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Coupled fixed point theorems in d-complete topological spaces

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This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

In this paper, we obtain prove two common coupled fixed point theorems in Hausdorff d- complete topological spaces.

Keywords: Coupled fixed points, d-complete space ,weakly compatible maps.
2010 MSC: Primary 47H10; Secondary 54H25.

1. Introduction and Preliminaries

In 1975, Kasahara [11, 12] introduced the notion of d-complete topological spaces as a generalization of complete metric spaces.

Definition 1.1. [11, 12]. Let $(X, \mathcal{T})$ be a topological space. Suppose $d : X \times X \to [0, \infty)$ satisfies
(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) for any sequence $\{x_n\}$ in $X, \Sigma_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies $\{x_n\}$ is convergent in $(X, \mathcal{T})$.
Then the triplet $(X, \mathcal{T}, d)$ is called a d- complete topological space.

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For details on d-complete topological spaces, we refer to Iseki [10] and Kasahara [11, 12, 13]. Hicks [6] and Hicks and Rhoades [7, 8] proved several fixed point theorems in d - complete topological spaces. Hicks and Saliga [9] and Saliga [19] obtained fixed point theorems for non - self maps in d - complete topological spaces.

In 2006, Bhaskar and Lakshmikantham [3] introduced the notion of a coupled fixed point in partially ordered metric spaces, also discussed some problems of the uniqueness of a coupled fixed point and applied their results to the problems of the existence and uniqueness of a solution for the periodic boundary value problems.

Later several authors proved coupled fixed and common coupled fixed point theorems in partial ordered metric spaces , partially ordered cone metric spaces and cone metric spaces for two maps(Refer to [1, 4, 5, 14, 15, 16, 17, 18, 20, 21, 22, 23]).

In this paper ,we prove a common coupled fixed point theorem for four mappings in d - complete topological spaces.

Definition 1.2. ([3]).Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.3. ([1]).Let X be a nonempty set. An element $(x, y) \in X \times X$ is called
(i) a coupled coincidence point of $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$.
(ii)a common coupled fixed point of $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.4. ([1]).Let X be a nonempty set. The mappings $F : X \times X \to X$ and $g : X \to X$ are called W-compatible if $g(F(x, y)) = F(gx, gy)$ and $g(F(y, x)) = F(gy, gx)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$ for some $(x, y) \in X \times X$.

In this paper,we obtain two common coupled and common fixed point theorems for two and four mappings satisfying a Berinde [2] type weak contraction conditions in Hausdorff d- complete topological spaces.

2. Main Results

Theorem 2.1. *Let (X, τ, d) be a Hausdorff topological space.Let $F, G : X \times X \to X$ and $f, g : X \to X$ be mappings satisfying*

$$(2.1.1) \qquad d(F(x,y),G(u,v)) \leq h \; max \left\{ \begin{array}{c} d(fx,gu), d(fy,gv), d(F(x,y),fx), d(G(u,v),gu) \end{array} \right\} + L \; min \left\{ \begin{array}{c} d(fx,gu), d(fy,gv), d(F(x,y),fx), \\ d(G(u,v),gu), d(F(x,y),gu), d(G(u,v),fx) \end{array} \right\},$$

for all $x, y, u, v \in X$, where $0 \leq h < 1$ and $L \geq 0$,
(2.1.2) *$F(X \times X) \subseteq g(X), G(X \times X) \subseteq f(X)$,*
(2.1.3) *one of $f(X)$ and $g(X)$ is d- complete,*
(2.1.4) *the pairs (F, f) and (G, g) are W-compatible,*
(2.1.5) *$d(x, y) = d(y, x)$ for all $x, y \in X$ and*
(2.1.6) *for each $y \in X, d(x_n, y) \to d(x, y)$, whenever $\{x_n\} \subseteq X, x \in X$ such that $x_n \to x$.*
Then F, G, f and g have a unique common coupled fixed point in $X \times X$ and also they have a unique common fixed point in X.

Proof. Let x_0 and y_0 be in X.
Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Since $G(X \times X) \subseteq f(X)$, we can choose $x_2, y_2 \in X$ such that $fx_2 = G(x_1, y_1)$ and $fy_2 = G(y_1, x_1)$. Continuing this process, we can construct the sequences $\{x_n\}$, $\{y_n\}$,$\{z_n\}$ and $\{p_n\}$ in X such that

$gx_{2n+1} = F(x_{2n}, y_{2n}) = z_{2n}$, say ;
$gy_{2n+1} = F(y_{2n}, x_{2n}) = p_{2n}$, say ;
$fx_{2n+2} = G(x_{2n+1}, y_{2n+1}) = z_{2n+1}$, say ; and
$fy_{2n+2} = G(y_{2n+1}, x_{2n+1}) = p_{2n+1}$, say ; for $n = 0, 1, 2, \ldots$. Now

$$\begin{aligned} d(z_{2n}, z_{2n+1}) &= d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})) \\ &\leq h \ max\{d(z_{2n-1}, z_{2n}), d(p_{2n-1}, p_{2n}), d(z_{2n}, z_{2n-1}), d(z_{2n+1}, z_{2n})\} \\ &\quad + L \ min \left\{ \begin{array}{l} d(z_{2n-1}, z_{2n}), d(p_{2n-1}, p_{2n}), d(z_{2n}, z_{2n-1}), \\ d(z_{2n+1}, z_{2n}), d(z_{2n}, z_{2n}), d(z_{2n+1}, z_{2n-1}) \end{array} \right\} \\ &= h \ max\{d(z_{2n-1}, z_{2n}), d(p_{2n-1}, p_{2n})\}. \end{aligned}$$

Also

$$\begin{aligned} d(p_{2n}, p_{2n+1}) &= d(F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1})) \\ &\leq h \ max\{d(p_{2n-1}, p_{2n}), d(z_{2n-1}, z_{2n}), d(p_{2n}, p_{2n-1}), d(p_{2n+1}, p_{2n})\} \\ &\quad + L \ min \left\{ \begin{array}{l} d(p_{2n-1}, p_{2n}), d(z_{2n-1}, z_{2n}), d(p_{2n}, p_{2n-1}), \\ d(p_{2n+1}, p_{2n}), d(p_{2n}, p_{2n}), d(p_{2n+1}, p_{2n-1}) \end{array} \right\} \\ &= h \ max\{d(p_{2n-1}, p_{2n}), d(z_{2n-1}, z_{2n})\}. \end{aligned}$$

Thus $max\{d(z_{2n}, z_{2n+1}), d(p_{2n}, p_{2n+1})\} \leq h \ max\{d(p_{2n-1}, p_{2n}), d(z_{2n-1}, z_{2n})\}$.

$$\begin{aligned} d(z_{2n-1}, z_{2n}) &= d(G(x_{2n-1}, y_{2n-1}), F(x_{2n}, y_{2n})) \\ &= d(F(x_{2n}, y_{2n}), G(x_{2n-1}, y_{2n-1})) \\ &\leq h \ max\{d(z_{2n-1}, z_{2n-2}), d(p_{2n-1}, p_{2n-2}), d(z_{2n}, z_{2n-1}), d(z_{2n-1}, z_{2n-2})\} \\ &\quad + L \ min \left\{ \begin{array}{l} d(z_{2n-1}, z_{2n-2}), d(p_{2n-1}, p_{2n-2}), d(z_{2n}, z_{2n-1}), \\ d(z_{2n-1}, z_{2n-2}), d(z_{2n}, z_{2n-2}), d(z_{2n-1}, z_{2n-1}) \end{array} \right\} \\ &= h \ max\{d(z_{2n-2}, z_{2n-1}), d(p_{2n-2}, p_{2n-1})\}. \end{aligned}$$

$$\begin{aligned} d(p_{2n-1}, p_{2n}) &= d(G(y_{2n-1}, x_{2n-1}), F(y_{2n}, x_{2n})) = d(F(y_{2n}, x_{2n}), G(y_{2n-1}, x_{2n-1})) \\ &\leq h \ max\{d(p_{2n-1}, p_{2n-2}), d(z_{2n-1}, z_{2n-2}), d(p_{2n}, p_{2n-1}), d(p_{2n-1}, p_{2n-2})\} \\ &\quad + L \ min \left\{ \begin{array}{l} d(p_{2n-1}, p_{2n-2}), d(z_{2n-1}, z_{2n-2}), d(p_{2n}, p_{2n-1}), \\ d(p_{2n-1}, p_{2n-2}), d(p_{2n}, p_{2n-2}), d(p_{2n-1}, p_{2n-1}) \end{array} \right\} \\ &= h \ max\{d(p_{2n-2}, p_{2n-1}), d(z_{2n-2}, z_{2n-1})\}. \end{aligned}$$

Thus $max\{d(z_{2n-1}, z_{2n}), d(p_{2n-1}, p_{2n})\} \leq h \ max\{d(z_{2n-2}, z_{2n-1}), d(p_{2n-2}, p_{2n-1})\}$. Hence

$$\begin{aligned} max\{d(z_n, z_{n+1}), d(p_n, p_{n+1})\} &\leq h \ max\{d(z_{n-1}, z_n), d(p_{n-1}, p_n)\} \\ &\leq h^2 \ max\{d(z_{n-2}, z_{n-1}), d(p_{n-2}, p_{n-1})\} \\ &\ \vdots \\ &\leq h^n \ max\{d(z_0, z_1), d(p_0, p_1)\}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} h^n$ is convergent, it follows that $\sum_{n=1}^{\infty} d(z_n, z_{n+1})$ and $\sum_{n=1}^{\infty} d(p_n, p_{n+1})$ are convergent. Hence $d(z_n, z_{n+1}) \to 0, d(p_n, p_{n+1}) \to 0$ as $n \to \infty$.
Suppose $f(X)$ is d-complete.Then $\{z_{2n+1}\} = \{fx_{2n+2}\} \subseteq f(X)$ and $\{p_{2n+1}\} = \{fy_{2n+2}\} \subseteq f(X)$ converge to some α and β in $f(X)$ respectively.

Hence there exist x and y in X such that $\alpha = fx$ and $\beta = fy$. Also the subsequences $\{z_{2n}\}$and $\{p_{2n}\}$ converge to α and β respectively.

$$\begin{aligned} d(F(x,y), z_{2n+1}) &= d(F(x,y), G(x_{2n+1}, y_{2n+1})) \\ &\le h\ max\,\{d(fx, z_{2n}), d(fy, p_{2n}), d(F(x,y), fx), d(z_{2n+1}, z_{2n})\} \\ &\quad + L\ min \left\{ \begin{array}{l} d(fx, z_{2n}), d(fy, p_{2n}), d(F(x,y), fx), \\ d(z_{2n+1}, z_{2n}), d(F(x,y), z_{2n}), d(z_{2n+1}, fx) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$ and using (2.1.5) and (2.1.6), we get
$d(F(x,y), fx) \le h\ d(F(x,y), fx) + L(0)$.
Hence $F(x,y) = fx = \alpha$.

$$\begin{aligned} d(F(y,x), p_{2n+1}) &= d(F(y,x), G(y_{2n+1}, x_{2n+1})) \\ &\le h\ max\,\{d(fy, p_{2n}), d(fx, z_{2n}), d(F(y,x), fy), d(p_{2n+1}, p_{2n})\} \\ &\quad + L\ min \left\{ \begin{array}{l} d(fy, p_{2n}), d(fx, z_{2n}), d(F(y,x), fy), \\ d(p_{2n+1}, p_{2n}), d(F(y,x), p_{2n}), d(p_{2n+1}, fy) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, we get $d(F(y,x), fy) \le h\ d(F(y,x), fy) + L(0)$.
Hence $F(y,x) = fy = \beta$.
Since the pair (f, S) is W-compatible, we have $F(\alpha, \beta) = F(fx, fy) = f(F(x,y)) = f\alpha$ and $F(\beta, \alpha) = F(fy, fx) = f(F(y,x)) = f\beta$.
Consider

$$\begin{aligned} d(F(\alpha,\beta), z_{2n+1}) &= d(F(\alpha,\beta), G(x_{2n+1}, y_{2n+1})) \\ &\le h\ max\,\{d(f\alpha, z_{2n}), d(f\beta, p_{2n}), d(F(\alpha,\beta), f\alpha), d(z_{2n+1}, z_{2n})\} \\ &\quad + L\ min \left\{ \begin{array}{l} d(f\alpha, z_{2n}), d(f\beta, p_{2n}), d(F(\alpha,\beta), f\alpha), \\ d(z_{2n+1}, z_{2n}), d(F(\alpha,\beta), z_{2n}), d(z_{2n+1}, f\alpha) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, we get $d(f\alpha, \alpha) \le h\ max\{d(f\alpha, \alpha), d(f\beta, \beta)\}$. Also consider

$$\begin{aligned} d(F(\beta,\alpha), p_{2n+1}) &= d(F(\beta,\alpha), G(y_{2n+1}, x_{2n+1})) \\ &\le h\ max\,\{d(f\beta, p_{2n}), d(f\alpha, z_{2n}), d(F(\beta,\alpha), f\beta), d(p_{2n+1}, p_{2n})\} \\ &\quad + L\ min \left\{ \begin{array}{l} d(f\beta, p_{2n}), d(f\alpha, z_{2n}), d(F(\beta,\alpha), f\beta), \\ d(p_{2n+1}, p_{2n}), d(F(\beta,\alpha), p_{2n}), d(p_{2n+1}, f\beta) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, we get $d(f\beta, \beta) \le h\ max\{d(f\beta, \beta), d(f\alpha, \alpha)\}$.
Thus $max\{d(f\alpha, \alpha), d(f\beta, \beta)\} \le h\ max\{d(f\alpha, \alpha), d(f\beta, \beta)\}$. Hence $f\alpha = \alpha$ and $f\beta = \beta$.
Thus $\alpha = f\alpha = F(\alpha, \beta)$.....(I) and $\beta = f\beta = F(\beta, \alpha)$...(II)
Since $F(X \times X) \subseteq gX$, there exist $\gamma, \delta \in X$ such that $g\gamma = F(\alpha, \beta) = f\alpha = \alpha$ and $g\delta = F(\beta, \alpha) = f\beta = \beta$.
Now

$$\begin{aligned} d(g\gamma, G(\gamma,\delta)) &= d(F(\alpha,\beta), G(\gamma,\delta)) \\ &\le h\ max\,\{0, 0, 0, d(G(\gamma,\delta), g\gamma)\} \\ &\quad + L\ min\,\{0, 0, 0, d(G(\gamma,\delta), g\gamma), 0, d(G(\gamma,\delta), g\gamma)\} \\ &= h\ d(G(\gamma,\delta), g\gamma). \end{aligned}$$

Hence $G(\gamma, \delta) = g\gamma$. Also

$$\begin{aligned} d(g\delta, G(\delta,\gamma)) &= d(F(\beta,\alpha), G(\delta,\gamma)) \\ &\le h\ max\,\{0, 0, 0, d(G(\delta,\gamma), g\delta)\} \\ &\quad + L\ min\,\{0, 0, 0, d(G(\delta,\gamma), g\delta), 0, d(G(\delta,\gamma), g\delta)\} \\ &= h\ d(G(\delta,\gamma), g\delta). \end{aligned}$$

Hence $G(\delta,\gamma)=g\delta$. Since the pair (G,g) is W-compatible, we have $g\alpha=g(g\gamma)=g(G(\gamma,\delta)=G(g\gamma,g\delta)=G(\alpha,\beta)$ and $g\beta=g(g\delta)=g(G(\delta,\gamma))=G(g\delta,g\gamma)=G(\beta,\alpha)$. Now consider

$$\begin{aligned} d(z_{2n},G(\alpha,\beta)) &= d(F(x_{2n},y_{2n}),G(\alpha,\beta)) \\ &\le h\ max\left\{d(z_{2n-1},g\alpha),d(p_{2n-1},g\beta),d(z_{2n},z_{2n-1}),0\right\} \\ &\quad + L\ min\left\{\begin{array}{l} d(z_{2n-1},g\alpha),d(p_{2n-1},g\beta),d(z_{2n},z_{2n-1}), \\ 0,d(z_{2n},g\alpha),d(G(\alpha,\beta),z_{2n-1}) \end{array}\right\}. \end{aligned}$$

Letting $n\to\infty$, we get $d(\alpha,g\alpha)\le h\ max\{d(\alpha,g\alpha),d(\beta,g\beta)\}$. Also consider

$$\begin{aligned} d(p_{2n},G(\beta,\alpha)) &= d(F(y_{2n},x_{2n}),G(\beta,\alpha)) \\ &\le h\ max\left\{d(p_{2n-1},g\beta),d(z_{2n-1},g\alpha),d(p_{2n},p_{2n-1}),0\right\} \\ &\quad + L\ min\left\{\begin{array}{l} d(p_{2n-1},g\beta),d(z_{2n-1},g\alpha),d(p_{2n},p_{2n-1}), \\ 0,d(p_{2n},g\beta),d(G(\beta,\alpha),p_{2n-1}) \end{array}\right\}. \end{aligned}$$

Letting $n\to\infty$, we get $d(\beta,g\beta)\le h\ max\{d(\beta,g\beta),d(\alpha,g\alpha)\}$. Hence $g\alpha=\alpha$ and $g\beta=\beta$.
Thus $\alpha=g\alpha=G(\alpha,\beta)......(III)$
and $\beta=g\beta=G(\beta,\alpha)$....(IV).
From (I),(II) (III) and (IV), we have
$f\alpha=g\alpha=\alpha=F(\alpha,\beta)=G(\alpha,\beta)$.......(V) and $f\beta=g\beta=\beta=F(\beta,\alpha)=G(\beta,\alpha)$...........(VI)
Thus (α,β) is a common coupled fixed point of F,G,f and g.
Suppose $(\alpha_1,\beta_1)\in X\times X$ is another common coupled fixed point of F,G,f and g.

$$\begin{aligned} d(\alpha_1,\alpha)=d(F(\alpha_1,\beta_1),G(\alpha,\beta)) &\le h\ max\left\{d(\alpha_1,\alpha),d(\beta_1,\beta),0,0\right\} \\ &\quad + L\ min\left\{d(\alpha_1,\alpha),d(\beta_1,\beta),0,0,d(\alpha_1,\alpha),d(\alpha,\alpha_1)\right\} \\ &= h\ max\left\{\mathrm{d}(\alpha_1,\alpha),\mathrm{d}(\beta_1,\beta)\right\}. \end{aligned}$$

Also

$$\begin{aligned} d(\beta_1,\beta)=d(F(\beta_1,\alpha_1),G(\beta,\alpha)) &\le h\ max\left\{d(\beta_1,\beta),d(\alpha_1,\alpha),0,0\right\} \\ &\quad + L\ min\left\{d(\beta_1,\beta),d(\alpha_1,\alpha),0,0,d(\beta_1,\beta),d(\beta,\beta_1)\right\} \\ &= h\ max\left\{d(\alpha_1,\alpha),d(\beta_1,\beta)\right\}. \end{aligned}$$

Thus $\max\{d(\alpha_1,\alpha),d(\beta_1,\beta)\}\le h\max\{d(\alpha_1,\alpha),d(\beta_1,\beta)\}$.
Hence $\alpha_1=\alpha$ and $\beta_1=\beta$.
Thus (α,β) is the unique common coupled fixed point of F,G,f and g.
Now, we will show that $\alpha=\beta$.

$$\begin{aligned} d(\alpha,\beta) &= d(F(\alpha,\beta),G(\beta,\alpha)) \\ &\le h\ max\left\{d(\alpha,\beta),d(\alpha,\beta),0,0\right\} + L\ min\left\{d(\alpha,\beta),d(\alpha,\beta),0,0,d(\alpha,\beta),d(\beta,\alpha)\right\} \\ &= h\ d(\alpha,\beta). \end{aligned}$$

Thus $\alpha=\beta$. Hence α is a common fixed point of F,G,f and g. Using (2.1.1), we can show that α is the unique common fixed point of F,G,f and g. □

The following example illustrates Theorem 2.1.

Example 2.2. Let $X=[0,1]$ and $d(x,y)=|x^2-y^2|,\forall x,y\in X$.

Define $F(x,y) = Sin(\frac{x^2+y^2}{4}) = G(x,y)$ and $fx = x = gx, \forall x \in X$. Then

$$\begin{aligned}
d(F(x,y),G(u,v)) &= d(Sin(\frac{x^2+y^2}{4}), Sin(\frac{u^2+v^2}{4})) \\
&= \left| Sin^2(\frac{x^2+y^2}{4}) - Sin^2(\frac{u^2+v^2}{4}) \right| \\
&= \left| Sin(\frac{x^2+y^2}{4} + \frac{u^2+v^2}{4}) Sin(\frac{x^2+y^2}{4} - \frac{u^2+v^2}{4}) \right| \\
&\leq \frac{1}{4}(\left|x^2-u^2\right| + \left|y^2-v^2\right|) \\
&\leq \frac{1}{2} max\{d(fx,gu), d(fy,gv)\} \\
&\leq \frac{1}{2}\ max \left\{\ d(fx,gu), d(fy,gv), d(F(x,y),fx), d(G(u,v),gu)\ \right\} \\
&\quad + L\ min \left\{ \begin{array}{c} d(fx,gu), d(fy,gv), d(F(x,y),fx), \\ d(G(u,v),gu), d(F(x,y),gu), d(G(u,v),fx) \end{array} \right\},
\end{aligned}$$

where $L = 0$.
One can verify all the other conditions easily. $(0,0)$ is the unique common fixed point of F, G, f and g.

Note: In Example 2.2, it is clear that (X,d) is a d-complete topological space and (X,d) is not a complete metric space .
Now we give another theorem for a pair of Jungck type maps without using symmetry of d.

Theorem 2.3. *Let (X,τ,d) be a Hausdorff topological space.Let $F : X \times X \to X$ and $f : X \to X$ be mappings satisfying*

(2.3.1) $$d(F(x,y),F(u,v)) \leq h\ max \left\{ \begin{array}{c} d(fx,fu), d(fy,fv), d(fx,F(x,y), d(fu,,F(u,v) \\ \end{array} \right\} \\ +L\ min \left\{ \begin{array}{c} d(fx,fu), d(fy,fv), d(fx,F(x,y)), \\ d(fu,F(u,v)), d(fx,F(u,v)), d(F(x,y),fu) \end{array} \right\},$$

for all $x,y,u,v \in X$, where $0 \leq h < 1$ and $L \geq 0$,
(2.3.2) $F(X \times X) \subseteq f(X)$,
(2.3.3) $f(X)$ is d- complete,
(2.3.4) the pair (F,f) is W-compatible,
(2.3.5) for each $y \in X, d(x_n,y) \to d(x,y)$,whenever $\{x_n\} \subseteq X, x \in X$ such that $x_n \to x$.
Then the mappings F and f have a unique common coupled fixed point in $X \times X$ and also they have a unique common fixed point in X.

Proof. Let x_0 and y_0 be in X. Since $F(X \times X) \subseteq f(X)$, we can choose $x_1, y_1 \in X$ such that $fx_1 = F(x_0,y_0)$ and $fy_1 = F(y_0,x_0)$.
Continuing this process, we can construct the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{p_n\}$ in X such that $fx_{n+1} = F(x_n,y_n) = z_n$, say and $fy_{n+1} = F(y_n,x_n) = p_n$, say for $n = 0,1,2,.....$

Consider

$$\begin{aligned} d(z_n, z_{n+1}) &= d(F(x_n, y_n), F(x_{n+1}, y_{n+1})) \\ &\leq h\ max\left\{d(z_{n-1}, z_n), d(p_{n-1}, p_n), d(z_{n-1}, z_n), d(z_n, z_{n+1})\right\} \\ &\quad + L\ min\left\{ \begin{array}{c} d(z_{n-1}, z_n), d(p_{n-1}, p_n), d(z_{n-1}, z_n), \\ d(z_n, z_{n+1}), d(z_{n-1}, z_{n+1}), d(z_n, z_n) \end{array} \right\} \\ &= h\ max\left\{d(z_{n-1}, z_n), d(p_{n-1}, p_n)\right\}. \end{aligned}$$

$$\begin{aligned} d(p_n, p_{n+1}) &= d(F(y_n, x_n), F(y_{n+1}, x_{n+1})) \\ &\leq h\ max\left\{d(p_{n-1}, p_n), d(z_{n-1}, z_n), d(p_{n-1}, p_n), d(p_n, p_{n+1})\right\} \\ &\quad + L\ min\left\{ \begin{array}{c} d(p_{n-1}, p_n), d(z_{n-1}, z_n), d(p_{n-1}, p_n), \\ d(p_n, p_{n+1}), d(p_{n-1}, p_{n+1}), d(p_n, p_n) \end{array} \right\} \\ &= h\ max\left\{d(p_{n-1}, p_n), d(z_{n-1}, z_n)\right\}. \end{aligned}$$

Thus

$$\begin{aligned} \max\left\{d(z_n, z_{n+1}), d(p_n, p_{n+1})\right\} &\leq h\ max\left\{d(p_{n-1}, p_n), d(z_{n-1}, z_n)\right\} \\ &\leq h^2\ max\left\{d(p_{n-2}, p_{n-1}), d(z_{n-2}, z_{n-1})\right\} \\ &: \\ &: \\ &\leq h^n\ max\left\{d(p_0, p_1), d(z_0, z_1)\right\}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} h^n$ is convergent, it follows that $\sum_{n=1}^{\infty} d(z_n, z_{n+1})$ and $\sum_{n=1}^{\infty} d(p_n, p_{n+1})$ are convergent. Hence $d(z_n, z_{n+1}) \to 0$, and $d(p_n, p_{n+1}) \to 0$ as $n \to \infty$. Suppose $f(X)$ is d-complete. Then there exist α and β in $f(X)$ such that $\{z_n\}$ and $\{p_n\}$ coverge to α and β respectively. Hence there exist $x, y \in X$ such that $\alpha = fx$ and $\beta = fy$.

$$\begin{aligned} d(z_n, F(x, y)) &= d(F(x_n, y_n), F(x, y)) \\ &\leq h\ max\left\{ \begin{array}{c} d(z_{n-1}, fx), d(p_{n-1}, fy), \\ d(z_{n-1}, z_n), d(fx, F(x, y)) \end{array} \right\} \\ &\quad + L\ min\left\{ \begin{array}{c} d(z_{n-1}, fx), d(p_{n-1}, fy), d(z_{n-1}, z_n), \\ d(fx, F(x, y)), d(z_{n-1}, F(x, y)), d(z_n, fx) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(fx, F(x, y)) &\leq h\ max\left\{0, 0, 0, d(fx, F(x, y))\right\} \\ &\quad + L\ min\left\{0, 0, 0, d(fx, F(x, y)), d(fx, F(x, y)), 0\right\} \\ &= hd(fx, F(x, y)). \end{aligned}$$

Hence $F(x, y) = fx = \alpha$.

$$\begin{aligned} d(p_n, F(y, x)) &= d(F(y_n, x_n), F(y, x))) \\ &\leq h\ max\left\{ \begin{array}{c} d(p_{n-1}, fy), d(z_{n-1}, fx), \\ d(p_{n-1}, p_n), d(fy, F(y, x)) \end{array} \right\} \\ &\quad + L\ min\left\{ \begin{array}{c} d(p_{n-1}, fy), d(z_{n-1}, fx), d(p_{n-1}, p_n), \\ d(fy, F(y, x)), d(p_{n-1}, F(y, x)), d(p_n, fy) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, we get $d(fy, F(y, x)) \leq h\ d(fy, F(y, x))$,
so that $F(y, x) = fy = \beta$.

Since (F, f) is W-compatible pair, we have
$f\alpha = ffx = f(F(x, y)) = F(fx, fy) = F(\alpha, \beta)$ and
$f\beta = ffy = f(F(y, x)) = F(fy, fx) = F(\beta, \alpha)$.

$$\begin{aligned} d(z_n, f\alpha) &= d(F(x_n, y_n), F(\alpha, \beta)) \\ &\leq h\ max\,\{d(z_{n-1}, f\alpha), d(p_{n-1}, f\beta), d(z_{n-1}, z_n), 0\} \\ &\quad + L\ min \left\{ \begin{array}{l} d(z_{n-1}, f\alpha), d(p_{n-1}, f\beta), d(z_{n-1}, z_n) \\ 0, d(z_{n-1}, f\alpha), d(z_n, f\alpha) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, we get $d(\alpha, f\alpha) \leq h\ max\{d(\alpha, f\alpha), d(\beta, f\beta)\}$.

$$\begin{aligned} d(p_n, f\beta) &= d(F(y_n, x_n), F(\beta, \alpha)) \\ &\leq h\ max\,\{d(p_{n-1}, f\beta), d(z_{n-1}, f\alpha), d(p_{n-1}, p_n), 0\} \\ &\quad + L\ min \left\{ \begin{array}{l} d(p_{n-1}, f\beta), d(z_{n-1}, f\alpha), d(p_{n-1}, p_n) \\ 0, d(p_{n-1}, f\beta), d(p_n, f\beta) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, we get $d(\beta, f\beta) \leq h\ max\{d(\beta, f\beta), d(\alpha, f\alpha)\}$.
Thus $max\{d(\alpha, f\alpha), d(\beta, f\beta\} \leq h\ max\{d(\alpha, f\alpha), d(\beta, f\beta)\}$
so that $f\alpha = \alpha$ and $f\beta = \beta$.
Thus $\alpha = f\alpha = F(\alpha, \beta)$——(I) and $\beta = f\beta = F(\beta, \alpha)$——(II).
Using (2.3.1) we can show that (α, β) is the unique pair in $X \times X$ satisfying (I) and (II) .
Now we will show that $\alpha = \beta$.

$$\begin{aligned} d(\alpha, \beta) &= d(F(\alpha, \beta), F(\beta, \alpha)) \\ &\leq h\ max\,\{d(\alpha, \beta), d(\beta, \alpha), 0, 0\} + L(0) \\ &= h\ max\,\{d(\alpha, \beta), d(\beta, \alpha)\}\,. \end{aligned}$$

$$\begin{aligned} d(\beta, \alpha) &= d(F(\beta, \alpha), F(\alpha, \beta)) \\ &\leq h\ max\,\{d(\beta, \alpha), d(\alpha, \beta)\}\,. \end{aligned}$$

Hence, $max\{d(\alpha, \beta), d(\beta, \alpha)\} \leq h\ max\{d(\alpha, \beta), d(\beta, \alpha)\}$. Thus $\alpha = \beta$. Hence α is a common fixed point of F and f. Using (2.3.1), we can show that α is the unique common fixed point of F and f. □

The following example illustrates Therorem 2.3.

Example 2.4. Let $X = \{0, 1\}$ and $d(x, y) = |x^2 - y|, \forall x, y \in X$. Define $F(x, y) = 1, \forall x, y \in X$ and $f1 = 1, f0 = 0$.

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