



# Uniformly normal structure and uniformly generalized Lipschitzian semigroups

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Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

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## Abstract

In this work, we introduce some condition on one-parameter semigroup of self-mappings it is called  $k$ -uniformly generalized Lipschitzian. The condition is weaker than Lipschitzian type conditions. Also, we show that a  $k$ -generalized Lipschitzian semigroup of nonlinear self-mappings of a nonempty closed convex subset  $C$  of real Banach space  $X$  admits a common fixed point if the semigroup has a bounded orbit and if  $k > 0$ . Our results extending the results due to L.C. Ceng, H. K. Xu and J.C. Yao [5]

*Keywords:* Uniformly normal structure, Uniformly generalized semigroup, Fixed point, Characteristic of convexity, Modulus of convexity.

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## 1. Introduction

Assume that  $X$  is a real Banach space with uniformly normal structure and  $C$  is a nonempty closed convex subset of  $X$ . A mapping  $T : C \rightarrow C$  is said to be a Lipschitzian mapping if, for each integer  $n \geq 1$ , there exist a constant  $k_n > 0$  such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y \in C.$$

A Lipschitzian mapping is said to be a  $k$ -uniformly Lipschitzian mapping if  $k_n \equiv k$  for all  $n \geq 1$ . These mappings were first studied by Goebel and Kirk [8]. They studied the existence of a fixed point of a uniformly  $k$ -Lipschitzian mapping  $T$  defined on a bounded closed convex subset of a uniformly convex Banach space

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$X$ . They showed that such mapping  $T$  has a fixed point if  $k < \gamma$  where  $\gamma > 1$  is the unique solution of the equation

$$(1 - \delta_X(1/\gamma))\gamma = 1, \quad (1.1)$$

$\delta_X$  being the modulus of convexity of  $X$ .

In 1973, Goebel and Kirk [8] then posed the question whether or not the constant  $\gamma > 1$  which solves the equation 1.1 is the largest number for which any  $k$ -uniformly Lipschitzian mapping  $T$  with  $k < \gamma$  has a fixed point.

In 1975, Lifschits [12] proved that in Hilbert space a  $k$ -uniformly Lipschitzian mapping with  $k < \sqrt{2}$  has a fixed point.

Casini and Maluta [4] and Ishihara and Takahashi [10] proved that a uniformly  $k$ -Lipschitzian semigroup in Banach space  $X$  has a common fixed point if  $k < \sqrt{N(X)}$  where  $N(X)$  is the uniformly normal structure coefficient.

Since then,  $k$ -uniformly Lipschitzian mapping have extensively been investigated by many authors. Moreover, some of results for uniformly Lipschitzian mapping have been extended to uniformly Lipschitzian semigroups, and even more general, to Lipschitzian semigroup; see [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26].

Particularly, in 1993, Tan and Xu [16] answered the question of Goebel and Kirk [8] mentioned above in the negative by proving the following:

**Theorem 1.1.** ([16], Theorem 3.5) *Let  $X$  be a real uniformly convex Banach space,  $C$  a nonempty closed convex subset of  $X$ , and  $\tau = \{T_s : s \in G\}$  a  $k$ -uniformly Lipschitzian semigroup on  $C$  with  $k < \alpha$ , where  $\alpha > 1$  is the unique solution of the equation*

$$\frac{\alpha^2}{N(X)} \delta_X^{-1}\left(1 - \frac{1}{\alpha}\right) = 1 \quad (1.2)$$

where  $N(X) > 1$  is the normal structure coefficient of  $X$ . Suppose there exists an  $x_0 \in C$  such that the orbit  $\{T_s x_0 : s \in G\}$  is bounded. then there exists  $z \in C$  such that  $T_s z = z$  for all  $s \in G$ .

*Remark 1.2.* One can prove that  $\gamma < \alpha$ , where  $\gamma$  and  $\alpha$  are the solution of equations 1.1 and 1.2, respectively. Consequently, the constant  $\gamma$  solving equation 1.1 is not the biggest number for which every  $k$ -uniformly Lipschitzian mapping  $T$  with  $k < \gamma$  has a fixed point. Indeed, the best possible number  $\gamma$  is still unknown, even in the setting of Hilbert spaces. It is therefore an interesting question to find another constant  $\alpha^*$  which is strictly bigger than  $\alpha$  and for which every  $k$ -uniformly Lipschitzian mapping  $T$  with  $k < \alpha^*$  has a fixed point.

Some years later, Zeng and Yang [23] proved a fixed point result for Lipschitzian semigroups as follows:

**Theorem 1.3.** ([23], theorem 3.1) *Let  $C$  be a nonempty bounded subset of a uniformly convex Banach Space  $X$ , and let  $\tau = \{T_s : s \in G\}$  be a  $k$ -uniformly Lipschitzian semigroup on  $C$  with*

$$\liminf_s |||T_s||| < \sqrt{\gamma_0 N(X)},$$

where

$$\gamma_0 = \inf\{\gamma : \gamma(1 - \delta_X(1/\gamma)) \geq 1/2\},$$

and  $|||T_s|||$  is the exact Lipschitzian constant of  $T_s$ . Suppose also there exists a nonempty bounded closed convex subset  $E$  of  $C$  with the following properties:

(P1)  $x \in E$  implies  $w_w(x) \subset E$ ; where  $w_w(x)$  is the weak  $w$ -limit set of  $\tau$  at  $x$ , i.e.,

$$w_w(x) = \{y \in X : y = \text{weak} - \lim_{t_\alpha} T_{t_\alpha} x \text{ for some subnet } \{t_\alpha\} \subset G\}.$$

(P2)  $\tau$  is asymptotically regular on  $E$ ; i.e.,  $\lim_t ||T_{t+s}x - T_t x|| = 0, \forall s \in G, x \in E$ .

Then there exists  $z \in C$  such that  $T_s z = z$  for all  $s \in G$ .

Recently, Ceng, Xu and Yao [5], studied the existence of fixed points of uniformly Lipschitzian semigroups  $\tau = \{T_s : s \in G\}$  in the setting of Banach spaces  $X$  under conditions weaker than uniform convexity. More precisely, their contributions were two fold: 1.1 they replaced the uniform convexity of  $X$  in Theorem 1.1 with the weaker condition of the uniformly normal structure of  $X$ ; and 1.2 they removed the asymptotic regularity on  $E$  of the semigroup  $\tau = \{T_s : s \in G\}$  in Theorem 1.2.

In this paper, we introduce a new  $k$ -generalized Lipschitzian one parameter semigroup of self mappings and by this condition we generalize the results due to Ceng, Xu and Yao [5].

## 2. Preliminaries

Let  $C$  be a closed convex subset of a Banach space  $X$ . Then the collection  $\tau = \{T_s : s \in G\}$  of mappings of  $C$  into itself is said to be generalized Lipschitzian semigroup on  $C$  if the following conditions are satisfied:

- (i)  $T_{st}x = T_s T_t x$  for all  $s, t \in G$  and  $x \in C$ ;
- (ii) for each  $x \in C$ , the mapping  $t \rightarrow T_t x$  from  $G$  into  $C$  is continuous;
- (iii) for each  $t \in G, T_t : X \rightarrow X$  is continuous.
- (iv) for each  $t \in G$ , there exists a constant  $k_t > 0$  such that

$$||T_t x - T_t y|| \leq k_t \max\{||x - y||, \frac{1}{2\omega} ||x - T_t x||, \frac{1}{2\omega} ||y - T_t y||\} \text{ for all } x, y \in C,$$

where  $\omega > k, \omega > 1$ .

In particular, if  $k_t \equiv k$  then  $\tau = \{T_s : s \in G\}$  is called  $k$ -uniformly generalized Lipschitzian semigroup on  $C$ . Recall that  $X$  is strictly convex if its unit sphere does not contain any line segments, that is,  $X$  is strictly convex if and only if the following implication holds:

$$x, y \in X, ||x|| = ||y|| = 1 \text{ and } ||(x + y)/2|| = 1 \Rightarrow x = y.$$

In order to measure the degree of convexity of  $X$ , we define its modulus of convexity  $\delta_X : [0, 2] \rightarrow [0, 1]$  by

$$\delta_X(\varepsilon) = \inf\{1 - ||(x + y)/2|| : ||x|| \leq 1, ||y|| \leq 1 \text{ and } ||x - y|| \geq \varepsilon\}.$$

The characteristic of convexity of  $X$  is the number  $\varepsilon_0(X) = \sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\}$ . It is easy to see [7] that  $X$  is uniformly convex iff  $\varepsilon_0(X) = 0$ ; uniformly nonsquare iff  $\varepsilon_0(X) < 2$ ; and strictly convex iff  $\delta(2) = 1$ . Moreover, if  $\varepsilon_0(X) < 1$ ; then  $X$  has a normal structure, that is, each bounded convex subset  $H$  of  $X$  which contains more than one point contains a point  $x_0$  such that  $\sup\{||x_0 - x|| : x \in H\} < \text{diam}(H)$ .

The following properties of modulus  $\delta_X$  of convexity of  $X$  are quite well-known (see [9])

- (a)  $\delta_X$  is increasing on  $[0, 2]$ , and moreover strictly increasing on  $[\varepsilon_0, 2]$ ;
- (b)  $\delta_X$  is continuous on  $[0, 2]$  (but not necessarily at  $\varepsilon = 2$ );
- (c)  $\delta_X(2) = 1$  iff  $X$  is strictly convex;
- (d)  $\delta_X(0) = 0$  and  $\lim_{\varepsilon \rightarrow 2^-} \delta_X(\varepsilon) = 1 - \varepsilon_0/2$

(e)  $[ \|a - x\| \leq r, \|a - y\| \leq r \text{ and } \|x - y\| \geq \varepsilon ] \Rightarrow \|a - (x + y)/2\| \leq r(1 - \delta_X(\varepsilon/r)).$

Recall that the normal structure coefficient  $N(X)$  of  $X$  is the number (see [3])

$$\inf \left\{ \frac{\text{diam}K}{r_K(K)} \right\},$$

where the infimum is taken over all bounded closed convex subsets  $K$  of  $X$  with more than one member, and  $r_K(K)$  and  $\text{diam}(K)$  are Chebyshev radii of  $K$  relative to it self and the diameter of  $K$ , respectively, i.e.,  $r_K(K) = \inf_{x \in K} \sup_{y \in K} \|x - y\|$  and  $\text{diam}K = \sup_{x,y \in K} \|x - y\|$ . A Banach space  $X$  is said to have uniformly normal structure if  $N(X) > 1$ . it is known that a Banach space with uniformly normal structure is reflexive and that all uniformly convex or uniformly smooth Banach spaces have uniformly normal structure (see, e.g., [26]). It is also been computed that  $N(H) = \sqrt{2}$  for a Hilbert spaces  $H$ . The computations of the normal structure coefficient  $N(X)$  for general Banach spaces look however complicated. No exact values of  $N(X)$  are known except for some special cases (e.g., Hilbert and  $L^p$  spaces). In general, we have the following lower bounded for  $N(X)$  (see [3, 14, 1])

$$N(X) \geq \frac{1}{1 - \delta_X(1)}.$$

Other lower bounds for  $N(X)$  in terms of some Banach space parameters or constants can be found in [11, 15].

Tan and Xu [16] have also proven that if  $X$  is uniformly convex and  $\gamma > 1$  is the unique solution of the equation 1.1, then  $N(X) > \gamma$ . Note that for a Hilbert space  $H$ , we have  $N(H) = \sqrt{2}$  and  $\gamma = \sqrt{5}/2$ . Suppose  $X$  is uniformly convex Banach space. Then it is easily seen that the equation

$$\alpha^2 \delta_X^{-1} \left(1 - \frac{1}{\alpha}\right) \tilde{N}(X) = 1 \tag{2.1}$$

has a unique solution  $\alpha > 1$ , where  $\tilde{N}(X) = 1/N(X)$ . Tan and Xu [16] proved that if  $\gamma > 1$  and  $\alpha > 1$  are the solution of 1.1 and 2.1, respectively, then  $\gamma < \alpha$ . Note that  $\gamma = \sqrt{5}/2$ , and  $\alpha = \frac{1}{\sqrt{\sqrt{3}-1}} > \gamma$ .

We need the notation of asymptotic centers, due to Edelstein [6]. Let  $C$  be a nonempty closed convex subset of a Banach space  $X$  and let  $\{x_t : t \in G\}$  be a bounded net of elements of  $X$ . Then the asymptotic radius and asymptotic center of  $\{x_t\}_{t \in G}$  with respect to  $C$  are the number

$$r_C\{x_t\} = \inf_{y \in C} \limsup_t \|x_t - y\|,$$

and respectively, the (possibly empty) set

$$A_C(\{x_t\}) = \{y \in C : \limsup_t \|x_t - y\| = r_C(\{x_t\})\}.$$

**Lemma 2.1.** ([16], Lemma 2.1) *If  $C$  is a nonempty closed convex subset of a reflexive Banach space  $X$ , then for every bounded net  $\{x_t\}_{t \in G}$  of elements of  $X$ ,  $A_C(\{x_t\})$  is a nonempty bounded closed convex subset of  $C$ . In particular, if  $X$  is a uniformly convex Banach space, then  $A_C(\{x_t\})$  consists of a single point.*

The following lemma can be proven in exactly the same way as in Lim [13] for sequences and the proof is thus omitted here.

**Lemma 2.2.** ([16], lemma 2.2) *Suppose  $X$  is a Banach space with uniformly normal structure. Then for every bounded net  $\{x_t\}_{t \in G}$  of elements of  $X$  there exists  $y \in \overline{\text{co}}(\{x_t : t \in G\})$  such that*

$$\limsup_t \|x_t - y\| \leq \tilde{N}(X)D(\{x_t\}),$$

where  $\tilde{N}(X) = 1/N(X)$ , and  $\overline{\text{co}}(E)$  is the closure of the convex hull of a set  $E \subset X$  and  $D(\{x_t\}) = \lim_t(\sup\{\|x_i - x_j\| : t \leq i, j \in G\})$  is the asymptotic diameter of  $\{x_t\}$ .

### 3. Main results

The following lemma plays an important role in proving our results.

**Lemma 3.1.** *Let  $\{T_s x_0; s \in G\}$  be a bounded for some  $x_0 \in C$  and  $\tau = \{T_s; s \in G\}$   $k$ -uniformly generalized Lipschitzian semigroup on  $C$ , then  $\{T_s x; s \in G\}$  is bounded for each  $x \in C$ .*

We next present the first result of this paper which weakens the uniform convexity assumption in theorem 1.1.

**Theorem 3.2.** *Suppose  $X$  is a real Banach space with  $N(X) > \max(1, \varepsilon_0)$ ,  $C$  is a nonempty closed convex subset of  $X$ , and  $\tau = \{T_s : s \in G\}$  is a  $k$ -uniformly generalized Lipschitzian semigroup on  $C$  which satisfy the condition (iv) with  $\omega < \alpha_*$ . Here  $\varepsilon_0$  is the characteristic of convexity of  $X$  and*

$$\alpha_* = \sup \left\{ \alpha : \alpha^2 \delta_X^{-1} \left( 1 - \frac{1}{\alpha} \right) N(X)^{-1} \leq 1 \text{ and } 1 - \frac{1}{\alpha} \in \left( 0, 1 - \frac{1}{2} \varepsilon_0 \right) \right\}. \tag{3.1}$$

*If  $\{T_s x_0 : s \in G\}$  is bounded for some  $x_0 \in C$ , then there exists  $z \in C$  such that  $T_s z = z$  for all  $s \in G$ .*

*Proof.* Put  $\tilde{N}(X) = N(X)^{-1}$ . Observe that the set

$$\left\{ \alpha : \alpha^2 \delta_X^{-1} \left( 1 - \frac{1}{\alpha} \right) N(X)^{-1} \leq 1 \text{ and } 1 - \frac{1}{\alpha} \in \left( 0, 1 - \frac{1}{2} \varepsilon_0 \right) \right\} \neq \phi. \tag{3.2}$$

Indeed, by properties (a),(b),(d) of the modulus  $\delta_x$  of convexity of  $X$ , we see that the mapping

$$\delta_x : [\varepsilon_0, 2) \rightarrow \delta_x([\varepsilon_0, 2)) = \left[ 0, 1 - \frac{1}{2} \varepsilon_0 \right)$$

is strictly increasing and continuous, and hence a bijection. Thus, we deduce that

$$\lim_{\alpha \rightarrow 1^+} \alpha^2 \delta_X^{-1} \left( 1 - \frac{1}{\alpha} \right) N(X)^{-1} = \delta_X^{-1}(0) \tilde{N}(X) = \varepsilon_0 \tilde{N}(X) < 1.$$

which implies that there exists  $\alpha_0 > 1$  such that  $\alpha_0^2 \delta_X^{-1} \left( 1 - \frac{1}{\alpha_0} \right) \tilde{N}(X) < 1$  and

$$1 - \frac{1}{\alpha_0} \in \delta_x([\varepsilon_0, 2)) = \left[ 0, 1 - \frac{1}{2} \varepsilon_0 \right).$$

This verifies our assertion 3.2.

Since  $X$  has a uniformly normal structure,  $X$  is reflexive. Due to the boundedness of  $\{T_s x_0 : s \in G\}$  and by Lemma 2.1, we get that  $A_C(\{T_t x_0\}_{t \in G})$  is nonempty bounded closed convex subset of  $C$ . Then we can choose  $x_1 \in A_C(\{T_t x_0\}_{t \in G})$  such that

$$\limsup_t \|T_t x_0 - x_1\| = \inf_{y \in C} \limsup_t \|T_t x_0 - y\|.$$

Since  $\tau$  is  $k$ -uniformly generalized Lipschitzian property, by lemma 3.1, we know that  $T_t x_1$  remains bounded. Consequently we can choose  $x_2 \in A_C(\{T_t x_1\}_{t \in G})$  such that

$$\limsup_t \|T_t x_1 - x_2\| = \inf_{y \in C} \limsup_t \|T_t x_1 - y\|.$$

Continuing this process, we can construct a sequence  $\{x_n\}_{n=0}^\infty$  in  $C$  with the properties:

- (i) for each  $n \geq 0$ ,  $\{T_t x_n\}_{t \in G}$  is bounded;
- (ii) for each  $n \geq 0$ ,  $x_{n+1} \in A_C(\{T_t x_n\}_{t \in G})$ ; that is  $x_{n+1}$  is a point in  $C$  such that

$$\lim_t \|T_t x_n - x_{n+1}\| = \inf_{y \in C} \lim_t \|T_t x_n - y\|.$$

Write  $r_n = r_C(\{T_t x_n\}_{t \in G})$ . Then by Lemma 2.2 we have

$$\begin{aligned} r_n &= \limsup_t \|T_t x_n - x_{n+1}\| \leq \tilde{N}(X)D(\{T_t x_n\}_{t \in G}) = \tilde{N}(X) \lim_t (\sup\{\|T_i x_n - T_j x_n\| : t \leq i, j \in G\}) \\ &\leq \tilde{N}(X)k \lim_t (\sup \max\{\|x_n - T_{j-i} x_n\|, \frac{1}{2\omega}\|x_n - T_i x_n\|, \frac{1}{2\omega}\|T_j x_n - T_{j-i} x_n\|\}) \\ &\leq \tilde{N}(X)k \lim_t (\sup \max\{d(x_n), \frac{1}{2\omega}d(x_n), \frac{1}{\omega}d(x_n)\}) \leq \tilde{N}(X).k.d(x_n), \end{aligned}$$

that is,

$$r_n \leq \tilde{N}(X).k.d(x_n) \leq \tilde{N}(X).\omega.d(x_n). \tag{3.3}$$

Where

$$d(x_n) = \sup\{\|x_n - T_t x_n\| : t \in G\}.$$

We may assume that  $d(x_n) > 0$  for all  $n \geq 0$  (since otherwise  $x_n$  is a common fixed point of the semigroup  $\tau$  and the proof is finished). Let  $n \geq 0$  be fixed and let  $\varepsilon > 0$  be small enough. We can choose  $j \in G$  such that

$$\|T_j x_{n+1} - x_{n+1}\| > d(x_{n+1}) - \varepsilon$$

and then choose  $s_0 \in G$  so large that

$$\|T_s x_n - x_{n+1}\| < r_n + \varepsilon < \omega(r_n + \varepsilon)$$

for all  $s \geq s_0$ . It turns out, for  $s \geq s_0 + j$ ,

$$\begin{aligned} \|T_s x_n - T_j x_{n+1}\| &\leq k \max\{\|T_{s-j} x_n - x_{n+1}\|, \frac{1}{2\omega}\|T_s x_n - T_{s-j} x_n\|, \frac{1}{2\omega}\|x_{n+1} - T_j x_{n+1}\|\} \\ &\leq k \max\{r_n + \varepsilon, \frac{1}{\omega}(r_n + \varepsilon), \frac{1}{2\omega}(r_n + \varepsilon) + \frac{1}{2\omega}\|T_s x_n - T_j x_{n+1}\|\}. \end{aligned}$$

If

$$\max\{r_n + \varepsilon, \frac{1}{\omega}(r_n + \varepsilon), \frac{1}{2\omega}(r_n + \varepsilon) + \frac{1}{2\omega}\|T_s x_n - T_j x_{n+1}\|\} = \frac{1}{2\omega}(r_n + \varepsilon) + \frac{1}{2\omega}\|T_s x_n - T_j x_{n+1}\|,$$

then we have:

$$\|T_s x_n - T_j x_{n+1}\| \leq \frac{k}{2\omega}(r_n + \varepsilon) + \frac{k}{2\omega}\|T_s x_n - T_j x_{n+1}\| \leq \frac{1}{2}(r_n + \varepsilon) + \frac{1}{2}\|T_s x_n - T_j x_{n+1}\|,$$

hence

$$\|T_s x_n - T_j x_{n+1}\| \leq r_n + \varepsilon < \omega(r_n + \varepsilon).$$

Then it follows from property (e) that

$$\|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\| \leq \omega(r_n + \varepsilon) \left(1 - \delta_X \left(\frac{d(x_{n+1}) - \varepsilon}{\omega(r_n + \varepsilon)}\right)\right)$$

for  $s \geq s_0 + j$  and hence

$$r_n \leq \limsup_s \|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\| \leq \omega(r_n + \varepsilon) \left(1 - \delta_X \left(\frac{d(x_{n+1}) - \varepsilon}{\omega(r_n + \varepsilon)}\right)\right).$$

Taking the limit as  $\varepsilon \rightarrow 0$  we obtain

$$r_n \leq \omega r_n \left(1 - \delta_X \left(\frac{d(x_{n+1})}{\omega r_n}\right)\right).$$

This implies that

$$\delta_X\left(\frac{d(x_{n+1})}{\omega r_n}\right) \leq 1 - \frac{1}{\omega} \tag{3.4}$$

$$d(x_{n+1}) \leq \omega r_n \delta_X^{-1}\left(1 - \frac{1}{\omega}\right) \tag{3.5}$$

Indeed, if  $d(x_{n+1})/(\omega r_n) \in [0, \varepsilon_0)$ , then noticing that  $\delta_X : [\varepsilon_0, 2) \rightarrow [0, 1 - \varepsilon_0/2)$  is a bijection and that  $1 - \frac{1}{\omega}$  lies in  $[0, 1 - \varepsilon_0/2)$  by assumption  $k < \omega < \alpha_*$ , we have  $\delta_X^{-1}\left(1 - \frac{1}{\omega}\right) \geq \varepsilon_0$ ; hence  $d(x_{n+1})/(\omega r_n) \leq \delta_X^{-1}\left(1 - \frac{1}{\omega}\right)$  and 3.5 follows. If  $d(x_{n+1})/(\omega r_n) \in [\varepsilon_0, 2]$ , then it is clear that  $d(x_{n+1})/(\omega r_n) \leq \delta_X^{-1}\left(1 - \frac{1}{\omega}\right)$ . This also shows that 3.6 is true.

Therefore, utilizing 3.3 and 3.5, we obtain

$$d(x_{n+1}) \leq \omega^2 \tilde{N}(X) \delta_X^{-1}\left(1 - \frac{1}{\omega}\right) d(x_n). \tag{3.6}$$

Write  $A = \omega^2 \tilde{N}(X) \delta_X^{-1}\left(1 - \frac{1}{\omega}\right)$ . Then  $A < 1$ . Indeed, from the assumption that  $\omega < \alpha_*$  it follows that there exists an  $\tilde{\alpha} > \omega$  such that

$$\tilde{\alpha}^2 \tilde{N}(X) \delta_X^{-1}\left(1 - \frac{1}{\tilde{\alpha}}\right) \leq 1 \quad \text{and} \quad \left(1 - \frac{1}{\tilde{\alpha}}\right) \in \delta_X((\varepsilon_0, 2)).$$

It then turns out that  $\delta_X^{-1}\left(1 - \frac{1}{\omega}\right) < \delta_X^{-1}\left(1 - \frac{1}{\tilde{\alpha}}\right)$ , and

$$A = \omega^2 \tilde{N}(X) \delta_X^{-1}\left(1 - \frac{1}{\omega}\right) < \tilde{\alpha}^2 \tilde{N}(X) \delta_X^{-1}\left(1 - \frac{1}{\tilde{\alpha}}\right) \leq 1.$$

Hence, it follows from 3.6 that

$$d(x_n) \leq A d(x_{n-1}) \leq \dots \leq A^n d(x_0). \tag{3.7}$$

Since

$$\|x_{n+1} - x_n\| \leq \limsup_t \|T_t x_n - x_{n+1}\| + \limsup_t \|T_t x_n - x_n\| \leq r_n + d(x_n) \leq 2d(x_n).$$

We get from 3.7 that  $\sum_{n=1}^\infty \|x_{n+1} - x_n\| < \infty$ , and hence  $\{x_n\}$  is a norm-Cauchy. Let  $z = \|\cdot\| - \lim_n x_n$ . Finally, we have for each  $s \in G$ ,

$$\begin{aligned} \|z - T_s z\| &\leq \|z - x_n\| + \|T_s x_n - x_n\| + \|T_s x_n - T_s z\| \leq \|z - x_n\| + d(x_n) + k \max\{\|z - x_n\|, \frac{1}{2\omega} d(x_n), \frac{1}{2\omega} \|z - T_s z\|\} \\ &\leq \|z - x_n\| + d(x_n) + k \|z - x_n\| + \frac{k}{2\omega} d(x_n) + \frac{k}{2\omega} \|z - T_s z\| < \|z - x_n\| + d(x_n) + k \|z - x_n\| + \frac{k}{2\omega} d(x_n) + \frac{1}{2} \|z - T_s z\| \\ &\|z - T_s z\| \leq 2(k + 1) \|z - x_n\| + (k + 2) d(x_n) \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Hence,  $T_s z = z$  for all  $s \in G$  and the proof is complete. □

**Theorem 3.3.** *Let  $C$  be a nonempty bounded subset of a uniformly convex Banach space  $X$ , and  $\tau = \{T_s : s \in G\}$  be a  $k$ -uniformly generalized Lipschitzian semigroup on  $C$  which satisfy the condition (iv) with*

$$\omega < \sqrt{\gamma_0 N(X)}, \text{ where } \gamma_0 = \inf\{\gamma \geq 1 : \gamma(1 - \delta_X(1/\gamma)) \geq 1/2\}. \tag{3.8}$$

*Suppose also there exists a nonempty bounded closed convex subset  $E$  of  $C$  with the following property  $(\mathfrak{R})$ :  $(\mathfrak{R})$   $x \in E$  implies  $w_w(x) \subset E$ .*

*Then there exists  $z \in E$  such that  $T_s z = z$  for all  $s \in G$ .*

*Proof.* Take an  $x_0 \in E$  and, consider for each  $t \in G$ , the bounded net  $\{T_s x_0 : t \leq s \in G\}$ . According to Lemma 2.2, we have a  $y_t \in \overline{\text{co}}\{T_s x_0 : t \leq s \in G\}$  such that

$$\limsup_s \|T_s x_0 - y_t\| \leq \tilde{N}(X)D(\{T_s x_0\}_{t \leq s \in G}), \tag{3.9}$$

where  $\tilde{N}(X) = 1/N(X)$  and  $D(\{T_s x_0\}_{t \leq s \in G})$  denotes the asymptotic diameter of the net  $\{z_l\}$  i.e, the number

$$\lim_t (\sup\{\|z_i - z_j\| : t \leq i, j \in G\}).$$

Since  $X$  is reflexive,  $\{y_t\}$  admits a subnet  $\{y_{t_\beta}\}$  converging weakly to some  $x_1 \in X$ . From 3.9 and the weakly lower semicontinuity of the functional  $\limsup_t \|T_t x_0 - y\|$ , it follows that

$$\limsup_t \|T_t x_0 - x_1\| \leq \tilde{N}(X)D(\{T_t x_0\}_{t \in G}). \tag{3.10}$$

It is also seen that  $x_1 \in \bigcap_{t \in G} \overline{\text{co}}\{T_s x_0 : t \leq s \in G\}$  and that

$$\|z - x_1\| \leq \limsup_t \|z - T_t x_0\| \text{ for all } z \in X. \tag{3.11}$$

Observing Property  $(\mathfrak{R})$  and the fact that  $\bigcap_{t \in G} \overline{\text{co}}\{T_s x_0 : t \leq s \in G\} = \overline{\text{co}}\{w_w(x_0)\}$  which is easy to prove by using the Separation Theorem(see[2]), we know that  $x_1$  actually lies in  $E$ . So we can repeat the above process and obtain a sequence  $\{x_n\}_{n=0}^\infty$  in  $E$  with the properties: for all nonnegative integers  $n \geq 0$ ,

$$\limsup_t \|T_t x_n - x_{n+1}\| \leq \tilde{N}(X)D(\{T_t x_n\}_{t \in G}). \tag{3.12}$$

and

$$\|z - x_{n+1}\| \leq \limsup_t \|z - T_t x_n\| \text{ for all } z \in X. \tag{3.13}$$

Write  $r_n = \limsup_t \|T_t x_n - x_{n+1}\|$  and  $d(x_n) = \sup\{\|x_n - T_t x_n\| : t \in G\}$ . This in view of 3.12, we have

$$\begin{aligned} r_n &= \limsup_t \|T_t x_n - x_{n+1}\| \leq \tilde{N}(X)D(\{T_t x_n\}_{t \in G}) = \tilde{N}(X) \lim_t (\sup\{\|T_i x_n - T_j x_n\| : t \leq i, j \in G\}) \\ &\leq \tilde{N}(X)k \lim_t (\sup \max\{\|x_n - T_{j-i} x_n\|, \frac{1}{2\omega}\|x_n - T_i x_n\|, \frac{1}{2\omega}\|T_j x_n - T_{j-i} x_n\|\}) \\ &\leq \tilde{N}(X)k \lim_t (\max\{d(x_n), \frac{1}{2\omega}d(x_n), \frac{1}{\omega}d(x_n)\}) \leq \tilde{N}(X).k.d(x_n) < \tilde{N}(X).\omega.d(x_n), \end{aligned}$$

that is,

$$r_n < \tilde{N}(X).\omega.d(x_n). \tag{3.14}$$

We may assume that  $d(x_n) > 0$  for all  $n \geq 0$ . Let  $n \geq 0$  be fixed and let  $\varepsilon > 0$  be small enough. First choose  $j \in G$  such that,

$$\|T_j x_{n+1} - x_{n+1}\| > d(x_{n+1}) - \varepsilon$$

and then choose  $s_0 \in G$  so large that

$$\|T_s x_n - x_{n+1}\| < r_n + \varepsilon < \omega(r_n + \varepsilon)$$

for all  $s \geq s_0$ . Then we have, for  $s \geq s_0 + j$ ,

$$\|T_s x_n - T_j x_{n+1}\| \leq k \max\{\|T_{s-j} x_n - x_{n+1}\|, \frac{1}{2\omega}\|T_s x_n - T_{s-j} x_n\|, \frac{1}{2\omega}\|x_{n+1} - T_j x_{n+1}\|\}$$



$$\leq k \max\{r_n + \varepsilon, \frac{1}{\omega}(r_n + \varepsilon), \frac{1}{2\omega}(r_n + \varepsilon) + \frac{1}{2\omega}\|T_s x_n - T_j x_{n+1}\|\}.$$

If

$$\max\{r_n + \varepsilon, \frac{1}{\omega}(r_n + \varepsilon), \frac{1}{2\omega}(r_n + \varepsilon) + \frac{1}{2\omega}\|T_s x_n - T_j x_{n+1}\|\} = \frac{1}{2\omega}(r_n + \varepsilon) + \frac{1}{2\omega}\|T_s x_n - T_j x_{n+1}\|,$$

then we have

$$\|T_s x_n - T_j x_{n+1}\| \leq (r_n + \varepsilon) < \omega(r_n + \varepsilon)$$

Then it follows from property (e) that, for  $s \geq s_0 + j$ ,

$$\|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\| \leq \omega(r_n + \varepsilon) \left(1 - \delta_X \left(\frac{d(x_{n+1}) - \varepsilon}{\omega(r_n + \varepsilon)}\right)\right).$$

Hence from 3.13 (taking  $z := (x_{n+1} + T_j x_{n+1})/2$ ) we obtain

$$\begin{aligned} \frac{1}{2}(d(x_{n+1}) - \varepsilon) &< \left\| \frac{1}{2}(T_j x_{n+1} - x_{n+1}) \right\| \leq \|T_j x_{n+1} - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\| \\ &\leq \limsup_t \|T_t x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\| \\ &\leq \omega(r_n + \varepsilon) \left(1 - \delta_X \left(\frac{d(x_{n+1}) - \varepsilon}{\omega(r_n + \varepsilon)}\right)\right). \end{aligned} \tag{3.15}$$

Taking the limit as  $\varepsilon \rightarrow 0$  we have

$$\frac{1}{2}d(x_{n+1}) \leq \omega r_n \left(1 - \delta_X \left(\frac{d(x_{n+1})}{\omega r_n}\right)\right). \tag{3.16}$$

On other hand, we easily find by 3.13 that, for each  $j \in G$ ,

$$\|T_j x_{n+1} - x_{n+1}\| \leq \limsup_t \|T_j x_{n+1} - T_t x_n\| < \omega r_n.$$

It turns out that

$$d(x_{n+1}) < \omega r_n \tag{3.17}$$

Combining 3.16 and 3.17 and using the definition of  $\gamma_0$  in 3.8, we infer that  $(\omega r_n)/d(x_{n+1}) \geq \gamma_0$ . It turns out from 3.14 that

$$d(x_{n+1}) \leq \frac{\omega}{\gamma_0} r_n < \frac{\omega^2}{\gamma_0 N(X)} d(x_n).$$

Consequently, we obtain

$$d(x_n) \leq A d(x_{n-1}) \leq \dots \leq A^n d(x_0),$$

where  $A = \omega^2[\gamma_0 N(X)]^{-1} < 1$  by assumption. Noticing that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \limsup_t \|T_t x_n - x_{n+1}\| + \limsup_t \|T_t x_n - x_n\| \\ &\leq r_n + d(x_n) \leq (1 + k\tilde{N}(X))d(x_n) \leq (1 + k\tilde{N}(X))A^n d(x_0), \end{aligned}$$

that  $\sum_{n=1}^\infty \|x_{n+1} - x_n\|$  is convergent. This implies that  $\{x_n\}$  is strongly convergent. Let  $z = \|\cdot\| - \lim_n x_n$ . Then, we have for each  $s \in G$ ,

$$\begin{aligned} \|z - T_s z\| &\leq \|z - x_n\| + \|T_s x_n - x_n\| + \|T_s x_n - T_s z\| \leq \|z - x_n\| + d(x_n) + k \max\{\|z - x_n\|, \frac{1}{2\omega}d(x_n), \frac{1}{2\omega}\|z - T_s z\|\} \\ &\leq \|z - x_n\| + d(x_n) + k\|z - x_n\| + \frac{k}{2\omega}d(x_n) + \frac{k}{2\omega}\|z - T_s z\| < \|z - x_n\| + d(x_n) + k\|z - x_n\| + \frac{k}{2\omega}d(x_n) + \frac{1}{2\omega}\|z - T_s z\| \\ &\|z - T_s z\| \leq 2(k + 1)\|z - x_n\| + (k + 2)d(x_n) \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Hence,  $T_s z = z$  for all  $s \in G$  and the proof is complete. □

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