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# $H(\cdot,\cdot)$-$\eta$-cococercive operators and variational-like inclusions in Banach spaces

Rais Ahmad[a], Mohammad Dilshad[a,*]

[a]Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India.

Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

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**Abstract**

In this paper, we define $H(\cdot,\cdot)$-$\eta$-cocoercive operators in $q$-uniformly smooth Banach spaces and its resolvent operator. We prove the Lipschitz continuity of the resolvent operator associated with $H(\cdot,\cdot)$-$\eta$-cocoercive operator and estimate its Lipschitz constant. By using the techniques of resolvent operator, an iterative algorithm for solving a variational-like inclusion problem is constructed. The existence of solution for the variational-like inclusions and the convergence of iterative sequences generated by the algorithm is proved. Some examples are given.

*Keywords:* $H(\cdot,\cdot)$-$\eta$-cocoercive, Algorithm, Inclusion, Banach spaces, Lipschitz continuity.
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## 1. Introduction

It is well known that variational inequality theory has become very effective and powerful tool for studying a wide range of problems arising in differential equations, mechanics, contact problems in elasticity, optimization and control problems, operation research, etc.. A useful and important generalization of variational inequalities is a generalized mixed type variational inequality containing nonlinear term. Due to the appearance of this nonlinear term, the projection method can not be used to study the existence and algorithm of solutions for the generalized mixed type variational inequalities. These facts motivated Hassouni and Moudafi [8] to suggest the resolvent operator technique which does not depend on the projection. He studied mixed type of variational inequalities, called variational inclusions.

*Corresponding author
*Email addresses:* `raisain_123@rediffmail.com` ( Rais Ahmad), `mdilshaad@gmail.com` (Mohammad Dilshad)

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The concept of $H$-monotone, $H$-accretive, $(H,\eta)$-accretive, $(H,\eta)$-monotone, $(A,\eta)$-accretive, $H(\cdot,\cdot)$-accretive, $(H(\cdot,\cdot)-\eta)$-monotone and $H(\cdot,\cdot)$-cocoercive operators are introduced and applied by Fang and Huang [4], Fang and Huang [5], Fang, Cho and Kim [6], Fang, Huang and Thompson [7], Lan, Cho and verma [9], Zou and Huang [13], Xu and Wong [11], Ahmad et al.[1]. The concept of $\eta$-cocoercivity, $\eta$-strong monotonicity and $\eta$-strong convexity of a mapping was introduced and studied by Ansari and Yao [2].

Impressed by the noble work mentioned above, in this paper, we introduce $H(\cdot,\cdot)$-$\eta$-cocoercive operators in $q$-uniformly smooth Banach spaces. The resolvent operator associated with $H(\cdot,\cdot)$-$\eta$-cocoercive operator is defined and its Lipschitz continuity is shown. With the help of resolvent operator an iterative algorithm is constructed for solving a variational-like inclusion problem in $q$-uniformly smooth Banach spaces. Some examples are constructed in support of definition of $H(\cdot,\cdot)$-$\eta$-cocoercive operators.

## 2. Preliminaries

Let $X$ is a real Banach space with dual space $X^\star$, $\langle\cdot,\cdot\rangle$ be the duality coupling between $X$ and $X^\star$, and $2^X$ denotes the family of all the non-empty subsets of $X$. The generalized duality mapping $J_q : X \to 2^{X^\star}$ is defined by

$$J_q(x) = \left\{ f^\star \in X^\star : \langle x, f^\star\rangle = \|x\|^q,\ \|f^\star\| = \|x\|^{q-1} \right\},\ \forall\ x \in X,$$

where $q > 1$ is a constant. The modulus of smoothness of $X$ is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{\ \frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \le 1, \|y\| \le t\ \right\}.$$

A Banach space $X$ is called uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0.$$

$X$ is called $q$-uniformly smooth if there exists a constant $C > 0$ such that $\rho_X(t) \le Ct^q,\ q > 1$.

Note that $J_q$ is single-valued if $X$ is uniformly smooth. In connection with the characteristic inequalities in $q$-uniformly smooth Banach spaces, Xu [12] proved the following nice result.

**Lemma 2.1.** *Let $X$ be a real uniformly smooth Banach space. Then $X$ is $q$-uniformly smooth if and only if there exists a constant $C_q > 0$ such that, for all $x, y \in X$,*

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q.$$

*We need the following definitions for proving our main result.*

**Definition 2.2.** Let $A, B : X \to X$, $\eta : X \times X \to X$ be the mappings and let $J_q : X \to 2^{X^\star}$ be the generalized duality mapping. Then $A$ is called

(i) $\eta$-cocoercive, if there exists a constant $\mu_1 > 0$ such that

$$\langle Ax - Ay,\ J_q(\eta(x,y))\rangle \ge \mu_1\|Ax - Ay\|^q,\quad \forall\ x, y \in X;$$

(ii) $\eta$-accretive, if

$$\langle Ax - Ay,\ J_q(\eta(x,y))\rangle \ge 0,\quad \forall\ x, y \in X;$$

(iii) $\eta$-strongly accretive, if there exists a constant $\beta_1 > 0$ such that

$$\langle Ax - Ay,\ J_q(\eta(x,y))\rangle \ge \beta_1\|x - y\|^q,\quad \forall\ x, y \in X;$$

if $\eta(x,y) = x - y$, then it is called strongly accretive.

(iv) $\eta$-relaxed cocoercive, if there exists a constant $\gamma_1 > 0$ such that

$$\langle Ax - Ay,\ J_q(\eta(x,y))\rangle \geq (-\gamma_1)\|Ax - Ay\|^q,\ \ \forall\, x, y \in X;$$

(v) $\alpha$-expansive, if there exists a constant $\alpha > 0$ such that

$$\|Ax - Ay\| \geq \alpha\|x - y\|,\ \ \forall\, x, y \in X;$$

(vi) $B$ is said to be $\beta$-Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|Bx - By\| \leq \beta\|x - y\|,\ \ \forall\, x, y \in X;$$

(vii) $\eta$ is said to Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \leq \tau\|x - y\|,\ \ \forall\, x, y \in X.$$

**Definition 2.3.** Let $A, B : X \to X$, $H : X \times X \to X$, $\eta : X \times X \to X$ be three single-valued mappings and $J_q : X \to 2^{X^\star}$ be the generalized duality mapping. Then

(i) $H(A, \cdot)$ is said to be $\eta$-cocoercive with respect to $A$, if there exists a constant $\mu > 0$ such that

$$\langle H(Ax, u) - H(Ay, u),\ J_q(\eta(x,y))\rangle \geq \mu\|Ax - Ay\|^q,\ \ \forall\, x, y \in X;$$

(ii) $H(\cdot, B)$ is said to $\eta$-relaxed cocoercive with respect to $B$, if there exists a constant $\gamma > 0$ such that

$$\langle H(u, Bx) - H(u, By),\ J_q(\eta(x,y))\rangle \geq (-\gamma)\|Bx - By\|^q,\ \ \forall\, x, y \in X;$$

(iii) $H(A, \cdot)$ is said to be $r_1$-Lipschitz continuous with respect to $A$, if there exists a constant $r_1 > 0$ such that

$$\|H(Ax, u) - H(Ay, u)\| \leq r_1\|x - y\|,\ \ \forall\, x, y \in X;$$

(iv) $H(\cdot, B)$ is said to be $r_2$-Lipschitz continuous with respect to $B$, if there exists a constant $r_2 > 0$ such that

$$\|H(u, Bx) - H(u, By)\| \leq r_2\|x - y\|,\ \ \forall\, x, y \in X.$$

**Definition 2.4.** A multi-valued mapping $M : X \to 2^X$ is said to be $\eta$-cocoercive, if there exists a constant $\mu_2 > 0$ such that

$$\langle u - v,\ J_q(\eta(x,y))\rangle \geq \mu_2\|u - v\|^q,\ \ \forall\, x, y \in X,\ u \in Mx,\ v \in My.$$

**Definition 2.5.** A multi-valued mapping $T : X \to CB(X)$ is said to be $\mathcal{D}$-Lipschitz continuous, if there exists a constant $\lambda_T > 0$ such that

$$\mathcal{D}(Tx, Ty) \leq \lambda_T\|x - y\|,\ \ \forall\, x, y \in X,$$

where $\mathcal{D}(\cdot,\cdot)$ is the Hausdörff metric on $CB(X)$.

**Definition 2.6.** Let $\eta : X \times X \to X$ be a mapping and let $T, Q : X \to CB(X)$ be two multi-valued mappings. A mapping $N : X \times X \to X$ is said to be $\eta$-strongly accretive with respect to $T$ and $Q$, if there exists a constant $t > 0$ such that

$$\langle N(x_1, y_1) - N(x_2, y_2),\ J_q(\eta(u_1, u_2))\rangle \geq t\|u_1 - u_2\|^q,\ \forall u_1, u_2 \in X, x_1 \in T(u_1), y_1 \in Q(u_1), x_2 \in T(u_2), y_2 \in Q(u_2).$$

**Definition 2.7.** Let $T, Q : X \to CB(X)$ be the multi-valued mappings. A mapping $N : X \times X \to X$ is said to be

(i) Lipschitz continuous for the first argument with respect to $T$, if there exists a constant $\lambda_{N_1} > 0$ such that
$$\|N(x_1,\cdot) - N(x_2,\cdot)\| \leq \lambda_{N_1}\|x_1 - x_2\|, \quad \forall\, u_1,u_2 \in X, x_1 \in T(u_1), x_2 \in T(u_2);$$

(ii) Lipschitz continuous for the second argument with respect to $Q$, if there exists a constant $\lambda_{N_2} > 0$ such that
$$\|N(\cdot,y_1) - N(\cdot,y_2)\| \leq \lambda_{N_2}\|y_1 - y_2\|, \quad \forall\, u_1,u_2 \in X, y_1 \in Q(u_1), y_2 \in Q(u_2).$$

**Example 2.8.** Let us consider the **2**-uniformly smooth Banach space $X = \mathbb{R}^2$. Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by
$$A(x_1,x_2) = (x_1, 3x_2), \quad B(y_1,y_2) = (-y_1, -y_1 - y_2), \quad \forall\, (x_1,x_2),(y_1,y_2) \in \mathbb{R}^2.$$
Suppose $H(A,B), \eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ are defined as
$$H(Ax,By) = Ax + By, \quad \eta(x,y) = x - y, \quad \forall\, x,y \in \mathbb{R}^2.$$

Then
$$\begin{aligned}\langle H(Ax,u) - H(Ay,u),\ \eta(x,y)\rangle &= \langle Ax + u - Ay - u,\ x - y\rangle \\ &= \langle Ax - Ay,\ x - y\rangle \\ &= \langle((x_1,3x_2) - (y_1,3y_2)),\ (x_1 - y_1, x_2 - y_2)\rangle \\ &= \langle(x_1 - y_1, 3(x_2 - y_2)),\ (x_1 - y_1, x_2 - y_2)\rangle \\ &= (x_1 - y_1)^2 + 3(x_2 - y_2)^2\end{aligned}$$

and
$$\begin{aligned}\|Ax - Ay\|^2 = \|(x_1 - y_1, 3(x_2 - y_2))\|^2 &= (x_1 - y_1)^2 + 9(x_2 - y_2)^2 \\ &\leq 3(x_1 - y_1)^2 + 9(x_2 - y_2)^2 \\ &= 3\{(x_1 - y_1)^2 + 3(x_2 - y_2)^2\} \\ &= 3\{\langle H(Ax,u) - H(Ay,u),\ \eta(x,y)\rangle\}\end{aligned}$$

i.e. $\langle H(Ax,u) - H(Ay,u),\ \eta(x,y)\rangle \geq \frac{1}{3}\|Ax - Ay\|^2$, which implies that $H$ is $\frac{1}{3}$-$\eta$-cocoercive with respect to $A$. Also
$$\begin{aligned}\langle H(u,Bx) - H(u,By), \eta(x,y)\rangle &= \langle Bx - By,\ x - y\rangle \\ &= \langle((-x_1, -x_1 - x_2) - (-y_1, -y_1 - y_2)),\ (x_1 - y_1, x_2 - y_2)\rangle \\ &= \langle(-(x_1 - y_1), -(x_1 - y_1) - (x_2 - y_2)),\ (x_1 - y_1, x_2 - y_2)\rangle \\ &= -(x_1 - y_1)^2 - (x_1 - y_1)(x_2 - y_2) - (x_2 - y_2)^2 \\ &= -\{(x_1 - y_1)^2 + (x_1 - y_1)(x_2 - y_2) + (x_2 - y_2)^2\}\end{aligned}$$

and
$$\begin{aligned}\|Bx - By\|^2 &= \|(-(x_1 - y_1), -(x_1 - y_1) - (x_2 - y_2))\|^2 \\ &= (x_1 - y_1)^2 + ((x_1 - y_1) + (x_2 - y_2))^2 \\ &= (x_1 - y_1)^2 + (x_1 - y_1)^2 + (x_2 - y_2)^2 + 2(x_1 - y_1)(x_2 - y_2) \\ &\leq 2(x_1 - y_1)^2 + 2(x_2 - y_2)^2 + 2(x_1 - y_1)(x_2 - y_2) \\ &= 2\{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_1 - y_1)(x_2 - y_2)\} \\ &= 2\{-\langle H(u,Bx) - H(u,By),\ \eta(x,y)\rangle\}\end{aligned}$$

i.e. $\langle H(u,Bx) - H(u,By),\ \eta(x,y)\rangle \geq -\frac{1}{2}\|Bx - By\|^2$, which implies that $H$ is $\frac{1}{2}$-$\eta$-relaxed cocoercive with respect to $B$.

## 3. $H(\cdot,\cdot)$-$\eta$-cocoercive operator

In this section, we introduce $H(\cdot,\cdot)$-$\eta$-cocoercive operator and discuss some of its properties.

**Definition 3.1.** Let $A, B : X \to X$, $H : X \times X \to X$, $\eta : X \times X \to X$ be the single-valued mappings. Let $M : X \to 2^X$ be a set-valued mapping. $M$ is said to be $H(\cdot,\cdot)$-$\eta$-cocoercive with respect to the mapping $A$ and $B$, if $M$ is $\eta$-cocoercive and $(H(A,B) + \lambda M)(X) = X$, for every $\lambda > 0$.

**Example 3.2.** Let $X = \mathbb{R}^2$ and $A$, $B$, $H(A,B)$ and $\eta$ are defined as in Example 2.8. Suppose that $M : X \to 2^X$ is defined by

$$M(x_1, x_2) = (x_1, 0), \quad \forall\ (x_1, x_2) \in \mathbb{R}^2.$$

Then it can be easily verify that $M$ is $\eta$-cocoercive and

$$(H(A,B) + \lambda M)(\mathbb{R}^2) = \mathbb{R}^2,\ \forall\ \lambda > 0,$$

which shows that $M$ is $H(\cdot,\cdot)$-$\eta$-cocoercive with respect to $A$ and $B$.

**Theorem 3.3.** *Let $H(A,B)$ be $\eta$-cocoercive with respect to $A$ with constant $\mu > o$ and $\eta$-relaxed cocoercive with respect to $B$ with constant $\gamma > 0$, $A$ is $\alpha$-expansive and $B$ is $\beta$-Lipschitz continuous and $\mu > \gamma, \alpha > \beta$. Let $M : X \to 2^X$ be $H(\cdot,\cdot)$-$\eta$-cocoercive operator with respect to $A$ and $B$. If the following inequality*

$$\langle x - y,\ J_q(\eta(u,v))\rangle \geq 0, \text{ holds for all } (v,y) \in \text{ Graph}(M),$$

*then $x \in Mu$, where* $\text{Graph}(M) = \{(u,x) \in X \times X : x \in Mu\}$.

*Proof.* Suppose that there exists some $(u_0, x_0)$ such that

$$\langle x_0 - y,\ J_q(\eta(u_0,v))\rangle \geq 0,\ \forall\ (v,y) \in \text{ Graph}(M). \tag{3.1}$$

Since $M$ is $H(\cdot,\cdot)$-$\eta$-cocoercive operator with respect to $A$ and $B$, we know that $(H(A,B) + \lambda M)(X) = X$, holds for every $\lambda > 0$ and so there exists $(u_1, x_1) \in$ Graph$(M)$ such that

$$H(Au_1, Bu_1) + \lambda x_1 = H(Au_0, Bu_0) + \lambda u_0 \in X. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned} 0 \leq \lambda\langle x_0 - x_1,\ J_q(\eta(u_0,u_1))\rangle &= -\langle H(Au_0,Bu_0) - H(Au_1,Bu_1),\ J_q(\eta(u_0,u_1))\rangle \\ &= -\langle H(Au_0,Bu_0) - H(Au_1,Bu_0),\ J_q(\eta(u_0,u_1))\rangle \\ &\quad - \langle H(Au_1,Bu_0) - H(Au_1,Bu_1),\ J_q(\eta(u_0,u_1))\rangle \\ &\leq -\mu\|Au_0 - Au_1\|^q + \gamma\|Bu_0 - Bu_1\|^q \\ &\leq -\mu\alpha^q\|u_0 - u_1\|^q + \gamma\beta^q\|u_0 - u_1\|^q \\ &= -(\mu\alpha^q - \gamma\beta^q)\|u_0 - u_1\|^q \leq 0, \end{aligned}$$

which gives $u_1 = u_0$, since $\mu > \gamma$ and $\alpha > \beta$. By (3.2), we have $x_1 = x_0$. Hence $(u_0, x_0) = (u_1, x_1) \in$ Graph$(M)$ and so $x_0 \in Mu_0$. □

**Theorem 3.4.** *Let $H(A,B)$ be $\eta$-cocoercive with respect to $A$ with constant $\mu > 0$ and $\eta$-relaxed cocoercive with respect to $B$ with constant $\gamma > 0$, $A$ is $\alpha$-expansive and $B$ is $\beta$-Lipschitz continuous, $\mu > \gamma$ and $\alpha > \beta$. Let $M$ be $H(\cdot,\cdot)$-$\eta$-cocoercive operator with respect to $A$ and $B$. Then the operator $(H(A,B) + \lambda M)^{-1}$ is single-valued.*

*Proof.* For any $u \in X$, let $x, y \in (H(A,B)+\lambda M)^{-1}(u)$. It follows that

$$-H(Ax,Bx)+u \in \lambda Mx$$

and

$$-H(Ay,By)+u \in \lambda My$$

as $M$ is $\eta$-cocoercive (thus $\eta$-accretive), we have

$$\begin{aligned}0 &\leq \langle -H(Ax,Bx)+u-(-H(Ay,By)+u),\ J_q(\eta(x,y))\rangle \\ &= -\langle H(Ax,Bx)-H(Ay,By),\ J_q(\eta(x,y))\rangle \\ &= -\langle H(Ax,Bx)-H(Ay,Bx)+H(Ay,Bx)-H(Ay,By),\ J_q(\eta(x,y))\rangle \\ &= -\langle H(Ax,Bx)-H(Ay,Bx),\ J_q(\eta(x,y))\rangle \\ &\quad -\langle H(Ay,Bx)-H(Ay,By),\ J_q(\eta(x,y))\rangle. \end{aligned} \tag{3.3}$$

Since $H$ is $\eta$-cocoercive with respect to $A$ with constant $\mu$ and $\eta$-relaxed cocoercive with respect to $B$ with constant $\gamma$, $A$ is $\alpha$-expansive and $B$ is $\beta$-Lipschitz continuous, thus (3.3) becomes

$$0 \leq -\mu\alpha^q\|x-y\|^q+\gamma\beta^q\|x-y\|^q = -(\mu\alpha^q-\gamma\beta^q)\|x-y\|^q \leq 0, \tag{3.4}$$

since $\mu > \gamma$ and $\alpha > \beta$. Thus, we have $x = y$ and so $(H(A,B)+\lambda M)^{-1}$ is single-valued. □

**Definition 3.5.** Let $H(A,B)$ be $\eta$-cocoercive with respect to $A$ with constant $\mu > 0$ and $\eta$-relaxed cocoercive with respect to $B$ with constant $\gamma > 0$, $A$ is $\alpha$-expansive and $B$ is $\beta$-Lipschitz continuous, $\mu > \gamma$ and $\alpha > \beta$. Let $M$ be $H(\cdot,\cdot)$-$\eta$-cocoercive operator with respect to $A$ and $B$. The resolvent operator $R^{H(\cdot,\cdot)-\eta}_{\lambda,M} : X \to X$ is defined by

$$R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(u) = (H(A,B)+\lambda M)^{-1}(u),\ \forall\ u \in X. \tag{3.5}$$

The following theorem shows that the resolvent operator is Lipschitz continuous.

**Theorem 3.6.** *Let $H(A,B)$ be $\eta$-cocoercive with respect to $A$ with constant $\mu > 0$ and $\eta$-relaxed cocoercive with respect to $B$ with constant $\gamma > 0$, $A$ is $\alpha$-expansive, $B$ is $\beta$-Lipschitz continuous and $\eta$ is $\tau$-Lipschitz continuous and $\mu > \gamma$, $\alpha > \beta$. Let $M$ be an $H(\cdot,\cdot)$-$\eta$-cocoercive operator with respect to $A$ and $B$. Then the resolvent operator $R^{H(\cdot,\cdot)-\eta}_{\lambda,M} : X \to X$ is $\dfrac{\tau^{q-1}}{\mu\alpha^q-\gamma\beta^q}$-Lipschitz continuous, that is*

$$\|R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(u) - R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(v)\| \leq \frac{\tau^{q-1}}{\mu\alpha^q-\gamma\beta^q}\|u-v\|,\quad \forall\ u,v \in X.$$

*Proof.* Let $u$ and $v$ be any given points in $X$. It follows from (3.5) that

$$R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(u) = (H(A,B)+\lambda M)^{-1}(u),$$

and

$$R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(v) = (H(A,B)+\lambda M)^{-1}(v).$$

This implies that

$$\frac{1}{\lambda}(u - H(A(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(u)),\ B(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(u)))) \in M(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(u)),$$

and

$$\frac{1}{\lambda}(v - H(A(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(v)),\ B(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(v)))) \in M(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(v)).$$

For the sake of clarity, we take

$$Pu = R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(u)\ \text{ and }\ Pv = R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(v).$$

Since $M$ is $\eta$-cocoercive (thus $\eta$-accretive), we have

$$\begin{aligned}&\frac{1}{\lambda}\langle u-H(A(Pu),B(Pu))-(v-H(A(Pv),B(Pv))),\ J_q(\eta(Pu,Pv))\rangle\\&=\frac{1}{\lambda}\langle u-v-(H(A(Pu),B(Pu))-H(A(Pv),B(Pv))),\ J_q(\eta(Pu,Pv))\rangle\geq 0.\end{aligned}$$

Therefore we have

$$\langle u-v,\ J_q(\eta(Pu,Pv))\rangle\geq\langle H(A(Pu),B(Pu))-H(A(Pv),B(Pv)),\ J_q(\eta(Pu,Pv))\rangle.$$

It follows that

$$\begin{aligned}\|u-v\|\|\eta(Pu,Pv)\|^{q-1}&\geq\langle u-v,\ J_q(\eta(Pu,Pv))\rangle\\&\geq\langle H(A(Pu),B(Pu))-H(A(Pv),B(Pu)),\ J_q(\eta(Pu,Pv))\rangle\\&\quad+\langle H(A(Pv),B(Pu))-H(A(Pv),B(Pv)),\ J_q(\eta(Pu,Pv))\rangle\\&\geq\mu\|A(Pu)-A(Pv)\|^q-\gamma\|B(Pu)-B(Pv)\|^q\\&\geq\mu\alpha^q\|Pu-Pv\|^q-\gamma\beta^q\|Pu-Pv\|^q\end{aligned}$$

and so

$$\|u-v\|\|\eta(Pu,Pv)\|^{q-1}\geq(\mu\alpha^q-\gamma\beta^q)\|Pu-Pv\|^q$$

or

$$\begin{aligned}(\mu\alpha^q-\gamma\beta^q)\|Pu-Pv\|^q&\leq\|u-v\|\|\eta(Pu,Pv)\|^{q-1}\\&\leq\|u-v\|\tau^{q-1}\|Pu-Pv\|^{q-1}\end{aligned}$$

$$\|Pu-Pv\|\leq\frac{\tau^{q-1}}{\mu\alpha^q-\gamma\beta^q}\|u-v\|,\quad\forall\, u,v\in X,$$

or

$$\|R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u)-R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v)\|\leq\frac{\tau^{q-1}}{\mu\alpha^q-\gamma\beta^q}\|u-v\|,\quad\forall\, u,v\in X.$$

This completes the proof. □

## 4. An application for solving variational-like inclusions

In this section, we shall show that under suitable assumption, $H(\cdot,\cdot)$-$\eta$-cocoercive operator plays an important role for solving a variational-like inclusion problem.

Let $\eta,N,W:X\times X\to X$, $g:X\to X$, $H:X\times X\to X$, $A,B:X\to X$ be the single-valued mappings and $T,Q,R,S:X\to CB(X)$, $M:X\to 2^X$ be the set-valued mappings such that $M$ is $H(\cdot,\cdot)$-$\eta$-cocoercive with respect to $A$ and $B$. Then we consider the following problem.

Find $u\in X$, $x\in T(u)$, $y\in Q(u)$, $z\in R(u)$, $v\in S(u)$ such that

$$0\in N(x,y)-W(z,v)+M(g(u)).\tag{4.1}$$

Problem (4.1) is called variational-like inclusion problem.
Below are some special cases of problem (4.1).

(i) If $W,R,S=0$, then problem (4.1) reduces to the problem of finding $u\in X$, $x\in T(u)$, $y\in Q(u)$ such that

$$0\in N(x,y)+M(g(u)).\tag{4.2}$$

Problem (4.2) is introduced and studied by Peng [10].

(ii) If $N(x,y) = N(x)$, $T$ is single-valued and $g = I$, the identity mapping, then problem (4.2) reduces to the problem considered by Bi et al.[3]. That is to find $u \in X$ such that

$$0 \in N(u) + M(u). \tag{4.3}$$

**Lemma 4.1.** *$(u, x, y, z, v)$, where $u \in X$, $x \in T(u)$, $y \in Q(u)$, $z \in R(u)$, $v \in S(u)$ is a solution of problem (4.1) if and only if $(u, x, y, z, v)$ is the solution of the following equation.*

$$g(u) = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}[H(A(gu), B(gu)) - \lambda\{N(x,y) - W(z,v)\}], \tag{4.4}$$

*where $\lambda > 0$ is a constant.*

*Proof.* Proof is direct consequence of definition of resolvent operator. □

Based on (4.4), we have the following iterative algorithm.

**Algorithm 4.2.** *For any given $u_0 \in X$, $x_0 \in T(u_0)$, $y_0 \in Q(u_0)$, $z_0 \in R(u_0)$, $v_0 \in S(u_0)$, compute the sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ by the following iterative procedure.*

$$g(u_{n+1}) = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}[H(A(gu_n), B(g(u_n)) - \lambda\{N(x_n, y_n) - W(z_n, u_n)\}], \tag{4.5}$$

$$\|x_{n+1} - x_n\| \leq \mathcal{D}(T(u_{n+1}), T(u_n)); \tag{4.6}$$

$$\|y_{n+1} - y_n\| \leq \mathcal{D}(Q(u_{n+1}), Q(u_n)); \tag{4.7}$$

$$\|z_{n+1} - z_n\| \leq \mathcal{D}(R(u_{n+1}), R(u_n)); \tag{4.8}$$

$$\|v_{n+1} - v_n\| \leq \mathcal{D}(S(u_{n+1}), S(u_n)); \tag{4.9}$$

*where $n$ is the iteration number, $\lambda > 0$ is a constant and $\mathcal{D}$ is the Hausdörff metric on $CB(X)$.*

**Theorem 4.3.** *Let $X$ be a $q$-uniformly smooth Banach space. Let $A, B, g : X \to X$, $H, N, \eta, W : X \times X \to X$ be the single-valued mappings. Let $T, Q, R, S : X \to CB(X)$ and $M : X \to 2^X$ be the multi-valued mappings such that $M$ is $H(\cdot,\cdot)$-$\eta$-cocoercive mapping with respect to $A$ and $B$. Suppose that*

*(i) $g$ is $\delta$-strongly accretive and $\lambda_g$-Lipschitz continuous,*

*(ii) $N$ is Lipschitz continuous for the first argument with constant $\lambda_{N_1}$ and $\lambda_{N_2}$ for the second argument, $\eta$-strongly accretive with respect to $T$ and $Q$ with constant $t$,*

*(iii) $W$ is Lipschitz continuous for the first argument with constant $\lambda_{W_1}$ and $\lambda_{W_2}$ for the second argument,*

*(iv) $\eta$ is $\tau$-Lipschitz continuous, $A$ is $\alpha$-expansive and $B$ is $\beta$-Lipschitz continuous,*

*(v) $H(A, B)$ is $\eta$-cocoercive with respect to $A$ with constant $\mu$ and $\eta$-relaxed-cocoercive with respect to $B$ with constant $\gamma$, $r_1$-Lipschitz continuous with respect to $A$ and $r_2$-Lipschitz continuous with respect to $B$,*

*(vi) $T, Q, R, S$ are $\mathcal{D}$-Lipschitz continuous mappings with constants $\lambda_T, \lambda_Q, \lambda_R$ and $\lambda_S$, respectively.*

*Suppose that the following condition is satisfied:*

$$\Big[\sqrt[q]{(r_1+r_2)^q\lambda_g^q - q\lambda t + q\lambda(\lambda_{N_1}\lambda_T + \lambda_{N_2}\lambda_Q)[(r_1+r_2)^{q-1}\lambda_g^{q-1} + \tau^{q-1}]}\Big] + \lambda^q C_q(\lambda_{N_1}\lambda_T + \lambda_{N_2}\lambda_Q)^q$$
$$< \big[\frac{\delta}{\tau^q}(\mu\alpha^q - \gamma\beta^q) - \lambda(\lambda_{W_1}\lambda_R + \lambda_{W_2}\lambda_S)\big], \tag{4.10}$$

$$\frac{\delta}{\tau^q}(\mu\alpha^q - \gamma\beta^q) > \lambda(\lambda_{W_1}\lambda_R + \lambda_{W_2}\lambda_S),\ \mu > \gamma,\ \alpha > \beta.$$

*Then there exist $u \in X$, $x \in T(u)$, $y \in Q(u)$, $z \in R(u)$ and $v \in S(u)$ satisfying the variational-like inclusion problem (4.1) and the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ generated by Algorithm 4.1 converge strongly to $u$, $x$, $y$, $z$, and $v$, respectively.*

*Proof.* Since $g$ is $\delta$-strongly accretive, we have

$$\begin{aligned}\|g(u_{n+1})-g(u_n)\|\|u_{n+1}-u_n\|^{q-1} &= \|g(u_{n+1})-g(u_n)\|\|J_q(u_{n+1}-u_n)\| \\ &\geq \langle g(u_{n+1})-g(u_n),\ J_q(u_{n+1}-u_n)\rangle \\ &\geq \delta\|u_{n+1}-u_n\|^q. \end{aligned} \tag{4.11}$$

From (4.11), we get

$$\|u_{n+1}-u_n\| \leq \frac{1}{\delta}\|g(u_{n+1})-g(u_n)\|. \tag{4.12}$$

By Algorithm 4.2 and Theorem 3.6, we have

$$\begin{aligned}\|g(u_{n+1})-g(u_n)\| &= \|R^{H(\cdot,\cdot)-\eta}_{\lambda,M}[H(A(gu_n),B(gu_n))-\lambda\{N(x_n,y_n)-W(z_n,v_n)\}] \\ &\quad - R^{H(\cdot,\cdot)-\eta}_{\lambda,M}[H(A(gu_{n-1}),B(gu_{n-1})) \\ &\quad - \lambda\{N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1})\}]\| \\ &\leq \frac{\tau^{q-1}}{\mu\alpha^q-\gamma\beta^q}\|H(A(gu_n),B(gu_n))-H(A(gu_{n-1}),B(gu_{n-1})) \\ &\quad - \lambda\{N(x_n,y_n)-N(x_{n-1},y_{n-1})\} \\ &\quad - \lambda\{W(z_n,v_n)-W(z_{n-1},v_{n-1})\}\| \\ &\leq \frac{\tau^{q-1}}{\mu\alpha^q-\gamma\beta^q}\|H(A(gu_n),B(gu_n))-H(A(gu_{n-1}),B(gu_{n-1})) \\ &\quad - \lambda\{N(x_n,y_n)-N(x_{n-1},y_{n-1})\}\| \\ &\quad + \frac{\tau^{q-1}\lambda}{\mu\alpha^q-\gamma\beta^q}\|W(z_n,v_n)-W(z_{n-1},v_{n-1})\|. \end{aligned} \tag{4.13}$$

Using Lipschitz continuity of $N$ with constant $\lambda_{N_1}$ for the first argument and $\lambda_{N_2}$ for the second argument and $\mathcal{D}$-Lipschitz continuity of $T$ and $Q$ with constants $\lambda_T$ and $\lambda_Q$, respectively, we have

$$\begin{aligned}\|N(x_n,y_n)-N(x_{n-1},y_{n-1})\| &= \|N(x_n,y_n)-N(x_{n-1},y_n) \\ &\quad + N(x_{n-1},y_n)-N(x_{n-1},y_{n-1})\| \\ &\leq \|N(x_n,y_n)-N(x_{n-1},y_n)\| \\ &\quad + \|N(x_{n-1},y_n)-N(x_{n-1},y_{n-1})\| \\ &\leq \lambda_{N_1}\|x_n-x_{n-1}\|+\lambda_{N_2}\|y_n-y_{n-1}\| \\ &\leq \lambda_{N_1}\mathcal{D}(T(u_n),T(u_{n-1}))+\lambda_{N_2}\mathcal{D}(Q(u_n),Q(u_{n-1})) \\ &\leq \lambda_{N_1}\lambda_T\|u_n-u_{n-1}\|+\lambda_{N_2}\lambda_Q\|u_n-u_{n-1}\| \\ &= (\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_Q)\|u_n-u_{n-1}\|. \end{aligned} \tag{4.14}$$

Also as $H(A,B)$ is $r_1$-Lipschitz continuous with respect to $A$ and $r_2$-Lipschitz continuous with respect to $B$ and $g$ is $\lambda_g$-Lipschitz continuous, we have

$$\|H(A(gu_n),B(gu_n))-H(A(gu_{n-1}),B(gu_{n-1}))\| \leq (r_1+r_2)\lambda_g\|u_n-u_{n-1}\|. \tag{4.15}$$

By using Lemma 2.1, (4.14), (4.15) and $\eta$-strongly accretiveness of $N$ with respect to $T$ and $Q$ with constant $t$ and $\tau$-Lipschitz continuity of $\eta$, we have

$$
\begin{aligned}
&\|H(A(gu_n),B(gu_n))-H(A(gu_{n-1}),B(gu_{n-1}))-\lambda\{N(x_n,y_n)-N(x_{n-1},y_{n-1})\}\|^q\\
&\le \|H(A(gu_n),B(gu_n))-H(A(gu_{n-1}),B(g(u_{n-1}))\|^q\\
&\quad -q\lambda\langle N(x_n,y_n)-N(x_{n-1},y_{n-1}),\ J_q(\eta(u_n,u_{n-1}))\rangle\\
&\quad -q\lambda\langle N(x_n,y_n)-N(x_{n-1},y_{n-1}),\ J_q[H(A(gu_n),B(gu_n))-H(A(gu_{n-1}),B(gu_{n-1}))]\\
&\quad -J_q(\eta(u_n,u_{n-1}))\rangle+\lambda^qC_q\|N(x_n,y_n)-N(x_{n-1},y_{n-1})\|^q\\
&\le (r_1+r_2)^q\lambda_g^q\|u_n-u_{n-1}\|^q-q\lambda t\|u_n-u_{n-1}\|^q+q\lambda\|N(x_n,y_n)-N(x_{n-1},y_{n-1})\|\times\\
&\big[\|H(A(gu_n)-,B(gu_n))-H(A(gu_{n-1}),B(gu_{n-1}))\|^{q-1}+\|\eta(u_n,u_{n-1})\|^{q-1}\big]\\
&\quad +\lambda^qC_q(\lambda_{N_1}\lambda_T+\lambda_{N_2})^q\|u_n-u_{n-1}\|^q\\
&\le (r_1+r_2)^q\lambda_g^q\|u_n-u_{n-1}\|^q-q\lambda t\|u_n-u_{n-1}\|^q+q\lambda(\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_Q)\|u_n-u_{n-1}\|\times\\
&\quad \big[(r_1+r_2)^{q-1}\lambda_g^{q-1}\|u_n-u_{n-1}\|^{q-1}+\tau^{q-1}\|u_n-u_{n-1}\|^{q-1}\big]\\
&\quad +\lambda^qC_q(\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_Q)^q\|u_n-u_{n-1}\|^q\\
&=\big[(r_1+r_2)^q\lambda_g^q-q\lambda t+q\lambda(\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_Q)[(r_1+r_2)^{q-1}\lambda_g^{q-1}+\tau^{q-1}]\\
&\quad +\lambda^qC_q(\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_Q)^q\big]\|u_n-u_{n-1}\|^q.
\end{aligned}
\tag{4.16}
$$

Using Lipschitz continuity of $W$ with constant $\lambda_{W_1}$ for the first argument and $\lambda_{W_2}$ for the second argument and $\mathcal{D}$-Lipschitz continuity of $R$ and $S$ with constants $\lambda_R$ and $\lambda_S$, respectively, we obtain

$$\|W(z_n,v_n)-W(z_{n-1},v_{n-1})\|\le(\lambda_{W_1}\lambda_R+\lambda_{W_2}\lambda_S)\|u_n-u_{n-1}\|. \tag{4.17}$$

In view of (4.16) and (4.17), (4.13) becomes

$$
\begin{aligned}
\|g(u_n)-g(u_{n-1})\|\le\Big[&\frac{\tau^{q-1}}{\mu\alpha^q-\gamma\beta^q}\Big(\sqrt[q]{(r_1+r_2)^q\lambda_g^q-q\lambda t+q\lambda(\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_Q)}\\
&\overline{[(r_1+r_2)^{q-1}\lambda_g^{q-1}+\tau^{q-1}]+\lambda^qC_q(\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_Q)^q}\Big)\\
&+\frac{\tau^{q-1}\lambda}{\mu\alpha^q-\gamma\beta^q}(\lambda_{W_1}\lambda_R+\lambda_{W_2}\lambda_S)\Big]\|u_n-u_{n-1}\|.
\end{aligned}
\tag{4.18}
$$

Using (4.18), (4.12) becomes

$$\|u_{n+1}-u_n\|\le\theta\|u_n-u_{n-1}\|. \tag{4.19}$$

where

$$
\begin{aligned}
\theta=\frac{1}{\delta}\Big[&\frac{\tau^{q-1}}{\mu\alpha^q-\gamma\beta^q}\Big(\sqrt[q]{(r_1+r_2)^q\lambda_g^q-q\lambda t+q\lambda(\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_Q)}\\
&\overline{[(r_1+r_2)^{q-1}\lambda_g^{q-1}+\tau^{q-1}]+\lambda^qC_q(\lambda_{N_1}\lambda_T+\lambda_{N_2}\lambda_Q)^q}\Big)\\
&+\frac{\tau^{q-1}\lambda}{\mu\alpha^q-\gamma\beta^q}(\lambda_{W_1}\lambda_R+\lambda_{W_2}\lambda_S)\Big]\|u_n-u_{n-1}\|.
\end{aligned}
$$

By condition (4.10), $0<\theta<1$ and hence $\{u_n\}$ is Cauchy sequence in $X$, so there exists $u\in X$ such that $u_n\to u$ as $n\to\infty$. By the $\mathcal{D}$-Lipschitz continuity of $T,Q,R$ and $S$, we have

$$\|x_{n+1}-x_n\|\le\lambda_T\|u_{n+1}-u_n\|;$$

$$\|y_{n+1}-y_n\|\le\lambda_Q\|u_{n+1}-u_n\|;$$

$$\|z_{n+1}-z_n\|\le\lambda_R\|u_{n+1}-u_n\|;$$

$$\|v_{n+1}-n_n\|\le\lambda_S\|u_{n+1}-u_n\|;$$

which shows that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ are all Cauchy sequences in $X$, so there exist $x$, $y$, $z$, and $v \in X$ such that $x_n \to x$, $y_n \to y$, $z_n \to z$ and $v_n \to v$, as $n \to \infty$. By the Lipschitz continuity of the operations $H$, $A$, $B$, $N$, $W$, $R_{\lambda,M}^{H(\cdot,\cdot)-\eta}$ and Algorithm 4.2, it follows that

$$g(u) = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}[H(A(gu), B(gu)) - \lambda\{N(x,y) - W(z,v)\}].$$

It remain to show that $x \in T(u)$. In fact, since $x_n \in T(u_n)$. we have

$$\begin{aligned} d(x, T(u)) &\leq \|x - x_n\| + d(x_n, T(u)) \\ &\leq \|x - x_n\| + \mathcal{D}(T(u_n), T(u)) \\ &\leq \|x - x_n\| + \lambda_T \|u_n - u\| \to 0, \quad \text{as } \ n \to \infty, \end{aligned}$$

which implies that $d(x, T(u)) = 0$, since $T(u) \in CB(X)$, it follows that $x \in T(u)$. Similarly, we can show that $y \in Q(u)$, $z \in R(u)$ and $v \in S(u)$. This completes the proof. □

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