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Jensen type inequalities for twice differentiable functions

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Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

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Abstract

In this paper, we give some Jensen-type inequalities for $\varphi : I \to \mathbb{R}$, $I = [\alpha, \beta] \subset \mathbb{R}$, where φ is a continuous function on I, twice differentiable on $\mathring{I} = (\alpha, \beta)$ and there exists $m = \inf\limits_{x \in \mathring{I}} \varphi''(x)$ or $M = \sup\limits_{x \in \mathring{I}} \varphi''(x)$. Furthermore, if φ'' is bounded on $\mathring{I}$, then we give an estimate, from below and from above of Jensen inequalities.

Keywords: Jensen inequality, Convex functions, Twice differentiable functions.
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1. Introduction and main results

Throughout this note, we write I and $\mathring{I}$ for the intervals $[\alpha, \beta]$ and (α, β) respectively $-\infty \leq \alpha < \beta \leq +\infty$. A function φ is said to be convex on I if $\lambda\varphi(x) + (1-\lambda)\varphi(y) \geq \varphi(\lambda x + (1-\lambda)y)$ for all $x, y \in I$ and $0 \leq \lambda \leq 1$. Conversely, if the inequality always holds in the opposite direction, the function is said to be concave on the interval. A function φ that is continuous function on I and twice differentiable on $\mathring{I}$ is convex on I if $\varphi''(x) \geq 0$ for all $x \in \mathring{I}$ (concave if the inequality is flipped).

The famous inequality of Jensen states that:

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Theorem 1.1. *([1], [3]) Let φ be a convex function on the interval $I \subset \mathbb{R}$, $x = (x_1, x_2, \cdots, x_n) \in I^n$ $(n \geq 2)$, let $p_i \geq 0$, $i = 1, 2, \cdots, n$ and $P_n = \sum_{i=1}^{n} p_i$. Then*

$$\varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi\left(x_i\right). \tag{1.1}$$

If φ is strictly convex, then inequality in (1.1) is strict except when $x_1 = x_2 = \cdots = x_n$. If φ is a concave function, then inequality in (1.1) is reverse.

Theorem 1.2. *[3] Let φ be a convex function on $I \subset \mathbb{R}$, and let $f : [0,1] \longrightarrow I$ be a continuous function on $[0,1]$. Then*

$$\varphi\left(\int_0^1 f\left(x\right) dx\right) \leq \int_0^1 \varphi\left(f\left(x\right)\right) dx. \tag{1.2}$$

If φ is strictly convex, then inequality in (1.2) is strict. If φ is a concave function, then inequality in (1.2) is reverse.

In [2], Malamud gave some complements to the Jensen and Chebyshev inequalities and in [4], Saluja gave some necessary and sufficient conditions for three-step iterative sequence with errors for asymptotically quasi-nonexpansive type mapping converging to a fixed point in convex metric spaces. In this paper, we give some inequalities of the above type for $\varphi : I \to \mathbb{R}$ such that φ is a continuous on I, twice differentiable on $\mathring{I}$ and there exists $m = \inf_{x\in\mathring{I}} \varphi''\left(x\right)$ or $M = \sup_{x\in\mathring{I}} \varphi''\left(x\right)$. We obtain the following results:

Theorem 1.3. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous function on I, twice differentiable on $\mathring{I}$, $x = (x_1, x_2, \cdots, x_n) \in I^n$ $(n \geq 2)$, let $p_i \geq 0$, $i = 1, 2, \cdots, n$ and $P_n = \sum_{i=1}^{n} p_i$.*
(i) If there exists $m = \inf_{x\in\mathring{I}} \varphi''\left(x\right)$, then

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi\left(x_i\right) - \varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)$$
$$\geq \frac{m}{2}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^2\right). \tag{1.3}$$

(ii) If there exists $M = \sup_{x\in\mathring{I}} \varphi''\left(x\right)$, then

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi\left(x_i\right) - \varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)$$
$$\leq \frac{M}{2}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^2\right). \tag{1.4}$$

Equality in (1.3) and (1.4) hold if $x_1 = x_2 = \cdots = x_n$ or if $\varphi\left(x\right) = \alpha x^2 + \beta x + \gamma$, $\alpha, \beta, \gamma \in \mathbb{R}$.

Theorem 1.4. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous function on I, twice differentiable on $\mathring{I}$. Suppose that $f : [a,b] \longrightarrow I$ and $p : [a,b] \longrightarrow \mathbb{R}^+$ are continuous functions on $[a,b]$.*
(i) If there exists $m = \inf_{x\in\mathring{I}} \varphi''\left(x\right)$, then

$$\frac{\int_a^b p\left(x\right)\varphi\left(f\left(x\right)\right) dx}{\int_a^b p\left(x\right) dx} - \varphi\left(\frac{\int_a^b p\left(x\right) f\left(x\right) dx}{\int_a^b p\left(x\right) dx}\right)$$

$$\geq \frac{m}{2}\left(\frac{\int_a^b p(x)(f(x))^2\,dx}{\int_a^b p(x)\,dx}-\left(\frac{\int_a^b p(x)f(x)\,dx}{\int_a^b p(x)\,dx}\right)^2\right). \tag{1.5}$$

(ii) If there exists $M=\sup\limits_{x\in\mathring{I}}\varphi''(x)$, *then*

$$\frac{\int_a^b p(x)\varphi(f(x))\,dx}{\int_a^b p(x)\,dx}-\varphi\left(\frac{\int_a^b p(x)f(x)\,dx}{\int_a^b p(x)\,dx}\right)$$

$$\leq \frac{M}{2}\left(\frac{\int_a^b p(x)(f(x))^2\,dx}{\int_a^b p(x)\,dx}-\left(\frac{\int_a^b p(x)f(x)\,dx}{\int_a^b p(x)\,dx}\right)^2\right). \tag{1.6}$$

Equality in (1.5) and (1.6) hold if $\varphi(x)=\alpha x^2+\beta x+\gamma,\ \alpha,\beta,\gamma\in\mathbb{R}$.

Corollary 1.5. *Let* $\varphi: I\longrightarrow\mathbb{R}$ *be a continuous function on* I, *twice differentiable on* $\mathring{I}$ *and let* $f:[a,b]\longrightarrow I$ *be a continuous function on* $[a,b]$.
(i) If there exists $m=\inf\limits_{x\in\mathring{I}}\varphi''(x)$, *then*

$$\frac{1}{b-a}\int_a^b\varphi(f(x))\,dx-\varphi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)$$

$$\geq\frac{m}{2}\left(\frac{1}{b-a}\int_a^b(f(x))^2\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2\right). \tag{1.7}$$

(ii) If there exists $M=\sup\limits_{x\in\mathring{I}}\varphi''(x)$, *then*

$$\frac{1}{b-a}\int_a^b\varphi(f(x))\,dx-\varphi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)$$

$$\leq\frac{M}{2}\left(\frac{1}{b-a}\int_a^b(f(x))^2\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2\right). \tag{1.8}$$

Equality in (1.7) and (1.8) hold if $\varphi(x)=\alpha x^2+\beta x+\gamma,\ \alpha,\beta,\gamma\in\mathbb{R}$.

Corollary 1.6. *Let* $\varphi: I\longrightarrow\mathbb{R}$ *be a continuous function on* I, *twice differentiable on* $\mathring{I}$, $x=(x_1,x_2,\cdots,x_n)\in I^n$ $(n\geq 2)$, *let* $p_i\geq 0$, $i=1,2,\cdots,n$ *and* $P_n=\sum_{i=1}^n p_i$. *If there exist* $m=\inf\limits_{x\in\mathring{I}}\varphi''(x)$ *and* $M=\sup\limits_{x\in\mathring{I}}\varphi''(x)$, *then we have*

$$\frac{m}{2}\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i^2-\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)^2\right)$$

$$\leq\frac{1}{P_n}\sum_{i=1}^n p_i\varphi(x_i)-\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)$$

$$\leq\frac{M}{2}\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i^2-\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)^2\right). \tag{1.9}$$

Equality in (1.9) occurs, if $\varphi(x)=\alpha x^2+\beta x+\gamma,\ \alpha,\beta,\gamma\in\mathbb{R}$.

Corollary 1.7. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous function on I, twice differentiable on $\mathring{I}$, and let $f : [0,1] \longrightarrow I$ be a continuous function on $[0,1]$. If there exist $m = \inf_{x\in\mathring{I}} \varphi''(x)$ and $M = \sup_{x\in\mathring{I}} \varphi''(x)$, then we have*

$$\frac{m}{2}\left(\int_0^1 (f(x))^2\,dx - \left(\int_0^1 f(x)\,dx\right)^2\right)$$

$$\leq \int_0^1 \varphi(f(x))\,dx - \varphi\left(\int_0^1 f(x)\,dx\right)$$

$$\leq \frac{M}{2}\left(\int_0^1 (f(x))^2\,dx - \left(\int_0^1 f(x)\,dx\right)^2\right). \tag{1.10}$$

Equality in (1.10) holds if $\varphi(x) = \alpha x^2 + \beta x + \gamma \quad \alpha, \beta, \gamma \in \mathbb{R}$.

Corollary 1.8. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a convex function on I, twice differentiable on $\mathring{I}$, and let $f : [a,b] \longrightarrow I$ be a continuous function on $[a,b]$. If there exists $m = \inf_{x\in\mathring{I}} \varphi''(x)$, then*

$$\frac{1}{b-a}\int_a^b \varphi(f(x))\,dx - \varphi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)$$

$$\geq \frac{m}{2}\left(\frac{1}{b-a}\int_a^b (f(x))^2\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2\right) \geq 0 \tag{1.11}$$

and

$$\frac{1}{P_n}\sum_{i=1}^n p_i\varphi(x_i) - \varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)$$

$$\geq \frac{m}{2}\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)^2\right) \geq 0. \tag{1.12}$$

Corollary 1.9. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a concave function on I, twice differentiable on $\mathring{I}$, and let $f : [a,b] \longrightarrow I$ be a continuous function on $[a,b]$. If there exists $M = \sup_{x\in\mathring{I}} \varphi''(x)$, then*

$$\frac{1}{b-a}\int_a^b \varphi(f(x))\,dx - \varphi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)$$

$$\leq \frac{M}{2}\left(\frac{1}{b-a}\int_a^b (f(x))^2\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2\right) \leq 0 \tag{1.13}$$

and

$$\frac{1}{P_n}\sum_{i=1}^n p_i\varphi(x_i) - \varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)$$

$$\leq \frac{M}{2}\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)^2\right) \leq 0. \tag{1.14}$$

Remark 1.10. In the above if $\varphi \in C^2([\alpha,\beta])$, then we can replace inf and sup by min and max respectively.

2. Lemma

Our proofs depend mainly upon the following lemma.

Lemma 2.1. *Let φ be a convex function on $I \subset \mathbb{R}$ and differentiable on $\mathring{I}$. Suppose that $f : [a,b] \longrightarrow I$ and $p : [a,b] \longrightarrow \mathbb{R}^+$ are continuous functions on $[a,b]$. Then*

$$\varphi\left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right) \le \frac{\int_a^b p(x)\,\varphi(f(x))\,dx}{\int_a^b p(x)\,dx}. \tag{2.1}$$

If φ is strictly convex, then inequality in (2.1) is strict. If φ is a concave function, then inequality in (2.1) is reverse.

Proof. Suppose that φ is a convex function on $I \subset \mathbb{R}$ and differentiable on $\mathring{I}$. Then for each $x,\, y \in \mathring{I}$, we have

$$\varphi(x) - \varphi(y) \ge (x-y)\,\varphi'(y). \tag{2.2}$$

Replace x by $f(x)$ and set $y = \frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}$ in (2.2), we obtain

$$\varphi(f(x)) - \varphi\left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right)$$

$$\ge \left(f(x) - \frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right)\varphi'\left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right). \tag{2.3}$$

Multiplying both sides of inequality (2.3) by $p(x)$ we obtain

$$p(x)\,\varphi(f(x)) - p(x)\,\varphi\left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right)$$

$$\ge \left(p(x) f(x) - p(x)\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right)\varphi'\left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right). \tag{2.4}$$

By integration in (2.4) we obtain (2.1). □

3. Proof of the Theorems

Proof of Theorem 1.3. Suppose that $\varphi : I \to \mathbb{R}$ is a continuous function on I and twice differentiable on $\mathring{I}$. Set $g(x) = \varphi(x) - \frac{m}{2}x^2$. Differentiating twice times both sides of g we get $g''(x) = \varphi''(x) - m \ge 0$. Then g is a convex function on I. By formula (1.1), we have

$$g\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i g(x_i) \tag{3.1}$$

which implies that

$$\varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{m}{2}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^2$$

$$\le \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) - \frac{m}{2}\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2. \tag{3.2}$$

Then, by (3.2) we can write

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi(x_i) - \varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \geq \frac{m}{2}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^2\right). \tag{3.3}$$

If we put $g(x) = -\varphi(x) + \frac{M}{2}x^2$, then by differentiating both sides of g we get $g''(x) = -\varphi''(x) + M \geq 0$. Hence g is a convex function on I and by similar proof as above, we obtain

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi(x_i) - \varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{M}{2}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^2\right). \tag{3.4}$$

□

Proof of Theorem 1.4. Suppose that $\varphi : I \to \mathbb{R}$ is a continuous function on I and twice differentiable on $\mathring{I}$. Set $g(x) = \varphi(x) - \frac{m}{2}x^2$. Differentiating both sides of g we get $g''(x) = \varphi''(x) - m \geq 0$. Hence g is a convex function on I and by formula (2.1) we have

$$g\left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right) \leq \frac{\int_a^b p(x)\, g(f(x))\,dx}{\int_a^b p(x)\,dx} \tag{3.5}$$

which implies that

$$\varphi\left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right) - \frac{m}{2}\left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right)^2 \leq \frac{\int_a^b p(x)\,\varphi(f(x))\,dx}{\int_a^b p(x)\,dx} - \frac{m}{2}\frac{\int_a^b p(x)(f(x))^2\,dx}{\int_a^b p(x)\,dx}. \tag{3.6}$$

Then by (3.6), we can write

$$\frac{\int_a^b p(x)\,\varphi(f(x))\,dx}{\int_a^b p(x)\,dx} - \varphi\left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right) \geq \frac{m}{2}\left(\frac{\int_a^b p(x)(f(x))^2\,dx}{\int_a^b p(x)\,dx} - \left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right)^2\right). \tag{3.7}$$

If we put $g(x) = \varphi(x) - \frac{M}{2}x^2$, then by differentiating both sides of g, we get $g''(x) = \varphi''(x) - M \leq 0$. Thus, g is a concave function on I and by a similar proof as above, we obtain

$$\frac{\int_a^b p(x)\,\varphi(f(x))\,dx}{\int_a^b p(x)\,dx} - \varphi\left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right) \leq \frac{M}{2}\left(\frac{\int_a^b p(x)(f(x))^2\,dx}{\int_a^b p(x)\,dx} - \left(\frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}\right)^2\right). \tag{3.8}$$

□

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