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Some further applications of KKM theorem in topological semilattices

Nguyen The Vinh[a,*]

[a]Department of Mathematical Analysis, University of Transport and Communications, Hanoi, Vietnam.

This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

In this paper, we obtain some further applications of KKM theorem in setting of topological semilattices such as Ky Fan-Kakutani type fixed point theorem, Sion-Neumann type set-valued minimax theorem, set-valued vector optimization problems.

Keywords: generalized Ky Fan minimax inequality, set-valued mapping, topological semilattices, C_Δ-quasiconvex, upper (lower) C-continuous, fixed point, Nash equilibrium.
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1. Introduction

In 1961, Ky Fan proved the following famous result:

Theorem 1.1. *Let C be a nonempty subset of a Hausdorff topological vector space X and let $T : C \to 2^X$ be such that*

1. *T is a KKM map, i.e,*
$$conv\{x_1, x_2, ..., x_n\} \subset \cup_{i=1}^n T(x_i)$$
for every finite subset $\{x_1, x_2, ..., x_n\} \subset C$;
2. *$T(x)$ is closed for all $x \in C$;*
3. *$T(x_0)$ is compact for some $x_0 \in C$.*

*Corresponding author
Email address: thevinhbn@gmail.com (Nguyen The Vinh)

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Then $\bigcap_{x\in C} T(x) \neq \emptyset$.

This important result includes several fundamental mathematical problems, like, Ky Fan minimax inequality, optimization, variational inequality problems and fixed point theorems (see [2]).

In 1996, Horvath and Llinares Ciscar [6] proved topological semilattices version of KKM theorem and gave some applications. Since then, KKM theory is continued in topological semilattices with some papers of Luo [10, 11], Vinh [17, 17, 18].

In this paper, we will continue to study some further applications of KKM theorem in some aspects as Sion-Neumann type set-valued minimax theorem, set-valued vector optimization problems.

The paper is organized as follows. After introduction and preliminaries, in section 3 we prove that Browder-Fan theorem is equivalent to KKM theorem. Section 4 is devoted to a set-valued form of Ky Fan minimax inequality and a set-valued form of Sion-Neumann type minimax theorem. In section 5 we prove an existence result of Pareto equilibria of constrained multiobjective games. The last section is concerned with a Kakutani-Ky Fan type fixed point theorem in topological semilattices with uniform structure.

2. Preliminaries

Definition 2.1. ([6]) A partially ordered set $(X, \leqslant)$ is called a sup-semilattice if any two elements x, y of X have a least upper bound, denoted by $\sup\{x, y\}$. The partially ordered set $(X, \leqslant)$ is a topological semilattice if X is a sup-semilattice equipped with a topology such that the mapping

$$X \times X \to X$$

$$(x, y) \mapsto \sup\{x, y\}$$

is continuous.

We have given the definition of a sup-semilattice, we could obviously also consider inf-semilattices. When no confusion can arise we will simply use the word semilattice. It is also evident that each nonempty finite set A of X will have a least upper bound, denoted by $\sup A$.

In a partially ordered set $(X, \leq)$, two arbitrary elements x and x' do not have to be comparable but, in the case where $x \leq x'$, the set

$$[x, x'] = \{y \in X : x \leq y \leq x'\}$$

is called an order interval or simply, an interval. Now assume that $(X, \leq)$ is a semilattice and A is a nonempty finite subset; then the set

$$\Delta(A) = \bigcup_{a\in A} [a, \sup A]$$

is well defined and it has the following properties:

1. $A \subseteq \Delta(A)$;
2. if $A \subset A'$, then $\Delta(A) \subseteq \Delta(A')$.

We say that a subset $E \subseteq X$ is Δ-convex if for any nonempty finite subset $A \subseteq E$ we have $\Delta(A) \subseteq E$.

Example 2.2. We consider $\mathbb{R}^2$ with usual order defined by

$$(a, b), (c, d) \in \mathbb{R}^2, (a, b) \leq (c, d) \Leftrightarrow a \leq c;\ b \leq d.$$

Clearly, $(\mathbb{R}^2, \leq)$ is a topological semilattice.

1. The set

$$X = \{(x, 1) : 0 \leq x \leq 1\} \cup \{(1, y) : 0 \leq y \leq 1\}$$

is Δ-convex but not convex in the usual sense.

2. The set
$$X = \{(x, y) : 0 \le x \le 1;\ y = 1 - x\}$$
is convex in the usual sense but not Δ-convex.

Definition 2.3. Let X be a topological semilattice or a Δ-convex subset of a topological semilattice, Y be a topological vector space, $C \subset Y$ be a closed, pointed and convex cone with $intC \neq \emptyset$. A mapping $F : X \to 2^Y \setminus \{\emptyset\}$ is said to be a

1. type I C_Δ-quasiconvex mapping if, for any pair $x_1, x_2 \in X$ and for any $x \in \Delta(\{x_1, x_2\})$, we have either
$$F(x) \subset F(x_1) - C$$
or
$$F(x) \subset F(x_2) - C;$$
2. type II C_Δ-quasiconvex mapping if, for any pair $x_1, x_2 \in X$ and for any $x \in \Delta(\{x_1, x_2\})$, we have either
$$F(x_1) \subset F(x) + C$$
or
$$F(x_2) \subset F(x) + C.$$

We use $\in$ instead of $\subset$ when F is single-valued.

Example 2.4. Let $X = [0, 1] \times [0, 1]$. We set $x^1 \le x^2$ denoting that $x^2 \in x^1 + \mathbb{R}^2_+, \forall x^1, x^2 \in X$, where $\mathbb{R}^2_+ = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \ge 0, y_2 \ge 0\}$. It is obvious that $(X, \le)$ is a topological semilattice, in which
$$x^1 \vee x^2 = (\max(x^1_1, x^2_1), \max(x^1_2, x^2_2)),\ \forall x^i = (x^i_1, x^i_2) \in X,\ i = 1, 2.$$

Let $F, G : X \to \mathbb{R}$ and $C = -\mathbb{R}_+$ such that
$$F(x) = [(1 - x_1)(1 - x_2), +\infty),\ \forall x = (x_1, x_2) \in X.$$
$$G(x) = (-\infty, (1 - x_1)(1 - x_2)],\ \forall x = (x_1, x_2) \in X.$$

Then F is type II C_Δ-quasiconvex mapping and it is not type I C_Δ-quasiconvex, G is type I C_Δ-quasiconvex mapping and it is not type II C_Δ-quasiconvex.

Remark 2.5. If $Y = \mathbb{R} = (-\infty, +\infty)$ and $C = [0, +\infty)$, and $F = \varphi$ is a real function, then the C-Δ-quasiconvexity of φ is equivalent to the Δ-quasiconvexity of φ (see [10]).

Definition 2.6. ([8], Definition 2.2) Let X be a topological space, Y a topological vector space with a cone C. Given a subset $D \subset X$, we consider a multi-valued mapping $F : D \to 2^Y$. The domain of F is defined to be the set $domF = \{x \in D : F(x) \neq \emptyset\}$.

1. F is said to be upper (lower) C-continuous at $\bar{x} \in domF$ if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that
$$F(x) \subset F(\bar{x}) + V + C \quad (F(\bar{x}) \subset F(x) + V - C,\ \text{respectively})$$
holds for all $x \in domF \cap U$.
2. If F is upper C-continuous and lower C-continuous at $\bar{x}$ simultaneously, we say that it is C-continuous at $\bar{x}$; and F is upper (respectively, lower) C-continuous on D if it is upper (respectively, lower) C-continuous at every point of D.
3. If F is single-valued, then the upper C-continuity and the lower C-continuity of F at $\bar{x}$ coincide and we say that F is C-continuous at $\bar{x}$.

Remark 2.7. If $Y = \mathbb{R}$ and $C = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ (or $C = \mathbb{R}_- = \{x \in \mathbb{R} : x \leq 0\}$) and F is C-continuous at $\bar{x}$, then F is lower semicontinuous (upper semicontinuous, respectively) at $\bar{x}$ in the usual sense.

Definition 2.8. (Luc [9]) Let Z be a real topological vector space, $C \subset Z$ be a pointed closed convex cone with $\mathrm{int}C \neq \emptyset$, and A be a nonempty subset of Z.

1. For $z_1, z_2 \in Z$, denote $z_1 \leq z_2$ if and only if $z_2 - z_1 \in C$, and $z_1 < z_2$ if and only if $z_2 - z_1 \in \mathrm{int}C$.
2. A point $\bar{z} \in A$ is said to be a vector minimal point (respectively, weakly vector minimal point) of A if for any $z \in A$, $z - \bar{z} \notin -C \setminus \{0\}$ (respectively, $z - \bar{z} \notin -\mathrm{int}C$). Moreover, the set of vector minimal points (respectively, weakly vector minimal points) of A is denoted by $\min\limits_C(A)$ (respectively, $\mathrm{w}\min\limits_C(A)$).

Lemma 2.9. *(Luc [9]) Let A be a nonempty compact subset of a real topological vector Z and $C \subset Z$ be a closed convex cone with $C \neq Z$. Then $\min\limits_C(A) \neq \emptyset$.*

Definition 2.10. Let X, Y be two topological spaces; $F : X \to 2^Y$ is said to have open lower sections if $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open for any $y \in Y$.

3. The equivalence of KKM theorem with Browder-Fan fixed point theorem

Let us recall two fundamental results of the KKM theory in topological semilattices.

Theorem 3.1. *(Horvath and Ciscar [6]) Let X be a topological semilattice with path-connected intervals, $C \subset X$ a nonempty subset of X, and $T : C \to 2^X$ be such that:*
(1) T has closed values;
(2) T is a KKM mapping;
(3) There exists $x_0 \in C$ such that the set $T(x_0)$ is compact.
Then we have the set $\cap_{x \in C} T(x)$ is not empty.

Theorem 3.2. *(Luo [10]) Let X be a topological semilattice with path-connected intervals and $T : X \to 2^X$ be such that:*
(1) For each $x \in X$, the set $T(x)$ is not empty and Δ-convex;
(2) For each $y \in X$, the set $T^{-1}(y)$ is open;
(3) There exists $x_0 \in C$ such that the set $X \setminus T^{-1}(x_0)$ is compact.
Then there exists $x^* \in X$ such that $x^* \in T(x^*)$.

To prove the equivalence of these theorems we need some auxiliary results. In what follows, we denote by $\langle B \rangle$ the family of all finite subsets of B.

Let $\mathcal{C}$ be the family of all convex subsets of a semilattice X and A is an arbitrary subset of X. We set $CO_\Delta(A) = \cap\{E \in \mathcal{C} : A \subseteq E\}$.

One can see without difficulty that a subset E of X is Δ-convex if and only if $CO_\Delta(E) = E$. The proof of Lemma 2.1 in [14] can be modified accordingly to obtain its version in semilattices as follows:

Lemma 3.3. *Let X be a semilattice and E be a nonempty subset of X. Then*
(1) $CO_\Delta(E)$ is a Δ-convex subset of X;
(2) $CO_\Delta(E)$ is the smallest Δ-convex of X containing E;
(3) $CO_\Delta(E) = \cup\{CO_\Delta(A) : A \in \langle E \rangle\}$.

Proof. (1) Let $A \in \langle CO_\Delta(E) \rangle$. Let D be any Δ-convex subset of X containing E. Then $A \subset CO_\Delta(E) \subset D$, so $A \in \langle D \rangle$ and hence $\Delta(A) \subset D$. Thus

$$\Delta(A) \subset \cap\{D : D \text{ is a } \Delta\text{-convex subset of } X \text{ containing } E\} = CO_\Delta(E).$$

Therefore $CO_\Delta(E)$ is Δ-convex.

(2) It is clear from the definition of $CO_\Delta(E)$ and (1).

(3) Let $M = \cup\{CO_\Delta(A) : A \in \langle E\rangle\}$. By (1), $M \subset CO_\Delta(E)$. On the other hand, it is clear that $E \subset M$. Thus to complete the proof, it suffices to show that M is Δ-convex. Indeed, let $B = \{x_1, x_2, ..., x_n\} \in \langle M\rangle$ be given. Then for each $i = 1, 2, ..., n$, there exists $A_i \in \langle E\rangle$ with $x_i \in \Delta(A_i)$. Let $A = \cup_{i=1}^n A_i$, then $A \in \langle E\rangle$ and $B \subset \cup_{i=1}^n \Delta(A_i)$. Since $\Delta(A)$ is Δ-convex, $\Delta(B) \subset \Delta(A) \subset M$. Hence M is Δ-convex. □

Lemma 3.4. *Let X be a topological space and Y be a semilattice. Suppose the mapping $\phi : X \to 2^Y \setminus \{\emptyset\}$ is such that for each $y \in Y$, $\phi^{-1}(y)$ is open in X. Define $\psi : X \to 2^Y \setminus \{\emptyset\}$ by $\psi(x) = CO_\Delta(\phi(x))$ for each $x \in X$. Then for each $y \in Y$, $\psi^{-1}(y)$ is open in X.*

Proof. Let $y \in Y$ be given. By Lemma 3.1, if $x \in \psi^{-1}(y)$, then

$$y \in \psi(x) = CO_\Delta(\phi(x)) = \cup\{\Delta(A) : A \in \langle\phi(x)\rangle\}.$$

Let $A = \{a_1, a_2, ..., a_n\} \in \langle\phi(x)\rangle$ be such that $y \in \Delta(A)$. Then $x \in \cap_{i=1}^n \phi^{-1}(a_i)$ which is an open neighbourhood of x. Let $U = \cap_{i=1}^n \phi^{-1}(a_i)$, then for each $z \in U$, $a_i \in \phi(z)$ for each $i = 1, 2, ..., n$ so that $y \in \Delta(A) \subset CO_\Delta(\phi(z)) = \psi(z)$. Hence $z \in \psi^{-1}(y)$ for each $z \in U$ and hence $x \in U \subset \psi^{-1}(y)$. Therefore $\psi^{-1}(y)$ is open in X. □

Now, we are in a position to state the first new result of this paper.

Theorem 3.5. *Theorems 3.1 and 3.2 are equivalent.*

Proof. **Theorem 3.1 $\Longrightarrow$ Theorem 3.2**: Let us assume that the conditions of Theorem 3.2 hold. We define $G : X \to 2^X$ by $G(y) = X \setminus T^{-1}(y)$ for each $y \in X$. We have

$$\bigcap_{y\in X} G(y) = X \setminus \bigcup_{y\in X} T^{-1}(y) = \emptyset,$$

Therefore, G is not a KKM mapping. Hence, there exists $A = \{x_1, x_2, ..., x_n\} \subset X$ such that $\Delta(A) \not\subset \cup_{x\in A} G(x)$. We infer that there exists $x^* \in \Delta(A)$ such that $x^* \notin G(x_i)$ for all $i = 1, 2, ..., n$. Thus $x^* \in T^{-1}(x_i)$ for all $i = 1, 2, ..., n$. It follows that $x_i \in T(x^*)$ for all $i = 1, 2, ..., n$. Then $x^* \in \Delta(A) \subset T(x^*)$.

Theorem 3.2 $\Longrightarrow$ Theorem 3.1: We assume that the conditions of Theorem 3.1 hold. For a contradiction, asumme that, $\cap_{x\in C} T(x) = \emptyset$. Then we can define a set valued mapping $\phi : X \to 2^X$ by $\phi(x) = \{y \in C : x \notin T(y)\}$. Clearly $\phi(x)$ is a nonempty subset of X for each $x \in X$. It follows that for each $y \in X$, $\phi^{-1}(y) = X \setminus T(y)$ is open in X. Let $\psi : X \to 2^X$ be the set-valued mapping defined by $\psi(x) = CO_\Delta\phi(x)$ for each $x \in X$. Thus for each $x \in C$, $\psi(x)$ is a nonempty Δ-convex subset of X and by Lemma 3.4, $\psi^{-1}(y)$ is open for each $y \in X$. Finally, $X \setminus \psi^{-1}(x_0) \subset X \setminus \phi^{-1}(x_0) = T(x_0)$ is compact. Hence by Theorem 3.2 there exists a point $x^* \in X$ such that

$$x^* \in \psi(x^*) = CO_\Delta\phi(x^*) = \cup\{\Delta(A) : A \in \langle\phi(x^*)\rangle\}.$$

This implies that there exists $A = \{x_1, x_2, ..., x_n\} \in \langle\phi(x^*)\rangle$ such that $x^* \in \Delta(A)$. Then $x^* \in \phi^{-1}(x_i) = X \setminus T(x_i)$ for $i = 1, 2, ..., n$. This means that $x^* \notin T(x_i)$ for $i = 1, 2, ..., n$, i.e, $x^* \notin \cup_{i=1}^n T(x_i)$, which contradicts the hypothesis (2) of Theorem 3.1. Hence $\cap_{x\in C} T(x) \neq \emptyset$. □

4. Ky Fan inequality and Sion-Neumann minimax theorem for set-valued mappings

We shall denote by $\sup A$ (resp. $\inf A$), where $A \subset Y$, the set of all efficient points of the set $\bar{A}$ (the closure of A) with respect to C (resp. with respect to $-C$), i.e.,

$$\sup A = \{a \in \bar{A} : (a + C) \cap \bar{A} = \{a\}\};$$
$$\inf A = \{a \in \bar{A} : (a - C) \cap \bar{A} = \{a\}\}.$$

Recall that A is bounded with respect to C, if the set $(a+C)\cap A$ is bounded for every $a \in A$. A classical lemma of R. Phelps [13], which we shall use in the sequel, states that if A is bounded with respect to C (resp. with respect to $-C$), then $\sup A \neq \emptyset$ (resp. $\inf A \neq \emptyset$) and

$$A \subset \sup A - C \quad \text{(resp. } A \subset \inf A + C).$$

We shall say that a set-valued mapping $F : X \to 2^Y$, where X is a topological space, is bounded with respect to C, if for every $x \in X$ and every $y \in F(x)$ the set $(y + C) \cap F(x)$ is bounded.

We have the following result (see [17, Theorem 3.5] for more general case).

Theorem 4.1. *Let K be a nonempty compact Δ-convex subset of a semilattice X with path-connected intervals, Y a topological vector space, C a closed convex pointed cone with int$C \neq \emptyset$ and $F : K \times K \to 2^Y$ a set-valued mapping. Assume that*

1. *For each $x \in K$, $F(x, x) \subset -C$;*
2. *For each $y \in K$, $F(., y)$ is lower C-continuous;*
3. *For each $x \in K$, $F(x, .)$ is type II $-C_\Delta$-quasiconvex.*

Then the solution set

$$S = \{x \in K : F(x, y) \subset -C, \text{ for all } y \in K\}$$

is a nonempty compact subset of K.

Proof. We define $T : K \to 2^K$ by

$$T(y) = \{x \in K : F(x, y) \subset -C\}, \text{ for each } y \in K.$$

We show that $T(y)$ is closed for each $y \in K$. Taking $\bar{x} \in \overline{T(y)}$, the closure of $T(y)$, we shall deduce that $\bar{x} \in T(y)$. By (2), the lower C-continuity of $F(., y)$ implies that for any neighborhood V of the origin in Y there is a neighborhood $U(\bar{x})$ of $\bar{x}$ such that

$$F(\bar{x}, y) \subset F(x, y) + V - C, \quad \text{for all } x \in U(\bar{x}).$$

Let $\{x_\alpha\}$ be any net in $T(y)$ converging to $\bar{x}$, hence there exists β such that $x_\alpha \in U(\bar{x})$, $\forall \alpha \geq \beta$ and then

$$F(\bar{x}, y) \subset F(x_\alpha, y) + V - C, \ \forall \alpha \geq \beta$$

and so

$$F(\bar{x}, y) \subset F(x_\alpha, y_i) + V - C \subset -C + V - C \subset -C + V \text{ for all } V.$$

Since C is closed, the last inclusion shows $F(\bar{x}, y) \subset -C$. Therefore, $\bar{x} \in T(y)$ and $T(y)$ is closed.
We shall show that for each $x \in K$, $P(x) = \{y \in K : F(x, y) \not\subset -C\}$ is Δ-convex. Suppose that there exists an $x' \in X$ such that $P(x')$ is not Δ-convex; then there exist $y^1, y^2 \in P(x')$ such that $\Delta(\{y^1, y^2\}) \not\subset P(x')$, i.e., there exists a $z \in \Delta(\{y^1, y^2\})$ and $z \notin P(x')$; hence $F(x', z) \subset -C$. By (3), we have either

$$F(x', y^1) \subset F(x', z) - C$$

or

$$F(x', y^2) \subset F(x', z) - C.$$

Consequently, we have either

$$F(x', y^1) \subset F(x', z) - C \subset -C - C \subset -C$$

or

$$F(x', y^2) \subset F(x', z) - C \subset -C - C \subset -C,$$

which is a contradiction. Therefore, for any $x \in X, P(x)$ is Δ-convex.

Finally, we prove that T is a KKM mapping. Suppose on the contrary that T is not KKM. Then there exists $A = \{y_1, y_2, ..., y_n\} \subset K$ such that

$$\Delta(A) \not\subset \bigcup_{i=1}^{n} T(y_i).$$

Thus there exists $z \in \Delta(A)$ such that $z \notin \bigcup_{i=1}^{n} T(y_i)$. Hence $z \notin T(y_i)$ for all $i = 1, 2, ..., n$. It follows that $y_i \in P(z)$ for all $i = 1, 2, ..., n$. Since $P(z)$ is Δ-convex, we have $z \in \Delta(A) \subset P(z)$, i.e., $F(z, z) \not\subset -C$, which contradicts the hypothesis (1). Then T is a KKM mapping. By Theorem 3.1, we infer that

$$\bigcap_{y \in K} T(y) \neq \emptyset$$

and the solution set $S = \{x \in K : F(x, y) \subset -C, \text{ for all } y \in K\}$ is a nonempty compact subset of K. □

Theorem 4.2. *Suppose that X, Y are compact topological semilattices with path-connected intervals, C is a closed convex pointed cone with $intC \neq \emptyset$ in a topological vector space and $F, G : X \times Y \to 2^E$ are two set-valued mappings such that the set $\cup_{y \in Y} \sup \cup_{x \in X} F(x, y)$ is bounded with respect to $-C$ and the set $\cup_{x \in X} \inf \cup_{y \in Y} G(x, y)$ is bounded with respect to C. Suppose that F and G satisfy the following conditions:*

1. *$F(x, y) - G(x, y) \subset -C$ for every $x \in X, y \in Y$;*
2. *$G(x, .)$ is C-Δ-quasiconcave on Y for every $x \in X$ and $F(., y)$ is $-C$-Δ-quasiconcave on X for every $y \in Y$;*
3. *$G(., y)$ is lower $-C$-continuous for every $y \in Y$ and $F(x, .)$ is lower C-continuous for every $x \in X$.*

Then there exist two points

$$z_1 \in \sup \cup_{x \in X} \inf \cup_{y \in Y} G(x, y)$$

and

$$z_2 \in \inf \cup_{y \in K} \sup \cup_{x \in X} F(x, y)$$

such that $z_1 - z_2 \in C$.

Proof. Define the mapping $H : X \times Y \times X \times Y \to 2^E$ by

$$H(\hat{x}, \hat{y}, x, y) = F(x, \hat{y}) - G(\hat{x}, y).$$

Applying Theorem 4.1 for H we obtain that there exist x_0, y_0 such that

$$H(x_0, y_0, x, y) \subset -C, \quad \forall x \in X, \ \forall y \in Y,$$

whence

$$\sup \cup_{x \in X} F(x, y_0) - \inf \cup_{y \in Y} G(x_0, y) \subset -C. \tag{4.1}$$

Using Phelps lemma stated at the beginning of this section, we have

$$\sup \cup_{x \in X} F(x, y_0) \subset \inf \cup_{y \in Y} \sup \cup_{x \in X} F(x, y) + C$$

and

$$\inf \cup_{y \in Y} G(x_0, y) \subset \sup \cup_{x \in X} \inf \cup_{y \in Y} G(x, y) - C.$$

Therefore, by (4.1) there exist

$$z_1 \in \sup \cup_{x \in X} \inf \cup_{y \in Y} G(x, y), \ c_1 \in C$$

and

$$z_2 \in \inf \cup_{y\in K} \sup \cup_{x\in X} F(x,y),\ c_2 \in C$$

such that

$$z_2 + c_2 - (z_1 - c_1) \in -C,$$

which implies

$$z_1 - z_2 \in C_1 + c_1 + c_2 \subset C.$$

□

Remark 4.3. Theorem 4.1 is a set-valued version of Ky Fan minimax inequality, while Theorem 4.2 is a set-valued form of Sion-Neumann type minimax theorem in topological semilattices.

5. The existence of (weak) Pareto equilibria

The following theorem, the proof of which is contained in the proof of Theorem 3 of Horvath and Llinares Ciscar in [6], will be the basic tool for our purpose.

Theorem 5.1. *Let X be a compact topological space, Y be a topological semilattice with path-connected intervals and $T : X \to 2^Y$ have nonempty Δ-convex values and open lower sections. Then there is a continuous selection $f : X \to Y$ of T such that $f = g \circ h$ where $g : \Delta_n \to Y$ and $h : X \to \Delta_n$ are continuous mappings and n is some positive integer.*

Lemma 5.2. *Let I be an index set and for each $i \in I$, let X_i be a nonempty, compact and Δ-convex subset of a topological semilattice with path-connected intervals and $X = \prod_{i\in I} X_i$. For each $i \in I$, let $T_i : X \to 2^{X_i}$ be a set-valued mapping such that*

1. *T_i has nonempty Δ-convex values;*
2. *T_i has open lower sections.*

Then there exists a point $x \in X$ such that $x \in T(x) := \prod_{i\in I} T_i(x)$; that is, $x_i \in T_i(x)$ for each $i \in I$, where $x_i = \pi_i(x)$ is the projection of x onto X_i for each $i \in I$.

Proof. By Theorem 5.1, for each $i \in I$, there exists continuous mappings $g_i : \Delta_{n_i} \to X_i$ and $h_i : X \to \Delta_{n_i}$ such that $f_i = g_i \circ h_i$ is a continuous selection of T_i, where n_i is some positive integer. Now let $S = \prod_{i\in I} \Delta_{n_i}$. For each $i \in I$, let E_i be the linear hull of the set $\{e_0, e_1, ..., e_{n_i}\}$, then E_i is a locally convex topological vector space as it is finite dimensional and Δ_{n_i} is a compact convex subset of E_i. Let $E = \prod_{i\in I} E_i$, then E is also a locally convex topological vector space and S is also a compact convex subset of E.

Now define continuous mappings $g : S \to X$ and $h : X \to S$ by

$$g(t) = \prod_{i\in I} g_i(\pi_i(t)),\ \forall t \in S \ \text{ and } \ h(x) = \prod_{i\in I} h_i(x),\ \forall x \in X,$$

where $\pi_i : S \to \Delta_{n_i}$ is the projection of S on Δ_{n_i} for each $i \in I$. By Tychonoff fixed point theorem [15], the continuous mapping $h \circ g : S \to S$ has a fixed point $t \in S$, i.e., $t = h \circ g(t)$. Let $\overline{x} = g(t)$, then we have

$$\begin{aligned}\overline{x} = g \circ h(\overline{x}) &= g\left(\prod_{i\in I} h_i(\overline{x})\right) \\ &= \prod_{i\in I} g_i\left(\pi_i\left(\prod_{i\in I} h_i(\overline{x})\right)\right) = \prod_{i\in I} g_i \circ h_i(\overline{x}).\end{aligned}$$

It follows that $\overline{x}_i = g_i \circ h_i(\overline{x}) \in T_i(\overline{x})$ for each $i \in I$. This completes the proof. □

From Lemma 5.2, we have the following fixed component theorem in topological semilattices.

Theorem 5.3 ([18]). *Let $\{X_i\}_{i\in I}$ be a family of compact Δ-convex sets each in a topological semilattice with path-connected intervals, $X = \prod_{i\in I} X_i$, and $\{T_i : X \to 2^{X_i}\}_{i\in I}$ a family of mappings satisfying the following conditions:*

1. *each T_i has Δ-convex values;*
2. *each T_i has open lower sections.*
3. *for each $x \in X$, there exists $i \in I$ such that $T_i(x) \neq \emptyset$.*

Then there exists $x = (x_i)_{i\in I} \in X$ and $i \in I$ such that $x_i \in T_i(x)$.

It is easy to see that Theorem 5.3 is equivalent to the following maximal element theorem for a family of mappings.

Theorem 5.4. *Let $\{X_i\}_{i\in I}$ be a family of compact Δ-convex sets each in a topological semilattice with path-connected intervals, $X = \prod_{i\in I} X_i$, and $\{T_i : X \to 2^{X_i}\}_{i\in I}$ a family of maps satisfying the following conditions:*

1. *each T_i has Δ-convex values;*
2. *each T_i has open lower sections.*
3. *for each $x = (x_i)_{i\in I} \in X$ and $i \in I, x_i \notin T_i(x)$.*

Then there exists $\bar{x} \in X$ such that $T_i(\bar{x}) = \emptyset$ for all $i \in I$.

The above theorem will be used in the main result of this section.
Let $(X_i, \leqslant_i), i \in I$, be a family of topological semilattices, and let X and X_{-i} be the product spaces with the product topology, i.e.,

$$X := \prod_{i\in I} X_i, \quad X_{-i} := \prod_{j\in I\setminus\{i\}} X_j,$$

For $x, x' \in X := \prod_{i\in I} X_i$, define $x \leq x'$ if and only if $x_i \leq_i x'_i$, then $(X, \leqslant)$ is a topological semilattice with $[\sup\{x, x'\}]_i = \sup\{x_i, x'_i\}$ for each $i \in I$ (see [6]). For any $x \in X$, $x = (x_{-i}, x_i)$, where $x_i \in X_i,\ x_{-i} \in X_{-i}$.

Let Y be a Hausdorff topological vector space. For each $i \in I$, let $A_i : X \to 2^{X_i}$ be the ith constraint correspondence and $F_i : X \to 2^Y$ the ith pay-off mapping. The following result is Theorem 4.1 in [18].

Theorem 5.5. *Let I be any index set and for each $i \in I$, X_i be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals,*

$$X := \prod_{i\in I} X_i, \quad X_{-i} := \prod_{j\in I\setminus\{i\}} X_j.$$

For each $i \in I$, let Y_i be a locally convex topological vector space and $A_i : X \to 2^{X_i}$, $F_i : X \to 2^{Y_i}$, C_i a closed, pointed and convex cone in Y_i with $intC_i \neq \emptyset$. Assume that

1. *$\forall i \in I$, A_i has open lower sections and nonempty Δ-convex values;*
2. *$\forall i \in I$, the set $B_i = \{x \in X : x_i \in A_i(x)\}$ is closed;*
3. *$\forall i \in I$, F_i is upper C_i-continuous with closed values;*
4. *$\forall i \in I$, $F_i(x_{-i}, u_i)$ is lower $-C_i$-continuous in x_{-i};*
5. *$\forall i \in I$, for any $x_{-i} \in X_{-i}$, the function $F_i(x_{-i}, .)$ is type II $C_{i\Delta}$-quasiconvex.*

Then there exists $x^* \in X$ such that for each $i \in I$,

$$x^*_i \in A_i(x^*), \quad F_i(x^*_{-i}, u_i) \subset F_i(x^*_{-i}, x^*_i) + C_i, \quad \forall u_i \in A_i(x^*).$$

Let I be any (finite or infinite) index set and for each $i \in I$, X_i be topological semilattices. We still use the following notations X, X_{-i} as in Theorem 5.5. For each $x \in X$, x_i and x_{-i} denote the projection of x on X_i and X_{-i} respectively. Write $x = (x_{-i}, x_i)$.

Let I be any set of players. Each player $i \in I$ has a strategy set X_i, a constrained correspondence $A_i : X \to 2^{X_i}$, a payoff $F_i : X \times X_i \to 2^{Y_i}$, where Y_i is a Hausdorff topological vector space, C_i is a pointed closed convex cone in Y_i with $\text{int}C_i \neq \emptyset$ and $C_i \neq Y_i$. A generalized constrained multiobjective game (GCMOG) $\Gamma = (X_i, A_i, F_i, C_i)_{i\in I}$ is a family of ordered quadruples (X_i, A_i, F_i, C_i). A point $x^* = (x^*_{-i}, x^*_i) \in X$ is said to be a Pareto (resp., weak Pareto) equilibrium point of Γ if for each $i \in I$, there exists a point $z^*_i \in F(x^*_{-i}, x^*_i)$ such that

$$x^*_i \in A_i(x^*), \quad z_i - z^*_i \notin -C_i \setminus \{0\}, \quad \forall z_i \in F_i(x^*_{-i}, u_i),\ u_i \in A_i(x^*)$$
$$(resp., x^*_i \in A_i(x^*), \quad z_i - z^*_i \notin -\text{int}C_i, \quad \forall z_i \in F_i(x^*_{-i}, u_i),\ u_i \in A_i(x^*))$$

Since $-\text{int}C_i \subset -C_i \setminus \{0\}$, it is easy to see that each Pareto equilibrium point of the GCMOG must be a weak Pareto equilibrium point of the GCMOG.

Theorem 5.6. *Let I be any index set and for each $i \in I$, X_i be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals, Y_i be a locally convex topological vector space, C_i be a closed, pointed and convex cone in Y_i with $intC_i \neq \emptyset$ and $C_i \neq Y_i$. Let $\Gamma = (X_i, A_i, F_i, C_i)$ be a generalized constrained multiobjective game. For each $i \in I$, let $A_i : X \to 2^{X_i}$, $F_i : X \to 2^{Y_i}$ satisfying the following conditions:*

1. *$\forall i \in I$, A_i has open lower sections and nonempty Δ-convex values;*
2. *$\forall i \in I$, the set $B_i = \{x \in X : x_i \in A_i(x)\}$ is closed;*
3. *$\forall i \in I$, F_i is upper C_i-continuous with compact values;*
4. *$\forall i \in I$, $F_i(x_{-i}, u_i)$ is lower $-C_i$-continuous in x_{-i};*
5. *$\forall i \in I$, for any $x_{-i} \in X_{-i}$, the function $F_i(x_{-i}, .)$ is type II $C_{i\Delta}$-quasiconvex.*

Then there exists $x^* \in X$ such that for each $i \in I$, there exists a point $z^*_i \in F(x^*)$ satisfying

$$x^*_i \in A_i(x^*), \quad z_i - z^*_i \notin -C_i \setminus \{0\}, \quad \forall z_i \in F_i(x^*_{-i}, u_i),\ u_i \in A_i(x^*)$$

i.e., $x^* \in X$ is a Pareto equilibrium point of the GCMOG and so $x^* \in X$ is also a weak Pareto equilibrium point of the GCMOG.

Proof. First, we prove that there exists $x^* = (x^*_{-i}, x^*_i) \in \prod_{i\in I} X_i$ such that for each $i \in I$,

$$x^*_i \in A_i(x^*), \quad F_i(x^*_{-i}, x^*_i) \cap \min_{C_i} F_i(x^*_{-i}, A_i(x^*)) \neq \emptyset. \tag{5.1}$$

If it is false, then for each $x \in \prod_{i\in I} X_i$, there exists $i \in I$ such that either

$$x_i \notin A_i(x)$$

or

$$F_i(x_{-i}, x_i) \cap \min_{C_i} F_i(x_{-i}, A_i(x)) = \emptyset.$$

But, by Theorem 5.5, there exists $x^* = (x^*_{-i}, x^*_i) \in \prod_{i\in I} X_i$ such that for each $i \in I$,

$$x^*_i \in A_i(x^*) \text{ and } F_i(x^*_{-i}, u_i) \subset F_i(x^*_{-i}, x^*_i) + C_i, \quad \forall u_i \in A_i(x^*). \tag{5.2}$$

Hence we have

$$F_i(x^*_{-i}, x^*_i) \cap \min_{C_i} F_i(x^*_{-i}, A_i(x^*)) = \emptyset. \tag{5.3}$$

By the condition (3), $F_i(x^*_{-i}, x^*_i)$ is compact in Y_i, it follows from Lemma 2.2, $\min\limits_{C_i} F_i(x^*_{-i}, x^*_i) \neq \emptyset$. Let $z^0_i \in \min_{C_i} F_i(x^*_{-i}, x^*_i) \subset F_i(x^*_{-i}, x^*_i)$. It follows from (5.3) that

$$z^0_i \notin \min_{C_i} F_i(x^*_{-i}, A_i(x^*)).$$

Hence, there exist $u^*_i \in A_i(x^*_{-i})$ and $z^*_i \in F_i(x^*_{-i}, u^*_i)$ such that

$$z^0_i \in z^*_i + C_i \setminus \{0\}. \tag{5.4}$$

By (5.5), there exists $z_i \in F_i(x^*_{-i}, x^*_i)$ such that

$$z^*_i \in z_i + C_i. \tag{5.5}$$

By (5.4) and (5.5), we have

$$z^0_i - z_i = z^0_i - z^*_i + z^*_i - z_i \in C_i \setminus \{0\} + C_i = C_i \setminus \{0\}.$$

which contradicts the fact that $z^0_i \in \min\limits_{C_i} F_i(x^*_{-i}, x^*_i)$. Therefore (5.1) is true. It follows from Definition 2.8 and (5.1) that there exists $x^* = (x^*_{-i}, x^*_i) \in X$ such that for each $i \in I$, there exists $z^*_i \in F_i(x^*_{-i}, x^*_i)$ satisfying

$$x^*_i \in A_i(x^*), \quad z_i - z^*_i \notin -C_i \setminus \{0\}, \quad \forall z_i \in F_i(x^*_{-i}, u_i),\ u_i \in A_i(x^*),$$

i.e., $x^* \in X$ is a Pareto equilibrium point of the GCMOG and so $x^* \in X$ is also a weak Pareto equilibrium point of the GCMOG. □

6. Ky Fan-Kakutani type fixed point theorem in topological semilattices

This section is concerned with a Kakutani-Ky Fan type fixed point theorem in topological semilattices with uniform structure.

Definition 6.1. (Kelly [7]) A uniformity for a set X is a non-void family $\mathcal{U}$ of subsets of $X \times X$ (called entourages) such that

1. each member of $\mathcal{U}$ contains the diagonal $\Omega = \{(x, x) \in X\}$,
2. if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$, where $U^{-1} = \{(y, x) \in X \times X : (x, y \in U)\}$,
3. if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in \mathcal{U}$, where

$$V \circ V = \{(x, z) : \exists y \in X \text{ such that } (x, y) \in V,\ (y, z) \in V\},$$

4. if U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$, and
5. if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

The pair $(X, \mathcal{U})$ is called a uniform space. For each $V \in \mathcal{U}$, we define a neighborhood of x as $V[x] := \{y \in X : (x, y) \in V\}$. An entourages V is called symmetric if $V = V^{-1}$. In this case, we have

$$y \in V[x] \Leftrightarrow x \in V[y].$$

Let

$$\mathcal{O} = \{G \subset X : \text{for each } x \in G \text{ there exists } V \in \mathcal{U} \text{ such that } V[x] \subset G\}.$$

Then $\mathcal{O}$ is a topology on X, and it called the topology induced by the uniformity $\mathcal{U}$. Moreover, $(X, \mathcal{O})$ is called a uniform topological space.
The uniform space $(X, \mathcal{U})$ is said to be separated if

$$\bigcap \{V : V \in \mathcal{U}\} = \Omega,$$

in this case $(X, \mathcal{O})$ becomes a Hausdorff space.

Definition 6.2. A topological semilattice X is said to be a locally Δ-convex space if X is a uniform topological space with uniformity $\mathcal{U}$ which has an open base $\beta := \{V_i : i \in I\}$ of symmetric entourages such that for each $V \in \beta$, the set $V[x]$ is a Δ-convex for each $x \in X$.

We shall assume that locally Δ-convex spaces also satisfy the following condition:
Condition (H): $\{x \in X : K \cap V[x] \neq \emptyset\}$ is Δ-convex for any Δ-convex subset K of X and $V \in \beta$ (see, Horvath [5, Definition 2, p. 345]).

Definition 6.3. (Berge [1]) Let X and Y be two Hausdorff topological spaces and $F : X \to 2^Y$ be a set-valued mapping, then F is upper semicontinuous at $x_0 \in X$ if for each open set U in Y with $U \supset F(x_0)$, there exists an open neighborhood $O(x_0)$ of x_0 such that $U \supset F(x)$ for any $x \in O(x_0)$; F is upper semicontinuous on X if F is upper semicontinuous at every point in X.

We need the following result.

Theorem 6.4. *(Horvath and Ciscar [6]) Let X be a topological semilattice with path-connected intervals, $C \subset X$ a nonempty subset of X, and $T : C \to 2^X$ be such that:*

1. *T has closed [resp., open] values;*
2. *T is a KKM mapping, i.e., for each $A \in \langle X \rangle$,*

$$\Delta(A) \subset \bigcup_{x \in A} T(x).$$

Then the family $\{T(x) : x \in C\}$ has the finite intersection property.

Theorem 6.5. *Let X be a separated compact locally Δ-convex space with path-connected intervals satisfying the condition (H) and $T : X \to 2^X$ be an upper semicontinuous set-valued mappings with nonempty closed Δ-convex values. Then T has a fixed point, i.e, there exists $x_0 \in X$ such that $x_0 \in T(x_0)$.*

Proof. Fix an element V of the base β, then for each $x \in X$, $V[x]$ is an open neighborhood of x. Since $T(X)$ is compact, there exists an $M = \{y_1, y_2, ..., y_n\} \subset X$ such that $T(X) \subset \cup_{y \in M} V[y]$.

For each $y_i \in M$, let $G(y_i) := \{x \in X : T(x) \cap \overline{V[y_i]} = \emptyset\}$. Since T is upper semicontinuous and $\overline{V[y_i]}$ is closed, by a standard argument, we can prove that each $G(y_i)$ is open. Moreover, since $T(X) \subset \cup_{i=1}^n V[y_i]$, we have

$$\bigcap_{i=1}^n G(y_i) = \left\{ x \in X : T(x) \cap \bigcup_{i=1}^n \overline{V[y_i]} = \emptyset \right\} = \emptyset.$$

Therefore, by Theorem 6.4, $G : M \to 2^X$ cannot be a KKM map; that is, there exist an $N \in \langle M \rangle$ and an $x_V \in \Delta(N)$ such that $x_V \notin G(N) = \cup_{y \in N} G(y)$. Hence $T(x_V) \cap \overline{V[y]} \neq \emptyset$ for all $y \in N$, and

$$N \subset L := \{y \in X : T(x_V) \cap \overline{V[y]} \neq \emptyset\}.$$

Since $T(x_V)$ is Δ-convex set and X satisfies the condition (H), L is Δ-convex. Therefore, $x_V \in \Delta(N) \subset L$ and hence $T(x_V) \cap \overline{V[x_V]} \neq \emptyset$.

So, for each basis element V, there exist $x_V, y_V \in X$ such that $y_V \in T(x_V)$ and $y_V \in \overline{V[x_V]}$. Since $T(X)$ is compact and β forms a directed set ordered by inclusion, we may assume that the net $\{y_V\}$ converges to some $x_0 \in K$. Since X is Hausdorff, x_V also converges to x_0. Since T is upper semicontinuous with closed values, the graph of T is closed in $X \times T(X)$, and hence we have $x_0 \in Tx_0$. This completes our proof. □

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References

[1] C. Berge, *Espaces Topologiques*, *Fonctions Multivoques*, Dunnod, Paris, 1959. 6.3

[2] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123-145. 1

[3] P. Deguire and M. Lassonde, *Familles sélectantes*, Topol. Methods Nonlinear Anal. **5** (1995), 261-269.

[4] K. Fan, *A minimax inequality and applications. In, O. Shisha, ed., Inequalities III,* Proceedings of the third Symposium on inequalities, Academic Press, New York, 1972.

[5] C. D. Horvath, *Contractibility and Generalized Convexity*, J. Math. Anal. Appl. **156** (1991), 341-357. 6

[6] C. D. Horvath and J. V. Llinares Ciscar, *Maximal elements and fixed points for binary relations on topological ordered spaces*, J. Math. Econom. **25** (1996), 291-306. 1, 2.1, 3.1, 5, 5, 6.4

[7] J. L. Kelly, *General Topology*, Van Nostrand, Princeton, NJ, 1955. 6.1

[8] L.-J. Lin and N. X. Tan, *On quasivariational inclusion problems of type I and related problems*, J. Glob. Optim. **39** (2007), 393-407. 2.6

[9] D. T. Luc, *Theory of Vector Optimization. In, Lecture Notes in Economics and mathematical systems*, Vol. 319, Springer-Verlag, Berlin, 1989. 2.8, 2.9

[10] Q. Luo, *KKM and Nash equilibria type theorems in topological ordered spaces*, J. Math. Anal. Appl. **264** (2001), 262-269. 1, 2.5, 3.2

[11] Q. Luo, *The applications of the Fan-Browder fixed point theorem in topological ordered spaces*, Appl. Math. Lett. **19** (2006), 1265-1271. 1

[12] J. W. Peng and X. M. Yang, *On existence of a solution for the system of generalized vector quasi-equilibrium problems with upper semicontinuous set-valued maps*, Inter. J. Math. Math. Sciences **15** (2005), 2409-2420.

[13] R. R. Phelps, *Convex functions, monotone operators and differentiablity*, *Lecture Notes in Mathematics*, Springer-Verlag, Vol. 1364, 1989. 4

[14] K.-K. Tan, X.-L. Zhang, *Fixed point theorems on G-convex spaces and applications*, Proc. Nonlinear Func. Anal. Appl. **1** (1996), 1-19. 3

[15] A. Tychonoff, *Ein Fixpunktsatz*, Math. Ann. **111** (1935), 767-776. 5

[16] N. T. Vinh, *Matching theorems, fixed point theorems and minimax inequalities in topological ordered spaces*, Acta Math. Vietnam. **30** (2005), 211-224.

[17] N. T. Vinh, *Some generalized quasi-Ky Fan inequalities in topological ordered spaces*, Vietnam J. Math. **36** (2008), 437-449. 1, 4

[18] N. T. Vinh,*Systems of generalized quasi-Ky Fan inequalities and Nash equilibrium points with set-valued maps in topological semilattices*, PanAmer. Math. J. **19** (2009), 79-92. 1, 5.3, 5

[19] G. X. Z. Yuan, *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker Inc., New York, 1999.