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CONVERGENCE OF IMPLICIT RANDOM ITERATION PROCESS WITH ERRORS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE RANDOM OPERATORS

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Abstract. In this paper, we prove that an implicit random iteration process with errors which is generated by a finite family of asymptotically quasi-nonexpansive random operators converges strongly to a common random fixed point of the random operators in uniformly convex Banach spaces.

1. Introduction

Random nonlinear analysis is an important mathematical discipline which is mainly concerned with the study of random nonlinear operators and their properties and is needed for the study of various classes of random equations. The study of random fixed point theory was initiated by the Prague school of Probabilities in the 1950s [12, 13, 25]. Common random fixed point theorems are stochastic generalization of classical common fixed point theorems. The machinery of random fixed point theory provides a convenient way of modeling many problems arising from economic theory (see e.g. [19]) and references mentioned therein. Random methods have revolutionized the financial markets. The survey article by Bharucha-Reid [9] attracted the attention of several mathematicians and gave wings to the theory. Itoh [15] extended Spacek's and Hans's theorem to multivalued contraction mappings. Now this theory has become the full fledged research area and various ideas associated with random fixed point theory are

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used to obtain the solution of nonlinear random system (see [4, 5, 8, 14, 21]). Papageorgiou [17, 18], Beg [2, 3] studied common random fixed points and random coincidence points of a pair of compatible random operators and proved fixed point theorems for contractive random operators in Polish spaces. Recently, Beg and Shahzad [7], Choudhury [11] and Badshah and Sayyed [1] used different iteration processes to obtain random fixed points. More recently, Beg and Abbas [6] studied common random fixed points of two asymptotically nonexpansive random operators through strong as well as weak convergence of sequence of measurable functions in the setup of uniformly convex Banach spaces. Also they construct different random iterative algorithms for asymptotically quasi-nonexpansive random operators on an arbitrary Banach space and established their convergence to random fixed point of the operators.

In 2001, Xu and Ori [27] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space H. Let C be a nonempty subset of H. Let $T_1, T_2, \ldots, T_N$ be self-mappings of C and suppose that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i, i = 1, 2, \ldots, N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows, with $\{t_n\}$ a real sequence in $(0, 1)$, $x_0 \in C$:

$$
\begin{aligned}
x_1 &= t_1 x_0 + (1 - t_1) T_1 x_1, \\
x_2 &= t_2 x_1 + (1 - t_2) T_2 x_2, \\
&\vdots \\
x_N &= t_N x_{N-1} + (1 - t_N) T_N x_N, \\
x_{N+1} &= t_{N+1} x_N + (1 - t_{N+1}) T_1 x_{N+1}, \\
&\vdots
\end{aligned}
$$

which can be written in the following compact form:

$$
x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \geq 1 \tag{1.1}
$$

where $T_k = T_{k \bmod N}$. (Here the mod N function takes values in $\mathcal{N}$). And they proved the weak convergence of the process (1.1).

In 2003, Sun [23] extend the process (1.1) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with $\{\alpha_n\}$ a real sequence in $(0, 1)$ and an initial point $x_0 \in C$, which is defined as follows:

$$
\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1) T_1 x_1, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_1^2 x_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N}) T_N^2 x_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1}) T_1^3 x_{2N+1}, \\
&\vdots
\end{aligned}
$$

which can be written in the following compact form:

$$
x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_i^k x_n, \quad n \geq 1 \tag{1.2}
$$

where $n = (k-1)N + i$, $i \in \mathcal{N}$.

Sun [23] proved the strong convergence of the process (1.2) to a common fixed point, requiring only one member T in the family $\{T_i : i \in \mathcal{N}\}$ to be semi-compact. The result of Sun [23] generalized and extended the corresponding main results of Wittmann [26] and Xu and Ori [27].

The purpose of this paper is to introduce and study an implicit random iteration process with errors which converges strongly to a common random fixed point of a finite family of asymptotically quasi-nonexpansive random operators in uniformly convex Banach spaces. Our results extend and improve the corresponding results of Beg and Abbas [5] and many others.

2. Preliminaries

Let (Ω, Σ) be a measurable space (Σ-sigma algebra) and let F be a nonempty subset of a Banach space X. We will denote by $C(X)$ the family of all compact subsets of X with Hausdorff metric H induced by the metric of X. A mapping $\xi\colon \Omega \to X$ is measurable if $\xi^{-1}(U) \in \Sigma$, for each open subset U of X. The mapping $T\colon \Omega \times X \to X$ is a random map if and only if for each fixed $x \in X$, the mapping $T(.,x)\colon \Omega \to X$ is measurable and it is continuous if for each $\omega \in \Omega$, the mapping $T(\omega,.)\colon X \to X$ is continuous. A measurable mapping $\xi\colon \Omega \to X$ is a random fixed point of a random map $T\colon \Omega \times X \to X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$, for each $\omega \in \Omega$. We denote the set of random fixed points of a random map T by $RF(T)$. Let $RF(T_i)$, $(i \in \mathcal{N})$ be the set of all random fixed points of T_i, where $\mathcal{N} = \{1, 2, \ldots, N\}$. The set of common random fixed points of T_i $(i \in \mathcal{N})$ denoted by $\mathcal{F}$, that is, $\mathcal{F} = \cap_{i=1}^N RF(T_i)$.

Let $B(x_0, r)$ denote the spherical ball centered at x_0 with radius r, defined as the set $\{x \in X : \|x - x_0\| \leq r\}$.

We denote the *nth* iterate $T(\omega, T(\omega, T(\omega, \ldots, T(\omega, x) \ldots,)))$ of T by $T^n(\omega, x)$. The letter I denotes the random mapping $I \colon \Omega \times X \to X$ defined by $I(\omega, x) = x$ and $T^0 = I$.

Definition 2.1. Let C be a nonempty separable subset of a Banach space X and $T \colon \Omega \times C \to C$ be a random map. The map T is said to be the following:

(a) A *nonexpansive random operator* if for $x, y \in C$, one has

$$\|T(\omega, x) - T(\omega, y)\| \ \leq \ \|x - y\|, \tag{2.1}$$

for each $\omega \in \Omega$.

(b) *Asymptotically nonexpansive random operator* if there exists a sequence of measurable mapping $k_n \colon \Omega \to [1, \infty)$ with $\lim_{n\to\infty} k_n(\omega) = 1$, such that for $x, y \in C$, one has

$$\|T^n(\omega, x) - T^n(\omega, y)\| \ \leq \ k_n(\omega) \|x - y\|, \tag{2.2}$$

for each $\omega \in \Omega$.

(c) *Asymptotically quasi-nonexpansive random operator* if for each $\omega \in \Omega$, $G(\omega) = \{x \in C : x = T(\omega, x)\} \neq \phi$ and there exists a sequence of measurable mapping $r_n \colon \Omega \to [0, \infty)$ with $\lim_{n\to\infty} r_n(\omega) = 0$, such that for $x \in C$ and $y \in G(\omega)$, the following inequality holds:

$$\|T^n(\omega, x) - y\| \ \leq \ (1 + r_n(\omega)) \|x - y\|, \tag{2.3}$$

for each $\omega \in \Omega$.

(d) Uniformly *L-Lipschitzian random operator* if for $x, y \in C$

$$\|T^n(\omega, x) - T^n(\omega, y)\| \ \leq \ L \|x - y\|, \tag{2.4}$$

for each $\omega \in \Omega$, where $n = 1, 2, \ldots$ and L is a positive constant.

(e) A *semicompact random operator* if for a sequence of measurable mappings $\{\xi_n\}$ from Ω to C, with $\lim_{n\to\infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0$, for every $\omega \in \Omega$, one has a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ and a measurable mapping $\xi \colon \Omega \to C$ such that $\{\xi_{n_k}\}$ converges pointwisely to ξ as $k \to \infty$.

Motivated and inspired by Xu and Ori [27], Sun [23] and some others, we introduce and study an implicit random iteration scheme with errors as follows:

Definition 2.2. Let $\{T_1, T_2, \ldots, T_N\}$ be a family of N-random operators from $\Omega \times C$ to C, where C is a nonempty closed convex subset of a separable Banach space X. Let $\{f_n\}$ be a bounded sequence of measurable mappings from Ω to C and let $\xi_0 \colon \Omega \to C$ be a measurable mapping. Define the random iteration process with errors $\{\xi_n\}$ as follows:

$$
\begin{aligned}
\xi_1(\omega) &= \alpha_1\xi_0(\omega) + \beta_1 T_1(\omega, \xi_1(\omega)) + \gamma_1 f_1(\omega), \\
\xi_2(\omega) &= \alpha_2\xi_1(\omega) + \beta_2 T_2(\omega, \xi_2(\omega)) + \gamma_2 f_2(\omega), \\
&\vdots \\
\xi_N(\omega) &= \alpha_N\xi_{N-1}(\omega) + \beta_N T_N(\omega, \xi_N(\omega)) + \gamma_N f_N(\omega), \\
\xi_{N+1}(\omega) &= \alpha_{N+1}\xi_N(\omega) + \beta_{N+1} T_1^2(\omega, \xi_{N+1}(\omega)) + \gamma_{N+1} f_{N+1}(\omega), \\
&\vdots \\
\xi_{2N}(\omega) &= \alpha_{2N}\xi_{2N-1}(\omega) + \beta_{2N} T_N^2(\omega, \xi_{2N}(\omega)) + \gamma_{2N} f_{2N}(\omega), \\
\xi_{2N+1}(\omega) &= \alpha_{2N+1}\xi_{2N}(\omega) + \beta_{2N+1} T_1^3(\omega, \xi_{2N+1}(\omega)) + \gamma_{2N+1} f_{2N+1}(\omega), \\
&\vdots
\end{aligned}
$$

which can be written in the following compact form:

$$\xi_n(\omega) = \alpha_n\xi_{n-1}(\omega) + \beta_n T_i^k(\omega, \xi_n(\omega)) + \gamma_n f_n(\omega), \tag{2.5}$$

where $n = (k-1)N + i$, $i \in \mathcal{N}$, and each $\{f_n(\omega)\}$ is bounded sequence in C, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three appropriate real sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for $n = 1, 2, \dots$. Process (2.5) is called the implicit random iteration process with errors for a finite family of random operators T_i $(i = 1, 2, \dots, N)$.

Recall that a mapping $T\colon C \to C$ where C is a subset of X with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) [22] if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

Definition 2.3. A family $\{T_i : i \in \mathcal{N}\}$ of N-mappings on C, where C be a nonempty subset of a separable Banach space X with $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy *condition* (B) on C if there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $a_1\|x - T_1x\| + a_2\|x - T_2x\| + \cdots + a_N\|x - T_Nx\| \geq f(d(x, \mathcal{F}))$ for all $x \in C$, where $d(x, \mathcal{F}) = \inf\{\|x - p\| : p \in \mathcal{F}\}$ and $a_1, a_2, \dots, a_N$ are N nonnegative real numbers such that $a_1 + a_2 + \cdots + a_N = 1$.

Remark 2.4. condition (B) reduces to condition (A) when $T_1 = T_2 = \cdots = T_N = T$.

In the sequel, we will need the following lemmas.

Lemma 2.5. *(Tan and Xu [24]): Let $\{a_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + r_n)a_n + \beta_n, \ \ \forall n \in N.$$

If $\sum_{n=1}^\infty r_n < \infty$, $\sum_{n=1}^\infty \beta_n < \infty$. Then

(i) $\lim_{n\to\infty} a_n$ exists.

(ii) If $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. *(Schu [20]) Let* E *be a uniformly convex Banach space and* $0 < a \le t_n \le b < 1$ *for all* $n \ge 1$. *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* E *satisfying*

$$\limsup_{n\to\infty} \|x_n\| \le r, \quad \limsup_{n\to\infty} \|y_n\| \le r, \quad \lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r,$$

for some $r \ge 0$. *Then*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

3. Main Results

In this section, we study an implicit random iteration scheme with errors to common random fixed point for a finite family of asymptotically quasi-nonexpansive random operators in uniformly convex Banach spaces. We also establish the necessary and sufficient condition for the convergence of this process to the common random fixed point of the above said finite family of random operators. Our results extend the corresponding results of Beg and Abbas [5] and Chang et al. [10].

Theorem 3.1. *Let* X *be a uniformly convex separable Banach space, and let* C *be a nonempty closed and convex subset of* X. *Let* $\{T_i : i \in \mathcal{N}\}$ *be a finite family of asymptotically quasi-nonexpansive random operators from* $\Omega \times C$ *to* C *with sequences of measurable mapping* $r_{n_i}\colon \Omega \to [0,\infty)$ *satisfying* $\sum_{n=1}^{\infty} r_{n_i}(\omega) < \infty$ *for each* $\omega \in \Omega$ *and for all* $i \in \mathcal{N}$ *and* $\mathcal{F} = \cap_{i=1}^{N} RF(T_i) \ne \emptyset$. *Let* ξ_0 *be a measurable mapping from* Ω *to* C, *then the sequence of random implicit iteration process with errors defined as by (2.5) converges to a common random fixed point of random operators* $\{T_i : i \in \mathcal{N}\}$ *in* C *if and only if* $\liminf_{n\to\infty} d(\xi_n(\omega), \mathcal{F}) = 0$, *where* $\{\beta_n\} \subset (s, 1-s)$ *for some* $s \in (0, \frac{1}{2})$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\{f_n(\omega)\}$ *is arbitrary bounded sequence in* C.

Proof. The necessity is obvious. Thus we will only prove the sufficiency. For any measurable mapping $\xi \in \mathcal{F}$, from (2.5), where $n = (k-1)N + i$, $T_n = T_{n(mod\ N)} = T_i,\ i \in \mathcal{N}$, it follows that

$$\begin{aligned}\|\xi_n(\omega)-\xi(\omega)\| =&\|\alpha_n\xi_{n-1}(\omega)+\beta_n T_i^k(\omega,\xi_n(\omega))+\gamma_n f_n(\omega)-\xi(\omega)\|\\ =&\|\alpha_n(\xi_{n-1}(\omega)-\xi(\omega))+\beta_n(T_i^k(\omega,\xi_n(\omega))-\xi(\omega))\\ &+\gamma_n(f_n(\omega)-\xi(\omega))\|\\ \leq&\alpha_n\|\xi_{n-1}(\omega)-\xi(\omega)\|+\beta_n\|T_i^k(\omega,\xi_n(\omega))-\xi(\omega)\|\\ &+\gamma_n\|f_n(\omega)-\xi(\omega)\|\\ \leq&\alpha_n\|\xi_{n-1}(\omega)-\xi(\omega)\|+\beta_n(1+r_{k_i}(\omega))\|\xi_n(\omega)-\xi(\omega)\|\\ &+\gamma_n\|f_n(\omega)-\xi(\omega)\|\\ \leq&\alpha_n\|\xi_{n-1}(\omega)-\xi(\omega)\|+(\beta_n+r_{k_i}(\omega))\|\xi_n(\omega)-\xi(\omega)\|\\ &+\gamma_n\|f_n(\omega)-\xi(\omega)\|\\ \leq&\alpha_n\|\xi_{n-1}(\omega)-\xi(\omega)\|+(1-\alpha_n+r_{k_i}(\omega))\|\xi_n(\omega)-\xi(\omega)\|\\ &+\gamma_n\|f_n(\omega)-\xi(\omega)\|. \end{aligned}\tag{3.1}$$

Since $\lim_{n\to\infty}\gamma_n=0$, there exists a natural number n_1 such that for $n>n_1$, $\gamma_n\leq\frac{s}{2}$. Hence

$$\alpha_n=1-\beta_n-\gamma_n\geq 1-(1-s)-\frac{s}{2}=\frac{s}{2},$$

for $n>n_1$. Thus, we have from (3.1) that

$$\begin{aligned}\alpha_n\|\xi_n(\omega)-\xi(\omega)\| \;\leq\;& \alpha_n\|\xi_{n-1}(\omega)-\xi(\omega)\|+r_{k_i}(\omega)\|\xi_n(\omega)-\xi(\omega)\|\\ &+\gamma_n\|f_n(\omega)-\xi(\omega)\|\end{aligned}$$

and

$$\begin{aligned}\|\xi_n(\omega)-\xi(\omega)\| \;\leq\;& \|\xi_{n-1}(\omega)-\xi(\omega)\|+\frac{r_{k_i}(\omega)}{\alpha_n}\|\xi_n(\omega)-\xi(\omega)\|\\ &+\frac{\gamma_n}{\alpha_n}\|f_n(\omega)-\xi(\omega)\|\\ \leq\;& \|\xi_{n-1}(\omega)-\xi(\omega)\|+\frac{2}{s}r_{k_i}(\omega)\|\xi_n(\omega)-\xi(\omega)\|\\ &+\frac{2}{s}\gamma_n\|f_n(\omega)-\xi(\omega)\|.\end{aligned}\tag{3.2}$$

Since $\sum_{n=1}^{\infty}r_{k_i}(\omega)<\infty$ for all $i\in\mathcal{N}$, $\lim_{n\to\infty}r_{n_i}(\omega)=0$ for each $i\in\mathcal{N}$. Hence there exists a natural number n_2, as $n>\frac{n_2}{N}+1$, that is, $n>n_2$ such that

$$r_{n_i}(\omega)\leq\frac{s}{4},\quad\forall\ i\in\mathcal{N}.$$

Then (3.2) becomes

$$\|\xi_n(\omega)-\xi(\omega)\|\leq\frac{s}{s-2r_{k_i}(\omega)}\|\xi_{n-1}(\omega)-\xi(\omega)\|+\frac{2\gamma_n}{s-2r_{k_i}(\omega)}\|f_n(\omega)-\xi(\omega)\|.\tag{3.3}$$

Let

$$1+\theta_{k_i}(\omega)=\frac{s}{s-2r_{k_i}(\omega)}=1+\frac{2r_{k_i}(\omega)}{s-2r_{k_i}(\omega)}.$$

Then

$$\theta_{k_i}(\omega)=\frac{2r_{k_i}(\omega)}{s-2r_{k_i}(\omega)}<\frac{4}{s}r_{k_i}(\omega).$$

Therefore

$$\sum_{k=1}^{\infty}\theta_{k_i}(\omega)<\frac{4}{s}\sum_{k=1}^{\infty}r_{k_i}(\omega)<\infty,\quad \forall\ i\in\mathcal{N}$$

and (3.3) becomes

$$\begin{aligned}\|\xi_n(\omega)-\xi(\omega)\| \leq&(1+\theta_{k_i}(\omega))\|\xi_{n-1}(\omega)-\xi(\omega)\|+\frac{2\gamma_n}{s-2r_{k_i}(\omega)}\|f_n(\omega)-\xi(\omega)\|\\ \leq&(1+\theta_{k_i}(\omega))\|\xi_{n-1}(\omega)-\xi(\omega)\|+\frac{4}{s}\gamma_n M,\end{aligned}\tag{3.4}$$

where, $M=\sup_{n\geq1}\|f_n(\omega)-\xi(\omega)\|$, since $\{f_n(\omega)\}$ is a bounded sequence in C. This implies that

$$d(\xi_n(\omega),\mathcal{F})\leq(1+\theta_{k_i}(\omega))d(\xi_{n-1}(\omega),\mathcal{F})+\frac{4}{s}\gamma_n M.$$

Since $\sum\limits_{k=1}^{\infty}\theta_{k_i}(\omega)<\infty$ and $\sum\limits_{k=1}^{\infty}\gamma_n<\infty$, it follows from Lemma 2.5, we know that $\lim\limits_{n\to\infty}d(\xi_n(\omega),\mathcal{F})=0$.

Next, we will prove that $\{\xi_n\}$ is a Cauchy sequence. Notice that when $x>0$, $1+x\leq e^x$, from (3.4) we have

$$\begin{aligned}
\|\xi_{n+m}(\omega)-\xi(\omega)\| \leq & (1+\theta_{k_i}(\omega))\|\xi_{n+m-1}(\omega)-\xi(\omega)\| + \frac{4M}{s}\gamma_{n+m} \\
\leq & (1+\theta_{k_i}(\omega))\Big[(1+\theta_{k_i}(\omega))\|\xi_{n+m-2}(\omega)-\xi(\omega)\| \\
& + \frac{4M}{s}\gamma_{n+m-1}\Big] + \frac{4M}{s}\gamma_{n+m} \\
\leq & (1+\theta_{k_i}(\omega))^2\Big[(1+\theta_{k_i}(\omega))\|\xi_{n+m-3}(\omega)-\xi(\omega)\| \\
& + \frac{4M}{s}\gamma_{n+m-2}\Big] + \frac{4M}{s}(1+\theta_{k_i}(\omega))(\gamma_{n+m-1}+\gamma_{n+m}) \\
\leq & (1+\theta_{k_i}(\omega))^3\|\xi_{n+m-3}(\omega)-\xi(\omega)\| \\
& + \frac{4M}{s}(1+\theta_{k_i}(\omega))^3\Big(\gamma_{n+m-2}+\gamma_{n+m-1}+\gamma_{n+m}\Big) \\
\leq & \dots \\
\leq & \ exp\left\{\sum_{i=1}^{N}\sum_{k=1}^{\infty}\theta_{k_i}(\omega)\right\}\|\xi_n(\omega)-\xi(\omega)\| \\
& + \frac{4M}{s}\ exp\left\{\sum_{i=1}^{N}\sum_{k=1}^{\infty}\theta_{k_i}(\omega)\right\}\sum_{j=n+1}^{n+m}\gamma_j \\
\leq & M'\|\xi_n(\omega)-\xi(\omega)\| + \frac{4MM'}{s}\sum_{j=n+1}^{n+m}\gamma_j, \qquad (3.5)
\end{aligned}$$

for all $\xi \in \mathcal{F}$ and $m, n \in \mathbb{N}$, where $M' = exp\left\{\sum_{i=1}^{N}\sum_{k=1}^{\infty}\theta_{k_i}(\omega)\right\} < \infty$. Since $\lim_{n\to\infty} d(\xi_n(\omega), \mathcal{F}) = 0$ and $\sum_{k=1}^{\infty} r_{k_i}(\omega) < \infty$ $(i \in \mathcal{N})$, there exists a natural number n_1 such that for $n \geq n_1$,

$$d(\xi_n(\omega), \mathcal{F}) < \frac{\varepsilon}{4M'} \ \ and \ \ \sum_{j=n_1+1}^{n+m}\gamma_j \leq \frac{s.\varepsilon}{16MM'}.$$

Thus there exists a point $\xi(\omega) \in \mathcal{F}$ such that $d(\xi_{n_1}(\omega), \xi(\omega)) < \frac{\varepsilon}{4M'}$ for each $\omega \in \Omega$. It follows from (3.5) that for all $n \geq n_1$ and $m \geq 1$, we have

$$\begin{aligned}
\|\xi_{n+m}(\omega)-\xi_n(\omega)\| \leq& \|\xi_{n+m}(\omega)-\xi(\omega)\|+\|\xi_n(\omega)-\xi(\omega)\| \\
\leq& M'\|\xi_{n_1}(\omega)-\xi(\omega)\|+\frac{4MM'}{s}\sum_{j=n_1+1}^{n+m}\gamma_j \\
&+M'\|\xi_{n_1}(\omega)-\xi(\omega)\|+\frac{4MM'}{s}\sum_{j=n_1+1}^{n+m}\gamma_j \\
<& M'.\frac{\varepsilon}{4M'}+\frac{4MM'}{s}.\frac{s.\varepsilon}{16MM'} \\
&+M'.\frac{\varepsilon}{4M'}+\frac{4MM'}{s}.\frac{s.\varepsilon}{16MM'} \\
=&\varepsilon. \qquad (3.6)
\end{aligned}$$

This implies that $\{\xi_n(\omega)\}$ is a Cauchy sequence for each $\omega \in \Omega$. Therefore, $\xi_n(\omega) \to p(\omega)$ for each $\omega \in \Omega$, where $p\colon \Omega \to \mathcal{F}$, being the limit of the measurable mappings, is also measurable. Now, $\lim_{n\to\infty} d(\xi_n(\omega), \mathcal{F}) = 0$, for each $\omega \in \Omega$, and the set $\mathcal{F}$ is closed; we have $p \in \mathcal{F}$, that is, p is a common random fixed point of $\{T_i : i \in \mathcal{N}\}$. This completes the proof. □

Lemma 3.2. *Let X be a uniformly convex separable Banach space, and let C be a nonempty closed and convex subset of X. Let $\{T_i : i \in \mathcal{N}\}$ be a finite family of uniformly L-Lipschitzian, asymptotically quasi-nonexpansive random operators from $\Omega\times C$ to C with sequences of measurable mapping $r_{n_i}\colon \Omega \to [0,\infty)$ satisfying $\sum_{n=1}^{\infty} r_{n_i}(\omega) < \infty$ for each $\omega \in \Omega$ and for all $i \in \mathcal{N}$ and $\mathcal{F} = \cap_{i=1}^{N} RF(T_i) \neq \emptyset$. Let ξ_0 be a measurable mapping from Ω to C, and let the sequence of random implicit iteration process with errors defined as by (2.5). If $\liminf_{n\to\infty} d(\xi_n(\omega), \mathcal{F}) = 0$, where $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$, $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\{f_n(\omega)\}$ is arbitrary bounded sequence in C, then*

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| = 0,$$

for each $\omega \in \Omega$.

Proof. It follows from (3.4), and Lemma 2.5, that $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$ exists for any $\xi \in \mathcal{F}$. Since $\{\xi_n(\omega) - \xi(\omega)\}$ is a convergent sequence, without loss of generality, we can assume that

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| \;=\; d_\omega, \qquad (3.7)$$

where $d_\omega \geq 0$. Observe that

$$\begin{aligned}
\|\xi_n(\omega)-\xi(\omega)\| =& \|\alpha_n\xi_{n-1}(\omega)+\beta_n T_i^k(\omega,\xi_n(\omega))+\gamma_n f_n(\omega)-\xi_(\omega)\| \\
=& \|\beta_n[T_i^k(\omega,\xi_n(\omega))-\xi(\omega)+\gamma_n(f_n(\omega)-\xi_{n-1}(\omega))] \\
&+(1-\beta_n)[\xi_{n-1}(\omega)-\xi(\omega)+\gamma_n(f_n(\omega)-\xi_{n-1}(\omega))]\|. \qquad (3.8)
\end{aligned}$$

From $\sum_{n=1}^{\infty} \gamma_n < \infty$ and (3.7), it follows that

$$\begin{aligned}\limsup_{n\to\infty} \|\xi_{n-1}(\omega) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_{n-1}(\omega))\| \\ \leq \limsup_{n\to\infty}[\|\xi_{n-1}(\omega) - \xi(\omega)\| \\ + \gamma_n \|f_n(\omega) - \xi_{n-1}(\omega)\|] \\ \leq d_\omega, \qquad (3.9)\end{aligned}$$

and hence

$$\begin{aligned}\limsup_{n\to\infty} \|T_n^k(\omega, \xi_n(\omega)) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_{n-1}(\omega))\| \\ \leq \limsup_{n\to\infty}[\|T_n^k(\omega, \xi_n(\omega)) - \xi(\omega)\| \\ + \gamma_n\|f_n(\omega) - \xi_{n-1}(\omega)\|] \\ \leq \limsup_{n\to\infty}[(1 + r_{k_n}(\omega))\|\xi_n(\omega)) - \xi(\omega)\| \\ + \gamma_n\|f_n(\omega) - \xi_{n-1}(\omega)\|] \\ \leq d_\omega, \qquad (3.10)\end{aligned}$$

where $n = (k-1)N + i$.

Therefore from (3.7) - (3.10) and Lemma 2.6, we have that

$$\lim_{n\to\infty} \left\|T_n^k(\omega, \xi_n(\omega)) - \xi_{n-1}(\omega)\right\| = 0, \qquad (3.11)$$

for each $\omega \in \Omega$.

Moreover, since

$$\begin{aligned}\|\xi_n(\omega) - \xi_{n-1}(\omega)\| &= \left\|\alpha_n\xi_{n-1}(\omega) + \beta_n T_n^k(\omega, \xi_n(\omega)) + \gamma_n f_n(\omega) - \xi_{n-1}(\omega)\right\| \\ &= \left\|\beta_n[T_n^k(\omega, \xi_n(\omega)) - \xi_{n-1}(\omega)] + \gamma_n[f_n(\omega) - \xi_{n-1}(\omega)]\right\| \\ &\leq \beta_n \left\|T_n^k(\omega, \xi_n(\omega)) - \xi_{n-1}(\omega)\right\| \\ &\quad + \gamma_n \|f_n(\omega) - \xi_{n-1}(\omega)\|, \qquad (3.12)\end{aligned}$$

hence by (3.11), we obtain

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi_{n-1}(\omega)\| = 0, \qquad (3.13)$$

for each $\omega \in \Omega$ and $\|\xi_n(\omega) - \xi_{n+l}(\omega)\| \to 0$, for each $\omega \in \Omega$ and $l < 2N$. Now, for $n > N$, we have

$$\begin{aligned}\|\xi_{n-1}(\omega)-T_n(\omega,\xi_n(\omega))\| \leq&\|\xi_{n-1}(\omega)-T_n^k(\omega,\xi_n(\omega))\| \\ &+\|T_n^k(\omega,\xi_n(\omega))-T_n(\omega,\xi_n(\omega))\| \\ \leq&\|\xi_{n-1}(\omega)-T_n^k(\omega,\xi_n(\omega))\| \\ &+L\|T_n^{k-1}(\omega,\xi_n(\omega))-\xi_n(\omega)\| \\ \leq&\|\xi_{n-1}(\omega)-T_n^k(\omega,\xi_n(\omega))\| \\ &+L\|T_n^{k-1}(\omega,\xi_n(\omega))-T_{n-N}^{k-1}(\omega,\xi_{n-N}(\omega))\| \\ &+L\Big[\|T_{n-N}^{k-1}(\omega,\xi_{n-N}(\omega))-\xi_{(n-N)-1}(\omega)\| \\ &+\|\xi_{(n-N)-1}(\omega)-\xi_n(\omega)\|\Big]. \end{aligned} \tag{3.14}$$

Since for each $n>N$, $n\equiv(n-N)\ mod\ N$. Thus $T_n=T_{n-N}$, therefore

$$\begin{aligned}\|\xi_{n-1}(\omega)-T_n(\omega,\xi_n(\omega))\| \leq&\|\xi_{n-1}(\omega)-T_n^k(\omega,\xi_n(\omega))\|+L^2\|\xi_n(\omega)-\xi_{n-N}(\omega)\| \\ &+L\|T_{n-N}^{k-1}(\omega,\xi_{n-N}(\omega))-\xi_{(n-N)-1}(\omega)\| \\ &+L\|\xi_{(n-N)-1}(\omega)-\xi_n(\omega)\|. \end{aligned} \tag{3.15}$$

This implies that

$$\lim_{n\to\infty}\|\xi_{n-1}(\omega)-T_n(\omega,\xi_n(\omega))\| \;=\; 0, \tag{3.16}$$

for each $\omega\in\Omega$. Now

$$\begin{aligned}\|\xi_n(\omega)-T_n(\omega,\xi_n(\omega))\| \leq&\|\xi_{n-1}(\omega)-\xi_n(\omega)\| \\ &+\|\xi_{n-1}(\omega)-T_n(\omega,\xi_n(\omega))\|. \end{aligned} \tag{3.17}$$

Hence

$$\lim_{n\to\infty}\|\xi_n(\omega)-T_n(\omega,\xi_n(\omega))\| \;=\; 0, \tag{3.18}$$

for each $\omega\in\Omega$. □

Theorem 3.3. *Let X be a uniformly convex separable Banach space, and let C be a nonempty closed and convex subset of X. Let $\{T_i : i\in\mathcal{N}\}$ be a finite family of uniformly L-Lipschitzian, asymptotically quasi-nonexpansive random operators from $\Omega\times C$ to C with sequences of measurable mapping $r_{n_i}\colon\Omega\to[0,\infty)$ satisfying $\sum\limits_{n=1}^{\infty} r_{n_i}(\omega)<\infty$ for each $\omega\in\Omega$ and for all $i\in\mathcal{N}$ and $\mathcal{F}=\cap_{i=1}^N RF(T_i)\neq\emptyset$. Suppose there is one member T in the family $\{T_i : i\in\mathcal{N}\}$ which is semi-compact random operator. Let ξ_0 be a measurable mapping from Ω to C. Then the sequence of random implicit iteration process with errors defined as by (2.5) converges to a common random fixed point of random operators $\{T_i : i\in\mathcal{N}\}$, where $\{\beta_n\}\subset(s,1-s)$ for some $s\in(0,\frac{1}{2})$, $\sum\limits_{n=1}^{\infty}\gamma_n<\infty$ and $\{f_n(\omega)\}$ is arbitrary bounded sequence in C.*

Proof. For any given $\xi(\omega) \in \mathcal{F}$, we note that

$$\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = d_\omega, \tag{3.19}$$

where $d_\omega \geq 0$. By Lemma 3.2, we know that

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| = 0, \tag{3.20}$$

for each $\omega \in \Omega$. Consequently, for any $j \in \mathcal{N}$,

$$\begin{aligned} \|\xi_n(\omega) - T_{n+j}(\omega, \xi_n(\omega))\| &\leq \|\xi_n(\omega) - \xi_{n+j}(\omega)\| \\ &\quad + \|\xi_{n+j}(\omega) - T_{n+j}(\omega, \xi_{n+j}(\omega))\| \\ &\quad + \|T_{n+j}(\omega, \xi_{n+j}(\omega)) - T_{n+j}(\omega, \xi_n(\omega))\| \\ &\leq (1+L)\|\xi_n(\omega) - \xi_{n+j}(\omega)\| \\ &\quad + \|\xi_{n+j}(\omega) - T_{n+j}(\omega, \xi_{n+j}(\omega))\| \\ &\quad \to 0, \end{aligned} \tag{3.21}$$

as $n \to \infty$ for each $\omega \in \Omega$ and $j \in \mathcal{N}$, where $\mathcal{N} = \{1, 2, \ldots, N\}$.

Consequently, $\|\xi_n(\omega) - T_j(\omega, \xi_n(\omega))\| \to 0$ as $n \to \infty$ for each $\omega \in \Omega$ and $j \in \mathcal{N}$. Assume that T_j is a semi-compact random operator. Therefore, there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ and a measurable mapping $\xi_0\colon \Omega \to C$ such that ξ_{n_k} converges pointwise to ξ_0. Now

$$\lim_{n\to\infty} \|\xi_{n_k}(\omega) - T_j(\omega, \xi_{n_k}(\omega))\| = \|\xi_0(\omega) - T_j(\omega, \xi_0(\omega))\| = 0,$$

for each $\omega \in \Omega$, and $j \in \mathcal{N}$. It implies that $\xi_0 \in \mathcal{F}$, and so $\liminf_{n\to\infty} d(\xi_n(\omega), \mathcal{F}) = 0$. Hence, by Theorem 3.1, we obtain that $\{\xi_n\}$ converges to a point in $\mathcal{F}$. This completes the proof. □

Theorem 3.4. *Let X be a uniformly convex separable Banach space, and let C be a nonempty closed and convex subset of X. Let $\{T_i : i \in \mathcal{N}\}$ be a finite family of uniformly L-Lipschitzian, asymptotically quasi-nonexpansive random operators from $\Omega \times C$ to C with sequences of measurable mapping $r_{n_i}\colon \Omega \to [0, \infty)$ satisfying $\sum_{n=1}^{\infty} r_{n_i}(\omega) < \infty$ for each $\omega \in \Omega$ and for all $i \in \mathcal{N}$ and $\mathcal{F} = \cap_{i=1}^{N} RF(T_i) \neq \emptyset$. Suppose the family $\{T_i : i \in \mathcal{N}\}$ satisfies the condition (B). Let ξ_0 be a measurable mapping from Ω to C. Then the sequence of random implicit iteration process with errors defined as by (2.5) converges to a common random fixed point of random operators $\{T_i : i \in \mathcal{N}\}$, where $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{f_n(\omega)\}$ is arbitrary bounded sequence in C.*

Proof. Let $\xi(\omega) \in \mathcal{F}$. Then it follows from (3.19) and the condition (B) that

$$\begin{aligned} d(\xi_n(\omega), \mathcal{F}) &\leq a_1 \|\xi_n(\omega) - T_1(\omega, \xi_n(\omega))\| + a_2 \|\xi_n(\omega) - T_2(\omega, \xi_n(\omega))\| \\ &\quad + \cdots + a_N \|\xi_n(\omega) - T_N(\omega, \xi_n(\omega))\| \end{aligned} \tag{3.22}$$

for each $n \geq 1$, and $\omega \in \Omega$. Since $\|\xi_n(\omega) - T_j(\omega, \xi_n(\omega))\| \to 0$ as $n \to \infty$, for each $\omega \in \Omega$ and $j \in \mathcal{N}$, we have

$$\lim_{n\to\infty} f(d(\xi_n(\omega), \mathcal{F})) \;=\; 0. \tag{3.23}$$

Since f is nondecreasing on $[0, \infty)$ with $f(0) = 0$ and $f(r) > 0$, for all $r \in (0, \infty)$, it follows that

$$\lim_{n\to\infty} d(\xi_n(\omega), \mathcal{F}) \;=\; 0. \tag{3.24}$$

Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} d(\xi_n(\omega), \mathcal{F}) = 0$, for each $\omega \in \Omega$, there exists a natural number n_1 such that for $n \geq n_1$, $d(\xi_n(\omega), \mathcal{F}) < \frac{\varepsilon}{3}$, for each $\omega \in \Omega$. In particular, there exists a point $\xi^*(\omega) \in \mathcal{F}$ such that $\|\xi_{n_1}(\omega) - \xi^*(\omega)\| < \frac{\varepsilon}{2}$. Now for $n \geq n_1$ and for all $m \geq 1$, we have

$$\begin{aligned}
\|\xi_{n+m}(\omega) - \xi_n(\omega)\| &\leq \|\xi_{n+m}(\omega) - \xi^*(\omega)\| + \|\xi_n(\omega) - \xi^*(\omega)\| \\
&\leq \|\xi_{n_1}(\omega) - \xi^*(\omega)\| + \|\xi_{n_1}(\omega) - \xi^*(\omega)\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This implies that $\{\xi_n(\omega)\}$ is a Cauchy sequence for each $\omega \in \Omega$. By the completeness of the space X, there exists a measurable mapping $p\colon \Omega \to C$ such that $\lim\limits_{n\to\infty} \xi_n(\omega) = p(\omega)$, for all $\omega \in \Omega$. Next, we prove that $p(\omega) \in \mathcal{F}$. Let $\varepsilon_1 > 0$ be given. Then there exists a natural number n_2 such that $\|\xi_n(\omega) - p(\omega)\| < \frac{\varepsilon_1}{4}$ for all $n \geq n_2$. Since $\lim\limits_{n\to\infty} d(\xi_n(\omega), \mathcal{F}) = 0$, there exists a natural number $n_3 \geq n_2$ such that for all $n \geq n_3$ we have $d(\xi_n(\omega), \mathcal{F}) < \frac{\varepsilon_1}{4}$ and in particular we have $d(\xi_{n_3}(\omega), \mathcal{F}) < \frac{\varepsilon_1}{4}$. This implies that there exists $q(\omega) \in \mathcal{F}$ such that $\|\xi_{n_3}(\omega) - q(\omega)\| < \frac{\varepsilon_1}{4}$. Then for each $i \in \mathcal{N}$ and $n \geq n_3$, we have

$$\begin{aligned}
\|T_i(\omega, p(\omega)) - p(\omega)\| &\leq \|T_i(\omega, p(\omega)) - q(\omega)\| + \|q(\omega) - p(\omega)\| \\
&\leq 2\,\|p(\omega) - q(\omega)\| \\
&\leq 2\Big[\,\|p(\omega) - \xi_{n_3}(\omega)\| + \|\xi_{n_3}(\omega) - q(\omega)\|\,\Big] \\
&< 2\Big(\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4}\Big) = \varepsilon_1.
\end{aligned}$$

Therefore, $T_i(\omega, p(\omega)) = p(\omega)$, for all $i \in \{1, 2, \ldots, N\} = \mathcal{N}$, that is, $p(\omega)$ is a common random fixed point of the random operators $\{T_i : i \in \mathcal{N}\}$. This completes the proof. □

Remark 3.5. Our results extend and improve the corresponding results of Beg and Abbas [5] to the case of implicit random iteration process with errors for a finite family of asymptotically quasi-nonexpansive random operators.

Remark 3.6. Theorem 3.1 is a stochastic version of Theorem 2.1 of Kim et al. [16].

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