EXACT CONTROLLABILITY OF SEMILINEAR THIRD ORDER DISPERSION EQUATION

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ABSTRACT. In this paper, a family of nonlinear functions is given for the exact controllability of semilinear third order dispersion equation. The obtained result has been illustrated by applying it on nonlinear Korteweg-de Vries (KdV) equation.

1. INTRODUCTION

Let us consider the semilinear third-order dispersion equation of the form

$$\frac{\partial \omega}{\partial t}(x,t) + \frac{\partial^3 \omega}{\partial x^3}(x,t) = (Gu)(x,t) + f(t, \omega(x,t)) \qquad (1.1)$$

in the domain $0 \leq x \leq 2\pi$, $t \geq 0$, such that the process is periodic with period 2π in the variable x with boundary conditions

$$\frac{\partial^k \omega}{\partial x^k}(0,t) = \frac{\partial^k \omega}{\partial x^k}(2\pi, t); k = 0, 1, 2, \qquad (1.2)$$

and initial condition

$$\omega(x,0) = 0, \qquad (1.3)$$

where u is the control function and $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function.

By taking $G = 0$ (i.e. without the control term) and nonlinear function $f(t, \omega) = -\omega_x - \omega \omega_x$, the equation (1.1) reduces into the partial differential equation model for the famous Korteweg-de Vries (KdV) equation (for derivation and more details see [1]), which describes the propagation of small amplitude long water waves in

a uniform channel, and in this context, ω is the amplitude or velocity, x is often proportional to distance in the direction of propagation and t is proportional to elapsed time. There is a vast amount of literature on the controllability of linear KdV equation with the controls either as distributed controls or through the boundary, see [2, 3, 4, 5] and references therein. The study of controllability of system (1.1)-(1.3) is important because this system is not only a good model for water waves but also a very useful model in nonlinear studies in many areas of engineering and science like laser theory, observations of the solitary waves, study of solitons in hydrodynamics and plasma theory etc. (see [6, 7, 8] and references therein). The controllability of an abstract semilinear control system in infinite dimensional spaces is an important property and has been studied by many authors in recent past via functional analysis approach, see [9, 10, 11, 12, 13].

For the sake of brevity, we shall write V and Z for the spaces $L_2(0, 2\pi)$ and $L_2[0, \tau; L_2(0, 2\pi)]$ respectively. In (1.1), the state $\omega(., t)$ takes values in the Banach space V and the control function $u \in Z$.

From the practical point of view, we restrict the distributed control $u(x, t)$ so that the quantity $[\omega] = \int_0^{2\pi} \omega(x, t)dx$ is conserved, and in order to conserve this quantity, we define the bounded linear operator G on Z as

$$(Gu)(x, t) = g(x)\{u(x, t) - \int_0^{2\pi} g(s)u(s, t)\mathrm{d}s\}, \qquad (1.4)$$

where $g(x)$ is a piece-wise continuous positive function on $[0, 2\pi]$ such that (see [14])

$$[g] = \int_0^{2\pi} g(s)\mathrm{d}s = 1. \qquad (1.5)$$

In [14], Russell and Zhang proved the controllability and stabilization of the third order linear dispersion equation. They discussed the exponential decay rates with distributed controls of restricted form and for the equation with boundary dissipation. George et. al. [15] proved the exact controllability of (1.1)-(1.5), in the case when f is a Lipschitz continuous function with sufficiently small Lipschitz constant.

2. Preliminaries and Hypotheses

The purpose of this paper is to study the exact controllability of the semilinear third order dispersion system (1.1)-(1.5).

Define an operator A on V with domain $D(A) \subseteq H^3(0, 2\pi)$ consisting of functions satisfying boundary conditions of the form (1.2) such that

$$A\omega = -\frac{\partial^3 \omega}{\partial x^3} \qquad (2.1)$$

It follow from Pazy [16] that A generates a C_0 group of isometries, $\{T(t)\}_{t \geq 0}$, on space V. By the variation of constant formula, we can write a mild solution of

(1.1)-(1.3) as

$$\omega(\cdot, t) = \int_0^t T(t-s)(Gu)(\cdot, s)\mathrm{d}s + \int_0^t T(t-s)f(s, \omega(\cdot, s))\mathrm{d}s \qquad (2.2)$$

Remark 2.1. Note that, some sufficient conditions are available in the literature for the existence of the mild solutions of system of the form (1.1)-(1.3), see [15, 16]. In the present paper we will show the exact controllability of the system (1.1)-(1.3) whose mild solution exists.

Let V_1 be a subspace of V defined as

$$V_1 = \{x \in V : [x_T] = \int_0^{2\pi} x(s)\mathrm{d}s = 0\}. \qquad (2.3)$$

It is proved in [15] that V_1 is a Hilbert space with respect to L_2-norm.
Let us analyze the exact controllability property:

Definition 2.2. The system (1.1)-(1.5) is said to be exactly controllable over a time interval $[0, \tau]$, if for any given $\omega_\tau \in V_1$ there exists a control $u \in Z$ such that the corresponding solution ω of (1.1)-(1.5) satisfies $\omega(\cdot, \tau) = \omega_\tau$.

In this paper we investigated a family of nonlinear functions $f(t, \omega(x, t)) = f_1(t, \omega(x, t)) + f_2(t, \omega(x, t)) + f_3(t, \omega(x, t))$ for that the system (1.1)-(1.5) is exact controllable, when

(H1) $R(F_1) \subseteq N(L)$,
(H2) $\int_0^{2\pi} f_2(t, \omega(x, t))\mathrm{d}x = 0$,
(H3) f_3 is Lipschitz continuous with sufficiently small Lipschitz constant, i.e., it satisfies the following inequality,

$$|f_3(t, x) - f_3(t, y)| \le l|x - y|; \ \forall x, y \in \mathbb{R} \text{ and } t \in [0, \tau] \qquad (2.4)$$

with small constant l.

where operators $L : Z \to V_1$ and $F_1 : Z \to Z$ are defined as

$$L\omega = \int_0^\tau T(\tau - s)\omega(\cdot, s))\mathrm{d}s \qquad (2.5)$$

$$(F_1\omega)(t) = f_1(t, \omega(\cdot, t)). \qquad (2.6)$$

3. MAIN RESULT

Before proving the main theorem, we shall prove the following lemma

Lemma 3.1. *If f_2 satisfies the condition (H2), then for each $\omega \in Z$ there exists $u \in Z$ such that, for any fixed t*

$$(Gu)(x, t) = f_2(t, \omega(x, t)). \qquad (3.1)$$

Proof. Eq. (3.1) is equivalent to the following integral equation

$$g(x)\{u(x, t) - \int_0^{2\pi} g(s)u(s, t)\mathrm{d}s\} = f_2(t, \omega(x, t)) \qquad (3.2)$$

or

$$u(x,t) - \int_0^{2\pi} g(s)u(s,t)\mathrm{d}s = \frac{f_2(t,\omega(x,t))}{g(x)} \tag{3.3}$$

Now, we define operator $H : L_2(0,2\pi) \to L_2(0,2\pi)$ as

$$(Hu)(x) = u(x) - \int_0^{2\pi} g(s)u(s)\mathrm{d}s \tag{3.4}$$

It is easy to see that the null space of operator H, i.e. $N(H)$, is one dimensional subspace of $L_2(0,2\pi)$ and is spanned by the set $\{1\}$. Now we shall find the operator H^* as follows:
Let $v \in L_2(0,2\pi)$, then

$$\begin{aligned}
\langle Hu, v \rangle_{L_2(0,2\pi)} &= \int_0^{2\pi} (Hu)(y)v(y)\mathrm{d}y \\
&= \int_0^{2\pi} u(y)v(y)\mathrm{d}y - \int_0^{2\pi} \left(\int_0^{2\pi} g(s)u(s)ds \right) v(y)\mathrm{d}y \\
&= \langle u, v \rangle_{L_2(0,2\pi)} - \int_0^{2\pi} \mathrm{d}s u(s) \left(\int_0^{2\pi} g(s)v(y)\mathrm{d}y \right) \\
&= \langle u, H^*v \rangle_{L_2(0,2\pi)} \tag{3.5}
\end{aligned}$$

Thus,

$$\begin{aligned}
(H^*v)(s) &= v(s) - \int_0^{2\pi} g(s)v(y)\mathrm{d}y \\
&= v(s) - \alpha g(s) \tag{3.6}
\end{aligned}$$

where $\alpha = \int_0^{2\pi} v(y)\mathrm{d}y$. Hence, the null space of operator H^*, i.e. $N(H^*)$, is the space spanned by the set $\{g(\cdot)\}$.
It is clear that H is a fredholm operator, it follows that (see [17])

$$R(H) = N(H^*)^\perp$$

Hence (3.3) has a solution $u(\cdot,t) \in L_2(0,2\pi)$ if

$$\langle \frac{f_2(t,\omega(\cdot,t))}{g(\cdot)}, \alpha g(\cdot) \rangle_{L_2(0,2\pi)} = 0, \tag{3.7}$$

that is, if

$$\int_0^{2\pi} f_2(t,\omega(x,t))\mathrm{d}x = 0 \tag{3.8}$$

Eq. (3.8) concludes that if condition (H2) holds true, eq. (3.3) has a solution $u(\cdot,t) \in L_2(0,2\pi)$ for any fixed $t \in [0,\tau]$. $\square$

Theorem 3.2. *[Main Theorem] Suppose that the nonlinear function f in Eq. (1.1) is expressed as $f(t,\omega(x,t)) = f_1(t,\omega(x,t)) + f_2(t,\omega(x,t)) + f_3(t,\omega(x,t))$, where f_1, f_2 and f_3 satisfy assumptions (H1), (H2) and (H3) respectively, then the system (1.1)-(1.5) is exact controllable on V_1.*

Proof. Let $v \in V_1$. Consider the following semilinear system (with the same conditions as given in (1.2)-(1.3))

$$\frac{\partial \omega}{\partial t}(x,t) + \frac{\partial^3 \omega}{\partial x^3}(x,t) = (Gu)(x,t) + f_3(t, \omega(x,t)) \tag{3.9}$$

From Theorem 4.1 in [15], it follows that above system is controllable, i.e., there exists a $u \in Z$ such that

$$v = \int_0^\tau T(\tau - s)(Gu)(\cdot, s)\mathrm{d}s + \int_0^\tau T(\tau - s)f_3(s, \omega(\cdot, s))\mathrm{d}s. \tag{3.10}$$

By condition (H1)

$$\int_0^\tau T(\tau - s)f_1(s, \omega(\cdot, s))\mathrm{d}s = 0. \tag{3.11}$$

By adding equations (3.10) and (3.11) we get

$$v = \int_0^\tau T(\tau - s)(Gu)(\cdot, s)\mathrm{d}s \;\; + \;\; \int_0^\tau T(\tau - s)f_1(s, \omega(\cdot, s))\mathrm{d}s$$

$$+ \;\; \int_0^\tau T(\tau - s)f_3(s, \omega(\cdot, s))\mathrm{d}s \tag{3.12}$$

By Lemma 3.1, we can find a $u_1 \in Z$ such that for all $t \in [0, \tau]$

$$(Gu_1)(\cdot, t) = f_2(t, \omega(\cdot, t)). \tag{3.13}$$

Using equations (3.12) and (3.13) we conclude that

$$v \;\; = \;\; \int_0^\tau T(\tau - s)(Gu - u_1)(\cdot, s)\mathrm{d}s + \int_0^\tau T(\tau - s)f_1(s, \omega(\cdot, s))\mathrm{d}s$$

$$+ \;\; \int_0^\tau T(\tau - s)f_2(s, \omega(\cdot, s))\mathrm{d}s + \int_0^\tau T(\tau - s)f_3(s, \omega(\cdot, s))\mathrm{d}s. \tag{3.14}$$

Eq. (3.14) implies that v is a solution of the system (1.1)-(1.5) with respect to control $u - u_1$. This proves the exact controllability of (1.1)-(1.5). Hence, the proof is complete. $\qquad\square$

Remark 3.3. Since $f_1 \equiv 0$ and $f_2 \equiv 0$ satisfies the conditions (H1) and (H2) respectively, the Theorem 4.1 of [15] follows as a particular case of above theorem.

Remark 3.4. If $f_1 \equiv 0$ and $f_3 \equiv 0$, conditions (H1) and (H3) are satisfied automatically. Now, take $f_2(t, \omega(x,t)) = -\omega(x,t)\omega_x(x,t)$, then problem (1.1)-(1.5) is converted into the exact controllability of nonlinear KdV equation. In this case, the condition (H2), $\int_0^{2\pi} f_2(t, \omega(x,t))\mathrm{d}x = 0$, is also satisfied. Thus, Theorem 3.2 gives the exact controllability of nonlinear KdV equation, which support the result of Theorem 1.1 of Russell and Zhang [2].

CONCLUSION

In this work, a family of nonlinear functions $f = f_1 + f_2 + f_3$ has been given, where f_1, f_2, f_3 satisfy the conditions (H1), (H2), (H3) respectively, for the exact controllability of system (1.1)-(1.5). Thus, Theorem 3.2 gives more choices of nonlinear functions than the paper [15], in which authors proved the exact controllability of system (1.1)-(1.5) by taking $f = f_3$, under the assumption

(H3). In [15], the imposed condition (H3) force us to consider the semilinear system (1.1)-(1.5) as a nonlinear perturbation of corresponding linear system. The exact controllability of nonlinear general KdV equation has also been given, as a consequence of Theorem 3.2, in Remark 3.4.

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