1 h(t)dt < +\infty;$$

(H_2) $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

Let

$$(Au)(t) = \int_0^1 G(t,s)h(s)f(s,u(s))ds, \quad t \in [0,1]. \tag{2.2}$$

We can verify that the nonzero fixed points of the operator A are positive solutions of the problem (1.1).

Define

$$P = \{u \in K | u(t) \ge \sigma\|u\|, \ t \in [0,1]\}.$$

Then P is subcone of K.

Lemma 2.1. *Suppose that (H_1) and (H_2) are satisfied. Then $A : P \to P$ is a completely continuous operator.*

Proof. Let $u \in K$. Since $G(t,s) \ge 0$, $(t,s) \in [0,1] \times [0,1]$, by the definition, we have $(Au)(t) \ge 0$, $t \in [0,1]$. On the other hand, by (2.1) we have

$$(Au)(t) = \int_0^1 G(t,s)h(s)f(s,u(s))ds \ge \sigma \int_0^1 \Phi(s)h(s)f(s,u(s))ds, \tag{2.3}$$

$$\|Au\| = \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)f(s,u(s))ds \le \int_0^1 \Phi(s)h(s)f(s,u(s))ds, \qquad (2.4)$$

for every $t \in [0,1]$, by (2.3) and (2.4) we have

$$(Au)(t) \ge \sigma\|Au\|.$$

Thus, we assert that $A : P \to P$. It follows from Arzela-Ascoli's theorem that if (H_1)–(H_2) are satisfied, $A : P \to P$ is completely continuous.

Our main result concerning three positive solutions of (1.1) will arise as applications of the following Leggett-Williams fixed point theorem [15].

A map $\alpha : P \to [0,+\infty)$ is said to be a nonnegative continuous concave functional on P if α is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in P$ and $t \in [0,1]$. Let a, b be two numbers such that $0 < a < b$ and α is a nonnegative continuous concave functional on P. We define the following convex sets:

$$P_a = \{x \in P | \|x\| < a\},$$
$$P(\alpha, a, b) = \{x \in P | a \le \alpha(x), \|x\| \le b\}.$$

Lemma 2.2. *Let $A : \overline{P_c} \to \overline{P_c}$ be completely continuous and α be a nonnegative continuous concave functional on P such that $\alpha(x) \le \|x\|$ for all $x \in \overline{P_c}$. Suppose there exist $0 < d < a < b \le c$ such that*

(i) $\{x \in P(\alpha,a,b) | \alpha(x) > a\} \ne \emptyset$ and $\alpha(Ax) > a$ for $x \in P(\alpha,a,b)$;

(ii) $\|Ax\| < d$ for $\|x\| \le d$;

(iii) $\alpha(Ax) > a$ for $x \in P(\alpha,a,c)$ with $\|Ax\| > b$.

Then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P_c}$ satisfying $x_1 < d, a < \alpha(x_2), \|x_3\| > d$ and $\alpha(x_3) < a$.

3. Existence result

In this section, we will impose growth conditions on f which allow us to apply Lemma 2.2 in regard to obtaining three solutions of (1.1).

Theorem 3.1. *Suppose that (H_1) and (H_2) are satisfied. If there exist constants a and d with $0 < d < a$ such that*

$$f(t,u) < \frac{d}{D},\ t \in [0,1],\ u \in [0,d], \qquad (3.1)$$

and

$$f(t,u) > \frac{a}{C},\ t \in [0,1],\ u \in [a, \frac{a}{\sigma}], \qquad (3.2)$$

where

$$D = \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)ds \quad \text{and} \quad C = \min_{t\in[0,1]} \int_0^1 G(t,s)h(s)ds.$$

Suppose further that one of the following conditions holds:

(H_3)

$$\lim_{u\to\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \frac{1}{D};$$

(H_4) *there exists a number c such that $c > \frac{a}{\sigma}$ and if $t \in [0,1]$ and $u \in [0,c]$, then $f(t,u) < \frac{c}{D}$.*
Then m-point BVP (1.1) has at least three nonnegative solutions.

Proof. For $u \in K$, let

$$\alpha(u) = \min_{t\in[0,1]} u(t).$$

It is easy to check that α is a nonnegative continuous concave functional on P with $\alpha(x) \le \|x\|$ for $x \in P$.

We first assert that if (H_3) holds, then there exists a number c such that $c > \frac{a}{\sigma}$ and $A : \overline{P_c} \to P_c$. To see this, suppose

$$\lim_{u\to\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} < \frac{1}{D},$$

then there exists $\tau > 0$ and $\gamma < \frac{1}{D}$ such that if $u > \tau$, then

$$\max_{t\in[0,1]} \frac{f(t,u)}{u} \le \gamma,$$

that is to say, $f(t,u) \le \gamma u$ for $t \in [0,1]$ and $u > \tau$. Set

$$\beta = \max_{t\in[0,1],u\in[0,\tau]} f(t,u).$$

Then $f(t,u) \le \gamma u + \beta$ for all $t \in [0,1]$ and $u \in [0,+\infty)$. Take

$$c > \max\left\{\frac{\beta D}{1-\gamma D}, \frac{a}{\sigma}\right\}.$$

If $u \in \overline{P_c}$, then

$$\begin{aligned}
\|Au\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)f(s,u(s))ds \\
&\le \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)[\gamma u(s)+\beta]ds \\
&\le \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)[\gamma\|u\|+\beta]ds \\
&\le (\gamma c+\beta)D < c.
\end{aligned}$$

Next we assert that if there exists a positive number r such that $f(t,u) < \frac{r}{D}$ for $t \in [0,1]$ and $u \in [0,r]$, then $A : \overline{P_r} \to P_r$. Indeed, if $u \in \overline{P_r}$, then

$$\|Au\| = \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)f(s,u(s))ds < \frac{r}{D}D = r.$$

Hence, we have shown that if either (H_3) or (H_4) holds, then there exists a number $c > \frac{a}{\sigma}$ such that A maps $\overline{P_c}$ into P_c.

Note that if $r = d$, then we may assert further that A maps $\overline{P_d}$ into P_d by (3.1).

Next, we assert that $\{u \in P(\alpha, a, \frac{a}{\sigma}) | \alpha(u) > a\} \neq \emptyset$ and $\alpha(Au) > a$ for all $u \in P(\alpha, a, \frac{a}{\sigma})$. Indeed, the constant function $\frac{a+\frac{a}{\sigma}}{2} \in \{u \in P(\alpha, a, \frac{a}{\sigma}) | \alpha(Au) > a\}$. Moreover, for $u \in P(\alpha, a, \frac{a}{\sigma})$, we have

$$\frac{a}{\sigma} \geq \|u\| \geq u(t) \geq \min_{t\in[0,1]} u(t) = \alpha(u) \geq a,$$

for all $t \in [0, 1]$. Thus, in view of (3.2), we see that

$$\begin{aligned} \alpha(Au) &= \min_{t\in[0,1]} \int_0^1 G(t,s)h(s)f(s,u(s))ds \\ &> \tfrac{a}{C} \min_{t\in[0,1]} \int_0^1 G(t,s)h(s)ds \\ &= a, \end{aligned}$$

as required.

Finally, we assert that if $u \in P(\alpha, a, c)$ and $\|Au\| > \frac{a}{\sigma}$, then $\alpha(Au) > a$. To see this, suppose $u \in P(\alpha, a, c)$ and $\|Au\| > \frac{a}{\sigma}$, then we have

$$\begin{aligned} \alpha(Au) &= \min_{t\in[0,1]} \int_0^1 G(t,s)h(s)f(s,u(s))ds \\ &\geq \sigma \int_0^1 \Phi(s)h(s)f(s,u(s))ds, \end{aligned}$$

for $t \in [0, 1]$. Thus,

$$\begin{aligned} \alpha(Au) &\geq \sigma \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)f(s,u(s))ds \\ &= \sigma\|Au\| > \sigma\tfrac{a}{\sigma} = a. \end{aligned}$$

To sum up, all the hypotheses of the Leggett-Williams theorem are satisfied by taking $b = \frac{a}{\sigma}$. Hence, A has at least three fixed points, i.e., the m-point BVP (1.1) has at least three nonnegative solutions u, v and w such that $\|u\| < d, a < \min_{t\in[0,1]} v(t), \|w\| > d$ and $\min_{t\in[0,1]} w(t) < a$. The proof is complete. □

4. Nonexistence results

In this section, we present our nonexistence results. Note that condition (H_1) holds throughout this section as well.

Theorem 4.1. If $Bf(t, u) < u$ for all $t \in [0, 1]$ and $u \in (0, +\infty)$, where $B = \int_0^1 \Phi(s)h(s)ds$. Then m-point BVP (1.1) has no positive solutions.

Proof. Assume, to the contrary, that $u(t)$ is a positive solution of (1.1). Then

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)h(s)f(s,u(s))ds \\ &< \tfrac{1}{B}\int_0^1 G(t,s)h(s)u(s)ds \\ &\leq \tfrac{1}{B}\int_0^1 \Phi(s)h(s)ds\|u\| \\ &= \|u\|, \end{aligned}$$

which is a contradiction. The proof is complete. □

Theorem 4.2. If $B'f(t,u) > u$ for all $t \in [0,1]$ and $u \in (0,+\infty)$, where $B' = \sigma^2 \int_0^1 \Phi(s)h(s)ds$. Then m-point BVP (1.1) has no positive solutions.

Proof. Assume, to the contrary, that $u(t)$ is a positive solution of (1.1). Then

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)h(s)f(s,u(s))ds \\ &> \tfrac{1}{B'}\int_0^1 G(t,s)h(s)u(s)ds \\ &\geq \tfrac{\sigma^2}{B'}\int_0^1 \Phi(s)h(s)ds\|u\| \\ &= \|u\|, \end{aligned}$$

which is a contradiction. The proof is complete. □

References

[1] A.V. Bitsadze, On the theory of nonlocal boundary value problems, Soviet Math. Dokl. 30 (1984) 8–10.

[2] A.V. Bitsadze and A.A. Samarskii, On a class of conditionally solvable nonlocal boundary value problems for harmonic functions, Soviet Math. Dokl. 31 (1985) 91–94.

[3] V.A. Il'in, E.I. Moiseev, Nonlocal boundary value problems of the second kind for a Sturm-CLiouville operator, J. Differential Equations 23 (1987) 979–987.

[4] C.P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation, J. Math. Anal. Appl. 168 (1992) 540–551.

[5] F. Zhang, Multiple positive solutions for nonlinear singular third order boundary value problem in abstract spaces., J. Nonlinear. Sci. Appl.1(1), (2008),36–44.

[6] X.R. Wu, F. Wang, Nonlinear solutions of singular second order three-point boundary value problem at resonance., J. Nonlinear. Sci. Appl. 1(1), (2008), 49–55.

[7] Y.Y. Pang, Z.B. Bai, Existence and multiplicity of positive solutions for p-Laplacian boundary value problem on time scales., J. Nonlinear. Sci. Appl. 3(1), (2010), 32–38.

[8] A. Guezane-Lakoud, S. Kelaiaia, Solvability of a nonlinear boundary value problem., J. Nonlinear. Sci. Appl. 4(4), (2011), 247–261.

[9] F. Wang, Y. Cui and F. Zhang, Existence of nonnegative solutions for second order m-point boundary value problems at resonance, Appl. Math. Comput. 217 (2011) 4849–4855.

[10] G. Zhang and J. Sun, Positive solutions of m-point boundary value problems, J. Math. Anal. Appl. 291 (2004) 406–418.

[11] G. Zhang and J. Sun, Multiple positive solutions of singular second-order m-point boundary value problems, J. Math. Anal. Appl. 317 (2006) 442–447.

[12] Y. Cui and Y. Zou, Nontrivial solutions of singular superlinear m-point boundary value problems, Appl. Math.Comp. 187 (2007) 1256–1264.
[13] X. Han, Positive solutions for a three-point boundary value problem, Nonlinear Anal., 66 (3), (2007) 679–688.
[14] L.X. Truong, L.T.P. Ngoc, N.T. Long, Positive solutions for an m-point boundary value problem, Electron. J. Differential Eqns., 111(2008), (2008) 1–11.
[15] R.W. Leggett and L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, Indiana University Math. J., 28, (1979) 673–688.

School of Mathematics and Physics, Changzhou University, Changzhou, China
E-mail address: fuliwang2011@163.com