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Existence results for impulsive differential equations with nonlocal conditions via measures of noncompactness

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This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

In this paper, we study the existence of integral solutions for impulsive evolution equations with nonlocal conditions where the linear part is nondensely defined. Some existence results of integral solutions to such problems are obtained under the conditions in respect of the Hausdorff's measure of noncompactness. Example is provided to illustrate the main result.

Keywords: Impulsive differential equations, nondensely defined, noncompact measures, nonlocal conditions, integral solutions, semigroup theory.
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1. Introduction

The purpose of this paper is to study the existence results for impulsive partial differential equations with nonlocal conditions in a real Banach space X of the form:

$$u'(t) = Au(t) + f(t, u(t)), \quad 0 \le t \le b, \quad t \ne t_i, \tag{1.1}$$

$$u(0) = g(u), \tag{1.2}$$

$$\Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \dots, p, \quad 0 < t_1 < t_2 < \cdots < t_p < b, \tag{1.3}$$

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where $A : D(A) \subset X$ is a nondensely defined operator, $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$, $u(t_i^+)$, $u(t_i^-)$ denote the right and left limit of u at t_i, respectively. f, g and I_i are appropriate functions to be specified later.

The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. It has seen considerable development in the last decade; see the monographs of Lakshmikantham et al. [22], Bainov and Simeonov [4], and Samoilenko and Perestyuk [27] where numerous properties of their solutions are studied, and detailed bibliographies are given.

The notion of "nonlocal condition" has been introduced to extend the study of the classical initial value problems; see e.g. [6, 7, 8, 13, 17, 18, 19, 23, 28]. It is more precise for describing nature phenomena than the classical condition since more information is taken into account, thereby decreasing the negative effects incurred by a possibly erroneous single measurement taken at the initial time. The study of abstract nonlocal initial value problems was initiated by Byszewski, we refer to some of the papers below. Byszewski [10, 11], Byszewski and Lakshmikantham [9] give the existence and uniqueness of mild solutions and classical solutions when f and g satisfy the Lipschitz -type conditions. Akca et al. [3] initiated the study of impulsive differential equations with nonlocal conditions in Banach spaces. For more details on nonlocal conditions and impulsive differential equations we refer [1, 2, 14, 15, 24].

Recently many authors [17, 23, 29] have been studied the case A is linear, densely defined operator on X which generates a C_0- semigroup. Very recently, Z. Fan [19] obtained the existence of mild solutions for the following impulsive semilinear differential equation

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)), \quad 0 \le t \le b, \quad t \ne t_i, \\ u(0) &= u_0 - g(u), \\ \Delta u(t_i) &= I_i(u(t_i)), \quad i = 1, 2, \dots, p, \quad 0 < t_1 < t_2 < \cdots < t_p < b, \end{aligned}$$

where the author proved the results under the assumptions of Hausdorff's measure of noncompactness. However, as indicated in [16], we some times need to study the nondensely defined operators. It occurs in many situations due to restrictions on the space where the equation is considered or due to boundary conditions. Recently, Z. Fan [18] proved the existence of integral solutions for the following partial differential equations with nonlocal conditions

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)), \quad t \in (0, b], \\ u(0) &= g(u), \end{aligned}$$

where the operator A is nondensely defined. The author established the results under the assumptions of the Hausdorff's measure of noncompactness. The present paper is motivated by the recent papers of Z. Fan [18, 19]. We use some hypothesis in [18, 19], and using the method of Hausdorff's measure of noncompactness, we give the existence of integral solution of impulsive differential equation with nonlocal conditions (1.1)-(1.3). The results obtained in this paper are generalizations of the results given by [12, 18, 19, 23, 24].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $(X, \|\cdot\|)$ be a real Banach space. We denote by $C(0, b; X)$ the space of $X-$valued continuous functions on $[0, b]$ with the norm

$$\|u\| = \sup\{\|u(t)\|, \ t \in [0, b]\}.$$

A measurable function $f : [0, b] \to X$ is Bochner integrable if and only if $|f|$ is Lebesgue integrable. For properties of the Bochner integrable, see for instance, Yosida [30]. By $L^1(0, b; X)$ the space of $X-$valued

Bochner integrable functions on $[0, b]$ with the norm

$$\|f\|_{L^1} = \int_0^b \|f(t)\| dt.$$

Let $\mathcal{PC}(0, b; X) = \{u : [0, b] \to X : u(t)$ be continuous at $t \neq t_i$ and left continuous at $t = t_i$ and the right limit $u(t_i^+)$ exists for $i = 1, 2, \dots, p\}$. Evidently $\mathcal{PC}(0, b; X)$ is a Banach space with the norm

$$\|u\|_{\mathcal{PC}} = \sup_{t\in[0,b]} \|u(t)\|.$$

Throughout this work, we suppose that

(H1) The linear operator $A : D(A) \subset X \to X$ satisfies the Hille-Yosida condition, if there exist $\overline{M} \geq 0$ and $\omega \in R$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\sup\{(\lambda - \omega)^n \|\mathbb{R}(\lambda, A)^n\| : n \in N, \lambda > \omega\} \leq \overline{M},$$

where $\mathbb{R}(\lambda, A) = (\lambda I - A)^{-1}$, $\rho(A)$ is the resolvent set of A.

(H2) The operator $T(t)$ generated by A_0 is compact in $\overline{D(A)}$ when $t > 0$ and $M = \sup\limits_{t\in[0,b]} \|T(t)\|$.

Definition 2.1. *A function $u : [0, b] \to X$ is said to be an integral solution of (1.1)-(1.3) on $[0, b]$ if the following conditions hold:*

(i) $u \in \mathcal{PC}(0, b; X)$;

(ii) $\int_0^s u(s)ds \in D(A)$ *for* $t \in [0, b]$;

(iii) $u(t) = g(u) + A\int_0^t u(s)ds + \int_0^t f(s, u(s))ds + \sum_{0<t_i<t} I_i(u(t_i))$, *for* $t \in [0, b]$.

From the closedness property of A, one can see that if u is an integral solution of (1.1)-(1.3) on $[0, b]$, then for all $t \in [0, b]$, $u(t) \in \overline{D(A)}$. In particular, $u(0) \in \overline{D(A)}$.

Let A_0 be the part of A in $\overline{D(A)}$ defined by

$$\begin{aligned} D(A_0) &= \{x \in D(A) : Ax \in \overline{D(A)}\}, \\ A_0 x &= Ax. \end{aligned}$$

Then A_0 generates a C_0- semigroup $\{T(t)\}_{t\geq 0}$ on $\overline{D(A)}$ (see Pazy [26] for semigroup theory) and the integral solution in Definition 2.1 (if it exists) is given by

$$u(t) = T(t)g(u) + \lim_{\lambda\to+\infty} \int_0^t T(t - s)B(\lambda)f(s, u(s))ds + \sum_{0<t_i<t} T(t - t_i)I_i(u(t_i)), \quad 0 \leq t \leq b,$$

where $B(\lambda) = \lambda\mathbb{R}(\lambda; A)$. For more details about nondensely defined operators and integrated semigroups we refer to [13, 16, 21].

Next, we introduce the Hausdorff's measure of noncompactness $\alpha(\cdot)$ defined on each bounded subset Ω of Banach space Y by

$$\alpha(\Omega) = \inf\{\epsilon > 0;\ \Omega \text{ has a finite } \epsilon - \text{net in } Y\}.$$

Some basic properties of $\alpha(\cdot)$ are given in the following Lemma.

Lemma 2.2. *([5]). Let Y be a real Banach space and $B, C \subseteq Y$ be bounded, the following properties are satisfied:*

(1) B is pre-compact if and only if $\alpha(B) = 0$;

(2) $\alpha(B) = \alpha(\bar{B}) = \alpha(convB)$, where $\bar{B}$ and conv B mean the closure and convex hull of B, respectively;

(3) $\alpha(B) \leq \alpha(C)$ when $B \subseteq C$;

(4) $\alpha(B + C) \leq \alpha(B) + \alpha(C)$, where $B + C = \{x + y; x \in B,\ y \in C\}$;

(5) $\alpha(B \cup C) \leq \max\{\alpha(B), \alpha(C)\}$;

(6) $\alpha(\lambda B) = |\lambda|\alpha(B)$ for any $\lambda \in R$;

(7) If the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant k, then $\alpha(QB) \leq k\alpha(B)$ for any bounded subset $B \subseteq D(Q)$, where Z be a Banach space.

(8) $\alpha(B) = \inf\{d_Y(B, C); C \subseteq Y$ be pre-compact$\}$=$\inf\{d_Y(B, C); C \subseteq Y$ be finite valued$\}$, where $d_Y(B, C)$ means the nonsymmetric (or symmetric) Hausdorff distance between B and C in Y;

(9) If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subsets of Y and $\lim_{n\to\infty} \alpha(W_n) = 0$, then $\cap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y.

The map $Q : W \subseteq Y \to Y$ is said to be an $\alpha-$contraction if there exists a positive constant $k < 1$ such that $\alpha(QC) \leq k\alpha(C)$ for any bounded closed subset $C \subseteq W$, where Y is a Banach space.

Lemma 2.3. *([5], Darbo-Sadovskii). If $W \subseteq Y$ is bounded closed and convex, the continuous map $Q : W \to W$ is an $\alpha-$contraction, then the map Q has at least one fixed point in W.*

Definition 2.4. *A countable set $\{f_n\}_{n=1}^{+\infty} \subset L^1(0, b; X)$ is said to be semicompact if the sequence $\{f_n(t)\}_{n=1}^{+\infty}$ is compact in X for a.e. $t \in [0, b]$ and if there is a function $\mu \in L^1(0, b; \mathbb{R})$ satisfying $\sup_{n\geq1} \|f_n(t)\| \leq \mu(t)$ for a.e. $t \in [0, b]$.*

Definition 2.5. *We call the operator $G : L^1(0, b; X) \to \mathcal{PC}(0, b; X)$ defined by*

$$Gf(t) = \lim_{\lambda\to+\infty} \int_0^t T(t - s)B(\lambda)f(s)ds, \quad t \in [0, b], \tag{2.1}$$

as the Cauchy operator.

Now, we give the following properties about the Cauchy operator G.

Proposition 2.6. *([20]) Let G be the Cauchy operator defiend by (2.1), $\{f_n\}_{n=1}^{+\infty}$ a sequence of functions in $L^1(0, b; X)$. Assume that there exist μ, η in $L^1(0, b; \mathbb{R})$ satisfying*

$$\sup_{n\geq1} \|f_n(t)\| \leq \mu(t) \quad \text{and} \quad \alpha(\{f_n(t)\}_{n=1}^{+\infty}) \leq \eta(t) \text{ a.e. } t \in [0, b].$$

Then for all $t \in [0, b]$, we have

$$\alpha(\{(Gf_n)(t)\}_{n=1}^{+\infty}) \leq 2M\overline{M} \int_0^t \eta(s)ds,$$

where α is the Hausdorff measure of noncompactness.

Proposition 2.7. *([20]) Let G be the Cauchy operator defined by (2.1). Then for every semicompact set $\{f_n\}_{n=1}^{+\infty} \subset L^1(0, b; X)$, the set $\{Gf_n\}_{n=1}^{+\infty}$ is relatively compact in $\mathcal{PC}(0, b; X)$.*

Proposition 2.8. *If $D \subseteq \mathcal{PC}(0, b; X)$ be bounded, then we have*

$$\sup_{t\in[0,b]} \alpha(D(t)) \le \alpha(D).$$

The follwoing fixed point theorem, a nonlinear alternative of Monch type, plays a crucial role in our existence of integral solutions of (1.1)-(1.3).

Theorem 2.9. *([25]) Let E be a Banach space, U an open subset of E and $0 \in U$. Suppose that $F : \overline{U} \to E$ is a continuous map which satisfies Monch's condition (that is, if $D \subseteq \overline{U}$ is countable and $D \subseteq \overline{co}(\{0\} \cup F(D))$, then $\overline{D}$ is compact) and assume that*

$$x \neq \lambda F(x) \quad \textit{for } x \in \partial U \textit{ and } \lambda \in (0, 1)$$

holds. Then F has a fixed point in $\overline{U}$.

3. Existence Results

In this section, we present and prove the existence of integral solutions for the impulsive nonlocal problem (1.1)-(1.3).

Let r be a finite positive constant. We consider the sets $B_r = \{x \in X : \|x\| \le r\}$, $W_r = \{u \in \mathcal{PC}(0, b; X) : u(t) \in B_r,\ \forall\ t \in [0, b]\}$.

Now, we list the following hypotheses:

(H3) The operator $T(t), 0 \leq t \leq b$ generated by A_0 is equicontinuous;

(H4) (i) $f : [0,b] \times X \to X$, for a.e. $t \in [0,b]$, the function $f(t,\cdot) : X \to X$ is continuous for all $x \in X$, the function $f(\cdot,x) : [0,b] \to X$ is measurable.

(ii) Moreover, for any $l > 0$, there exists a function $\rho_l \in L^1(0,b;\mathbb{R})$ such that

$$\|f(t,x)\| \leq \rho_l(t)$$

for a.e. $t \in [0,b]$ and all $x \in B_l$;

(iii) There exists a function $m \in L^1(0,b;\mathbb{R})$ and a nondecreasing continuous function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\|f(t,x)\| \leq m(t)\psi(\|x\|)$$

for all $x \in X, t \in [0,b]$.

(H5) There exists a function $h \in L^1(0,b;\mathbb{R})$ such that for every bounded $D \subseteq W_r$,

$$\alpha(f(t,D)) \leq h(t)\alpha(D)$$

for a.e. $t \in [0,b]$, where α is the Hausdorff measure of noncompactness.

(H6) $g : \mathcal{PC}(0,b;X) \to \overline{D(A)}$ is Lipschitz continuous with Lipschitz constant k;

(H7) $I_i : X \to \overline{D(A)}$ is Lipschitz continuous with Lipschitz constant $k_i, i = 1,2,\ldots,p$.

Theorem 3.1. *Assume that the conditions (H1),(H2), (H4)(i)(ii) and (H6)-(H7) are satisfied. Then the nonlocal impulsive problem (1.1)-(1.3) has at least one integral solution on* $[0,b]$ *provided that*

$$M\Big[(k + \sum_{i=1}^{p} k_i)r + \|g(0)\| + \sum_{i=1}^{p} \|I_i(0)\| + \overline{M}\|\rho_r\|_{L^1}\Big] \leq r \tag{3.1}$$

Proof. Define the operator $Q : \mathcal{PC}(0,b;X) \to \mathcal{PC}(0,b;X)$ by

$$(Qu)(t) = (Q_1u)(t) + (Q_2u)(t) + (Q_3u)(t), \tag{3.2}$$

with

$$(Q_1u)(t) = T(t)g(u), \tag{3.3}$$

$$(Q_2u)(t) = \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)), \tag{3.4}$$

$$(Q_3u)(t) = \lim_{\lambda \to +\infty} \int_0^t T(t-s)B(\lambda)f(s,u(s))ds, \tag{3.5}$$

for all $t \in [0,b]$. It is easy to see that the fixed point of Q is the integral solution of the nonlocal impulsive problem (1.1)-(1.3). Subsequently, we will prove that Q has a fixed point, by using Lemma 2.2.

Firstly, we prove that the mapping Q is continuous on $\mathcal{PC}(0,b;X)$. For this purpose, let $\{u_n\}_{n=1}^{+\infty}$ be a sequence in $\mathcal{PC}(0,b;X)$ with $\lim_{n\to\infty} u_n = u$ in $\mathcal{PC}(0,b;X)$. By the continuity of f with respect to the second argument, we deduce that for each $s \in [0,b], f(s,u_n(s))$ converges to $f(s,u(s))$ in X; and we have

$$\begin{aligned}\|Qu_n - Qu\|_{\mathcal{PC}} \leq\ & M\Big[\|g(u_n) - g(u)\| + \sum_{i=1}^{p} \|I_i(u_n(t_i)) - I_i(u(t_i))\|\Big] \\ & + M\overline{M}\int_0^b \|f(s,u_n(s)) - f(s,u(s))\|ds.\end{aligned}$$

Then by continuity of g, I_i and using dominated convergence theorem, we get $\lim_{n\to\infty} Qu_n = Qu$ in $\mathcal{PC}(0,b;X)$, which implies that the mapping Q is continuous on $\mathcal{PC}(0,b;X)$.

Secondly, we claim that $QW_r \subseteq W_r$. In fact, for any $u \in W_r \subseteq \mathcal{PC}(0,b;X)$, by (H4)(ii), (3.1) and

$$\|B(\lambda)\| \leq \frac{\lambda\overline{M}}{\lambda-\omega} \to \overline{M}, \quad \text{as } \lambda \to +\infty.$$

We have

$$\begin{aligned}
\|(Qu)(t)\| &\leq \|T(t)g(u)\| + \|\lim_{\lambda\to+\infty}\int_0^t T(t-s)B(\lambda)f(s,u(s))ds\| + \|\sum_{0<t_i<t} T(t-t_i)I_i(u(t_i))\| \\
&\leq M[\|g(u)-g(0)\| + \|g(0)\|] + M\overline{M}\int_0^b \|f(s,u(s))\|ds \\
&\quad + M\Big[\sum_{i=1}^p \|I_i(u(t_i)) - I_i(0)\| + \|I_i(0)\|\Big] \\
&\leq M\Big[(k+\sum_{i=1}^p k_i)r + \|g(0)\| + \sum_{i=1}^p \|I_i(0)\| + \overline{M}\|\rho_r\|_{L^1}\Big] \\
&\leq r
\end{aligned}$$

It implies that $QW_r \subseteq W_r$.

Now, according to Lemma 2.2, it remains to prove that Q is an $\alpha-$contraction in W_r. By using conditions (H6) and (H7), we get that $Q_1+Q_2 : W_r \to \mathcal{PC}(0,b;X)$ is Lipschitz continuous with constant $M(k+\sum_{i=1}^p k_i)$. In fact, for any $u, v \in W_r$, we have

$$\begin{aligned}
\|(Q_1+Q_2)u - (Q_1+Q_2)v\|_{\mathcal{PC}} &\leq \sup_{t\in[0,b]}\Big[\|T(t)[g(u)-g(v)]\| \\
&\quad + \sum_{i=1}^p \|T(t-t_i)[I_i(u(t_i)) - I_i(v(t_i))]\|\Big] \\
&\leq M\Big[k\|u-v\|_{\mathcal{PC}} + \sum_{i=1}^p k_i\|u-v\|_{\mathcal{PC}}\Big] \\
&= M[k+\sum_{i=1}^p k_i]\|u-v\|_{\mathcal{PC}}
\end{aligned}$$

Thus, Lemma 2.1 (7), we obtain that

$$\alpha((Q_1+Q_2)W_r) \leq M[k+\sum_{i=1}^p k_i]\alpha(W_r).$$

Finally, we prove that $Q_3 : W_r \to \mathcal{PC}(0,b;X)$ is a compact operator by using Arzela- Ascoli's theorem. From [18, Theorem 3.1], we see that Q_3 is compact. Thus $\alpha(Q_3W_r) = 0$. Consequently,

$$\begin{aligned}
\alpha(QW_r) &\leq \alpha((Q_1+Q_2)W_r) + \alpha(Q_3W_r) \\
&\leq M[k+\sum_{i=1}^p k_i]\alpha(W_r)
\end{aligned}$$

Since the condition (3.1), $M[k+\sum_{i=1}^p k_i] < 1$, the mapping Q is an $\alpha-$contraction in W_r. By Darbo-Sadovskii's fixed point theorem, the operator Q has a fixed point in W_r, which is the integral solution of the nonlocal impulsive problem (1.1)-(1.3). This completes the proof. □

Theorem 3.2. *Assume that the conditions (H1),(H2), (H4)(i)(iii) and (H5)-(H7) are satisfied. Then the nonlocal impulsive problem (1.1)-(1.3) has at least one integral solution on* $[0, b]$ *provided that there exists a constant* $N > 0$ *with*

$$\frac{[1 - M(k + \sum_{i=1}^{p} k_i)]N}{M\Big[\|g(0)\| + \overline{M}\psi(N)\|m\|_{L^1} + \sum_{i=1}^{p}\|I_i(0)\|\Big]} > 1 \tag{3.6}$$

and that

$$M\Big[(k + \sum_{i=1}^{p} k_i) + 2\overline{M}\|h\|_{L^1}\Big] < 1. \tag{3.7}$$

Proof. We define the operator Q as defined in (3.2)-(3.5) for all $t \in [0, b]$. It is easy to see that the fixed point of Q is the integral solution of nonlocal impulsive problem (1.1)-(1.3). Subsequently, we will prove that Q has a fixed point by using Theorem 2.1. For better readability, we break the proofs into three steps.
Step 1: The operator Q is continuous on $\mathcal{PC}(0, b; X)$.

For this purpose, let $\{u_n\}_{n=1}^{+\infty}$ be a sequence in $\mathcal{PC}(0, b; X)$ with $\lim_{n\to\infty} u_n = u$ in $\mathcal{PC}(0, b; X)$. Then by (H4)(i), we have that

$$f(s, u_n(s)) \to f(s, u(s)), \quad (n \to +\infty) \quad \forall\ s \in [0, b].$$

Since $I_i, i = 1, 2, \ldots, p$ is continuous and $\|f(s, u_n(s)) - f(s, u(s))\| \le 2\psi(N)m(s)$ for some integer N, by (H4)(iii) and (H6)-(H7) together with the dominated convergence theorem, we have

$$\begin{aligned}\|Qu_n - Qu\|_{\mathcal{PC}} \le{}& M\Big[\|g(u_n) - g(u)\| + \sum_{i=1}^{p}\|I_i(u_n(t_i)) - I_i(u(t_i))\|\Big]\\ &+ M\overline{M}\int_0^b \|f(s, u_n(s)) - f(s, u(s))\|ds\\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Thus, Q is continuous on $\mathcal{PC}(0, b; X)$.
Step 2: The Monch's condition holds.

For this purpose, let $D \subseteq W_r$ be countable and $D \subseteq \overline{co}(\{0\} \cup Q(D))$. We show that $\alpha(D) = 0$. Without loss of generality, we may assume that $D = \{u_n\}_{n=1}^{+\infty}$. By using the condition (H3), we can easily verify that $\{Q_3 u_n\}_{n=1}^{+\infty}$ is equicontinuous. Moreover, $Q_1 + Q_2 : D \to \mathcal{PC}(0, b; X)$ is Lipschitz continuous with constant $M(k + \sum_{i=1}^{p} k_i)$ due to the conditions (H6) and (H7). In fact, $u, v \in D$, we have

$$\begin{aligned}\|(Q_1 + Q_2)u - (Q_1 + Q_2)v\|_{\mathcal{PC}} &\le \sup_{t\in[0,b]}\Big[\|T(t)[g(u) - g(v)]\|\\ &\qquad + \sum_{i=1}^{p}\|T(t - t_i)[I_i(u(t_i)) - I_i(v(t_i))]\|\Big]\\ &\le M\Big[k\|u - v\|_{\mathcal{PC}} + \sum_{i=1}^{p} k_i\|u - v\|_{\mathcal{PC}}\Big]\\ &= M[k + \sum_{i=1}^{p} k_i]\|u - v\|_{\mathcal{PC}}\end{aligned}$$

So, from Proposition (2.1),(2.3) and Lemma 2.1, we have

$$\begin{aligned}
\alpha\left(\{Qu_n\}_{n=1}^{+\infty}\right) &\le \alpha\left(\{(Q_1+Q_2)u_n\}_{n=1}^{+\infty}\right)+\alpha\left(\{Q_3u_n\}_{n=1}^{+\infty}\right)\\
&\le M(k+\sum_{i=1}^{p}k_i)\alpha\left(\{u_n\}_{n=1}^{+\infty}\right)\\
&\quad+\sup_{t\in[0,b]}\alpha\left(\{\lim_{\lambda\to+\infty}\int_0^t T(t-s)B(\lambda)f(s,u_n(s))ds\}_{n=1}^{+\infty}\right)\\
&\le M(k+\sum_{i=1}^{p}k_i)\alpha\left(\{u_n\}_{n=1}^{+\infty}\right)\\
&\quad+2M\overline{M}\int_0^b h(s)\sup_{t\in[0,b]}\alpha\left(\{u_n(t)\}_{n=1}^{+\infty}\right)ds\\
&\le M(k+\sum_{i=1}^{p}k_i)\alpha\left(\{u_n\}_{n=1}^{+\infty}\right)+2M\overline{M}\|h\|_{L^1}\alpha\left(\{u_n\}_{n=1}^{+\infty}\right)\\
&\le M\Big[(k+\sum_{i=1}^{p}k_i)+2\overline{M}\|h\|_{L^1}\Big]\alpha\left(\{u_n\}_{n=1}^{+\infty}\right)
\end{aligned}$$

Thus, we get that

$$\alpha(D)\le\alpha(\overline{co}\{0\}\cup Q(D))=\alpha(Q(D))\le M\Big[(k+\sum_{i=1}^{p}k_i)+2\overline{M}\|h\|_{L^1}\Big]\alpha(D)$$

which implies that $\alpha(D)=0$, since the condition (3.7) holds.
Step 3: Now, let $\mu\in(0,1)$ and $u=\mu Q(u)$. Then for $t\in[0,b]$, we have

$$u(t)=\mu T(t)g(u)+\mu\lim_{\lambda\to+\infty}\int_0^t T(t-s)B(\lambda)f(s,u(s))ds+\mu\sum_{0<t_i<t}T(t-t_i)I_i(u(t_i))$$

and one has

$$\begin{aligned}
\|u(t)\| &\le M[\|g(u)-g(0)\|+\|g(0)\|]+M\overline{M}\int_0^b\|f(s,u(s))\|ds\\
&\quad+M\Big[\sum_{i=1}^{p}\|I_i(u(t_i))-I_i(0)\|+\|I_i(0)\|\Big]\\
&\le M[k\|u\|+\|g(0)\|]+M\overline{M}\int_0^b m(s)\psi(\|u\|)ds+M\sum_{i=1}^{p}[k_i\|u\|+\|I_i(0)\|]
\end{aligned}$$

Consequently,

$$\frac{[1-M(k+\sum_{i=1}^{p}k_i)]\|u\|}{M\Big[\|g(0)\|+\overline{M}\psi(\|u\|)\|m\|_{L^1}+\sum_{i=1}^{p}\|I_i(0)\|\Big]}\le 1$$

Then by (3.6) there exists N such that $\|u\|\neq N$. Set

$$U=\{u\in\mathcal{PC}(0,b;X):\|u\|\le N\}.$$

From the choice of U there is no $u\in\partial U$ such that $u=\mu Q(u)$ for some $\mu\in(0,1)$. Thus, we get a fixed point Q in $\overline{U}$ due to Theorem 2.1, which is a integral solution of (1.1)-(1.3). □

4. Example

As an application of Theorem 3.1, we consider the following system

$$\frac{\partial}{\partial t}w(t,x) = \nabla w(t,x) + F(t, w(t,x)),\ 0 \le t \le b,\ t \ne t_i,\ i = 1,2,\dots,p,\ x \in \Omega, \tag{4.1}$$

$$w(t_i^+, x) - w(t_i^-, x) = I_i(w(t_i, x)), \quad i = 1,2,\dots,p, \tag{4.2}$$

$$w(t,x) = 0, \quad 0 \le t \le b, \quad x \in \partial\Omega, \tag{4.3}$$

$$w(0,x) + \sum_{j=1}^{p} c_j w(t_j, x) = u_0(x), \quad 0 \le t_j \le b, \quad x \in \Omega, \tag{4.4}$$

where Ω is a bounded open set in R^n with regular boundary $\partial\Omega, u_0 \in C(\overline{\Omega}; R^n)$, $\nabla = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$, $F : [0,b] \times \overline{\Omega} \to R^n$, $c_j,\ j = 1,2,\dots,p$ and $I_i,\ i = 1,2,\dots,p$ are given real numbers.

We choose $X = C(\overline{\Omega}, R^n)$ and we consider the operator $A : D(A) \subseteq X \to X$ defined by

$$D(A) = \{z \in X : \nabla z \in X \quad \text{and } z = 0 \ \text{ on } \ \partial\Omega\},$$
$$Az = \nabla z.$$

Now, we have

(i) $\overline{D(A)} = C_0(\overline{\Omega}; R^n) = \{z \in X : z = 0 \text{ on } \ \partial\Omega\} \ne X$;

(ii) $(0, +\infty) \subseteq \rho(A)$;

(iii) $\|\mathbb{R}(\lambda; A)\| \le \frac{1}{\lambda}$, for $\lambda > 0$.

This implies that the operator A satisfies the conditions (H1) and (H2) with $M = \overline{M} = 1$.

We assume that

(1) $f : [0,b] \times X \to X$ is a continuous function defined by

$$f(t,z)(x) = F(t, z(x)), \quad x \in \overline{\Omega}.$$

(2) $g : \mathcal{PC}(0,b;X) \to X$ is a continuous function defined by

$$g(u)(x) = u_0(x) - \sum_{j=1}^{p} c_j u(t_j)(x), \quad t \in [0,b], \quad x \in \overline{\Omega},$$

where $u(t)(x) = w(t,x)$, $t \ge 0, x \in \overline{\Omega}$.

(3) $I_i : X \to X$ is a continuous function defined by

$$\Delta w(t_i, x) = w(t_i^+, x) - w(t_i^-, x), \quad i = 1,2,\dots,p,$$

where $u(t_i)(x) = w(t_i, x)$, $t \ge 0,\ x \in \overline{\Omega}$.

Under these assumptions, the problem (4.1)-(4.4) can be reformulated as the abstract problem (1.1)-(1.3).

Thus, under the appropriate conditions on the functions f, g and I_i as those in (H4)(i)(ii) and (H6)-(H7), the problem (1.1)-(1.3) has an integral solution.

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