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Weak and strong convergence of an explicit iteration process for an asymptotically quasi-i-nonexpansive mapping in Banach spaces

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Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

Communicated by Professor D. Turkoglu

Abstract

In this paper, we prove the weak and strong convergence of an explicit iterative process to a common fixed point of an asymptotically quasi-I-nonexpansive mapping T and an asymptotically quasi-nonexpansive mapping I, defined on a nonempty closed convex subset of a Banach space.

Keywords: Asymptotically quasi-I-nonexpansive self-mappings, explicit iterations, common fixed point, uniformly convex Banach space.
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1. Introduction

Let K be a nonempty subset of a real normed linear space X and let $T : K \to K$ be a mapping. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$ and we denote by $D(T)$ the domain of a mapping T. Throughout this paper, we assume that X is a real Banach space and $F(T) \neq \emptyset$. Now, we recall some well-known concepts and results.

Definition 1.1. *A mapping $T : K \to K$ is said to be*

1. *nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$;*

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2. *asymptotically nonexpansive, if there exists a sequence* $\{\lambda_n\} \subset [1,\infty)$ *with* $\lim_{n\to\infty} \lambda_n = 1$ *such that* $\|T^n x - T^n y\| \leq \lambda_n \|x-y\|$ *for all* $x, y \in K$ *and* $n \in N$*;*
3. *quasi-nonexpansive, if* $\|Tx - p\| \leq \|x-p\|$ *for all* $x \in K,\ p \in F(T)$*;*
4. *asymptotically quasi-nonexpansive, if there exists a sequence* $\{\mu_n\} \subset [1,\infty)$ *with* $\lim_{n\to\infty} \mu_n = 1$ *such that* $\|T^n x - p\| \leq \mu_n \|x-p\|$ *for all* $x \in K,\ p \in F(T)$ *and* $n \in N$*.*

The first nonlinear ergodic theorem was proved by Baillon [1] for general nonexpansive mappings in Hilbert space H : if K is a closed and convex subset of H and T has a fixed point, then every $x \in K$, $\{T^n x\}$ is weakly almost convergent, as $n \to \infty$, to a fixed point of T. It was also shown by Pazy [2] that if H is a real Hilbert space and $\left(\frac{1}{n}\right) \sum_{i=0}^{n-1} T^i x$ converges weakly, as $n \to \infty$, to $y \in K$, then $y \in F(T)$.
In [3], [4] Browder studied the iterative construction for fixed points of nonexpansive mappings on closed and convex subsets of a Hilbert space. The concept of a quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Kiziltunc et al. [5] studied common fixed points of two nonself nonexpansive mappings in Banach Spaces. Khan [8] presented a two-step iterative process for two asymptotically quasi-nonexpansive mappings. Fukhar-ud-din and Khan [9] studied convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications. Diaz and Metcalf [7] and Dotson [10] studied quasi-nonexpansive mappings in Banach spaces. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces was discussed by Ghosh and Debnath [11]. The iterative approximation problems for nonexpansive mapping, were studied extensively by Goebel and Kirk [12], Liu [13], Wittmann [14], Reich [15], Gornicki [16], Schu [17], Shioji and Takahashi [18], and Tan and Xu [19] in the settings of Hilbert spaces and uniformly convex Banach spaces.

There are many concepts which generalize a notion of nonexpansive mapping. One of such concepts is I-nonexpansivity of a mapping T [20]. Let us recall some notions.

Definition 1.2. *Let* $T : K \to K,\ I : K \to K$ *be two mappings of nonempty subset* K *of a real normed linear space* X*. Then* T *is said to be*

1. *I-nonexpansive, if* $\|Tx - Ty\| \leq \|Ix - Iy\|$ *for all* $x, y \in K$*;*
2. *asymptotically I-nonexpansive, if there exists a sequence* $\{\lambda_n\} \subset [1,\infty)$ *with* $\lim_{n\to\infty} \lambda_n = 1$ *such that* $\|T^n x - T^n y\| \leq \lambda_n \|I^n x - I^n y\|$ *for all* $x, y \in K$ *and* $n \geq 1$*;*
3. *quasi I-nonexpansive, if* $\|Tx - p\| \leq \|Ix - p\|$ *for all* $x \in K$, $p \in F(T) \cap F(I)$*;*
4. *asymptotically quasi I-nonexpansive, if there exists a sequance* $\{\mu_n\} \subset [1,\infty)$ *with* $\lim_{n\to\infty} \mu_n = 1$ *such that* $\|T^n x - p\| \leq \mu_n \|I^n x - p\|$ *for all* $x \in K,\ p \in F(T) \cap F(I)$ *and* $n \geq 1$*.*

Remark 1.3. *If* $F(T) \cap F(I) \neq \emptyset$ *then an asymptotically* I*-nonexpansive mapping is asymptotically quasi* I*-nonexpansive.*

Best approximation properties of I-nonexpansive mappings were investigated in [20]. In [21] strong convergence of Mann iterations of I-nonexpansive mapping has been proved. In [22] the weak and strong convergence of implicit iteration process to a common fixed point of a finite family of I-asymptotically nonexpansive mappings were proved. In [23] the weak convergence theorems of three-step iterative scheme for an I-quasi-nonexpansive mappings in a Banach space has been studied. In [24] a weakly convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. Mukhamedov and Saburov [27] studied weak and strong convergence of an implicit iteration process for an asymptotically quasi-I-nonexpansive mapping in Banach space. In [28] Mukhamedov and Saburov studied strong convergence of an explicit iteration process for a totally asymptotically I-nonexpansive mapping in Banach spaces. This iteration scheme is defined as follows.

Let K be a nonemty closed convex subset of a real Banach space X. Consider $T : K \to K$ an asymptotically quasi I-nonexpansive mapping, where $I : K \to K$ an asymptotically quasi-nonexpansive mapping. Then for two given sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $[0, 1]$ we shall consider the following iteration scheme:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n y_n,\ n \geq 0, \\ y_n = (1-\beta_n)\, x_n + \beta_n I^n x_n. \end{cases} \tag{1.1}$$

Inspired and motivated by these facts, we study the convergence of an explicit iterative involving an asymptotically quasi-I-nonexpansive mapping in nonempty closed convex subset of uniformly convex Banach spaces.

In this paper, we prove weak and strong convergences of an explicit iterative process (1.1) to a common fixed point of T and I.

2. Preliminaries

Recall that a Banach space X is said to satisfy *Opial condition* [25] if, for each sequence $\{x_n\}$ in X such that $\{x_n\}$ converges weakly to x implies that

$$\lim_{n\to\infty}\inf \|x_n - x\| < \lim_{n\to\infty}\inf \|x_n - y\| \tag{2.1}$$

for all $y \in X$ with $y \neq x$. It is weel known that (see [26]) inequality (2.1) is equivalent to

$$\lim_{n\to\infty}\sup \|x_n - x\| < \lim_{n\to\infty}\sup \|x_n - y\|. \tag{2.2}$$

Definition 2.1. *Let K be a closed subset of a real Banach space X and let $T : K \to K$ be a mapping.*

1. *A mapping T is said to be semiclosed (demiclosed) at zero, if for each bounded sequence $\{x_n\}$ in K, the conditions x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.*
2. *A mapping T is said to be semicompact, if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$, $n \to \infty$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ strongly.*
3. *T is called a uniformly L-Lipschitzian mapping, if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $x, y \in K$ and $n \geq 1$.*

Lemma 2.2. *[17] Let X be a uniformly convex Banach space and let b, c be two constant with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ and $\{x_n\}$, $\{y_n\}$ are two sequence in X such that*

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = d, \quad \lim_{n\to\infty}\sup \|x_n\| \leq d, \quad \lim\sup \|y_n\| \leq d, \tag{2.3}$$

holds some $d \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3. *[19] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$. If one of the following conditions is satisfied:*

1. $a_{n+1} \leq a_n + b_n, \quad n \geq 1,$
2. $a_{n+1} \leq (1 + b_n) a_n, \quad n \geq 1,$

then the limit $\lim_{n\to\infty} a_n$ exists.

3. Main Results

In this section, we prove convergence theorems of an explicit iterative scheme (1.1) for an asymptotically quasi-I-nonexpansive mapping in Banach spaces. In order to prove our main results, the following lemmas are needed.

Lemma 3.1. *Let X be a real Banach space and let K be a nonempty closed convex subset of X. Let $T : K \to K$ be an asymptotically quasi-I-nonexpansive mapping with a sequence $\{\lambda_n\} \subset [1, \infty)$ and $I : K \to K$ be an asymptotically quasi-nonexpansive mapping with a sequence $\{\mu_n\} \subset [1, \infty)$ such that $\mathcal{F} = F(T) \cap F(I) \neq \emptyset$. Suppose $N = \lim_n \lambda_n \geq 1$, $M = \lim_n \mu_n \geq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1) \alpha_n < \infty$. If $\{x_n\}$ is an explicit iterative sequence defined by (1.1), then for each $p \in \mathcal{F} = F(T) \cap F(I)$ the limit $\lim_{n\to\infty} \|x_n - p\|$ exists.*

Proof. Since $p \in \mathcal{F} = F(T) \cap F(I)$, for any given $p \in F$, it follows (1.1) that

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|T^n y_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \lambda_n \|I^n y_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \lambda_n \mu_n \|y_n - p\|. \end{aligned} \tag{3.1}$$

Again from (1.1) we derive that

$$\begin{aligned} \|y_n - p\| &\leq (1-\beta_n)\|x_n - p\| + \beta_n \|I^n x_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n \mu_n \|x_n - p\| \\ &\leq (1-\beta_n)\mu_n \|x_n - p\| + \beta_n \mu_n \|x_n - p\| \\ &\leq \mu_n \|x_n - p\|, \end{aligned} \tag{3.2}$$

which means

$$\|y_n - p\| \leq \mu_n \|x_n - p\| \leq \lambda_n \mu_n \|x_n - p\|. \tag{3.3}$$

Then from (3.3) we have

$$\|x_{n+1} - p\| \leq \left[1 + \alpha_n\left(\lambda_n^2 \mu_n^2 - 1\right)\right] \|x_n - p\|. \tag{3.4}$$

By putting $b_n = \alpha_n\left(\lambda_n^2 \mu_n^2 - 1\right)$ the last inequality can be rewritten as follows:

$$\|x_{n+1} - p\| \leq (1 + b_n)\|x_n - p\|. \tag{3.5}$$

By hypothesis we find

$$\begin{aligned} \sum_{n=1}^{\infty} b_n &= \sum_{n=1}^{\infty} \alpha_n\left(\lambda_n^2 \mu_n^2 - 1\right) \\ &= \sum_{n=1}^{\infty} (\lambda_n \mu_n + 1)(\lambda_n \mu_n - 1)\alpha_n \\ &\leq (NM + 1)\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1)\alpha_n < \infty. \end{aligned}$$

Defining $a_n = \|x_n - p\|$ in (3.5) we have

$$a_{n+1} \leq (1 + b_n)a_n, \tag{3.6}$$

and Lemma 2.3 implies the existence of the limit $\lim_{n\to\infty} a_n$. The means the limit

$$\lim_{n\to\infty} \|x_n - p\| = d \tag{3.7}$$

exists, where $d \geq 0$ constant. This completes the proof. □

Theorem 3.2. *Let X be a real Banach space and let K be a nonempty closed convex subset of X. Let $T : K \to K$ be a unifornly L_1-Lipschitzian asymptotically quasi-I-nonexpansive mapping with a sequence $\{\lambda_n\} \subset [1, \infty)$ and $I : K \to K$ be a unifornly L_2-Lipschitzian asymptotically quasi-nonexpansive mapping with a sequence $\{\mu_n\} \subset [1, \infty)$ such that $\mathcal{F} = F(T) \cap F(I) \neq \emptyset$. Suppose $N = \lim_n \lambda_n \geq 1$, $M = \lim_n \mu_n \geq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1)\alpha_n < \infty$. Then an explicit iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to a common fixed point in $\mathcal{F} = F(T) \cap F(I)$ if and only if*

$$\lim_{n\to\infty} \inf d(x_n, F) = 0. \tag{3.8}$$

Proof. The necessity of condition (3.7) is obvious. Let us prove the sufficiency part of theorem. Since $T, I : K \to K$ are uniformly L-Lipschitzian mappings, so T and I are continuous mappings. Therefore the sets $F(T)$ and $F(I)$ are closed. Hence $\mathcal{F} = F(T) \cap F(I)$ is a nonempty closed set.

For any given $p \in F$, we have

$$\|x_{n+1} - p\| \leq (1 + b_n) \|x_n - p\|, \tag{3.9}$$

as before where $b_n = \alpha_n \left(\lambda_n^2 \mu_n^2 - 1\right)$ with $\sum_{n=1}^{\infty} b_n < \infty$. Hence, we have

$$d(x_{n+1}, F) \leq (1 + b_n) d(x_n, F). \tag{3.10}$$

From (3.9) due to Lemma 2.3 we obtain the existence of the limit $\lim_{n\to\infty} d(x_n F)$. By condition (3.7), we get

$$\lim_{n\to\infty} d(x_n, F) = \lim_{n\to\infty} \inf d(x_n, F) = 0. \tag{3.11}$$

Let us prove that the sequence $\{x_n\}$ converges to a common fixed point of T and I. In fact, due to $1 + t \leq \exp(t)$ for all $t > 0$, and from (3.8), we obtain

$$\|x_{n+1} - p\| \leq \exp(b_n) \|x_n - p\|. \tag{3.12}$$

Hence, for any positive integers m, n from (3.11) with $\sum_{n=1}^{\infty} b_n < \infty$ we find

$$\begin{aligned} \|x_{n+m} - p\| &\leq \exp(b_{n+m-1}) \|x_{n+m-1} - p\| \\ &\leq \exp\left(\sum_{i=n}^{n+m-1} b_i\right) \|x_n - p\| \\ &\leq \exp\left(\sum_{i=1}^{\infty} b_i\right) \|x_n - p\|, \end{aligned} \tag{3.13}$$

which means that

$$\|x_{n+m} - p\| \leq W \|x_n - p\| \tag{3.14}$$

for all $p \in F$, where $W = \exp\left(\sum_{i=1}^{\infty} b_i\right) < \infty$.

Since $\lim_{n\to\infty} d(x_n F) = 0$, then for any given $\varepsilon > 0$, there exists a positive integer number n_0 such that

$$d(x_{n_0}, F) < \frac{\varepsilon}{W}. \tag{3.15}$$

Therefore there exists $p_1 \in F$ such that

$$\|x_{n_0} - p_1\| < \frac{\varepsilon}{W}. \tag{3.16}$$

Consequently, for all $n \geq n_0$ from (3.14) we derive

$$\begin{aligned} \|x_n - p_1\| &\leq W \|x_{n_0} - p_1\| \\ &< W \cdot \frac{\varepsilon}{W} \\ &= \varepsilon, \end{aligned} \tag{3.17}$$

which means that the strong convergence limit of the sequence $\{x_n\}$ is a common fixed point p_1 of T and I. This completes the proof. □

Lemma 3.3. *Let X be a real uniformly Banach space and let K be a nonempty closed convex subset of X. Let $T : K \to K$ be a unifornly L_1-Lipschitzian asymptotically quasi-I-nonexpansive mapping with a sequence $\{\lambda_n\} \subset [1, \infty)$ and $I : K \to K$ be a unifornly L_2-Lipschitzian asymptotically quasi-nonexpansive mapping with a sequence $\{\mu_n\} \subset [1, \infty)$ such that $\mathcal{F} = F(T) \cap F(I) \neq \emptyset$. Suppose $N = \lim_n \lambda_n \geq 1$, $M = \lim_n \mu_n \geq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[t, 1-t]$ for some $t \in (0, 1)$ such that $\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1) \alpha_n < \infty$. Then an explicit iterative sequence $\{x_n\}$ defined by (1.1) satisfies the following:*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0, \quad \lim_{n\to\infty} \|x_n - Ix_n\| = 0. \tag{3.18}$$

Proof. First, we will prove that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0, \quad \lim_{n\to\infty} \|x_n - I^n x_n\| = 0. \tag{3.19}$$

According to Lemma 3.1 for any $p \in \mathcal{F} = F(T) \cap F(I)$ we have $\lim_{n\to\infty} \|x_n - p\| = d$. It follows from (1.1) that

$$\|x_{n+1} - p\| = \|(1-\alpha_n)(x_n - p) + \alpha_n (T^n y_n - p)\| \to d, \quad n \to \infty. \tag{3.20}$$

By means of asymptotically quasi-I-nonexpansivity of T and asymptotically quasi-nonexpansivity of I from (3.3) we get

$$\lim_{n\to\infty} \sup \|T^n y_n - p\| \le \lim_{n\to\infty} \sup \lambda_n \mu_n \|y_n - p\| \le \lim_{n\to\infty} \sup \lambda_n^2 \mu_n^2 \|x_n - p\| = d. \tag{3.21}$$

Now using

$$\lim_{n\to\infty} \sup \|x_n - p\| = d, \tag{3.22}$$

with (3.21) and applying Lemma 2.2 to (3.20) we obtain

$$\lim_{n\to\infty} \|x_n - T^n y_n\| = 0. \tag{3.23}$$

Now from (1.1) and (3.22) we infer that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|\alpha_n (T^n y_n - x_n)\| = 0. \tag{3.24}$$

From (3.23) and (3.24) we get

$$\lim_{n\to\infty} \|x_{n+1} - T^n y_n\| \le \lim_{n\to\infty} \|x_{n+1} - x_n\| + \lim_{n\to\infty} \|x_n - T^n y_n\| = 0. \tag{3.25}$$

On the other hand, we have

$$\begin{aligned} \|x_n - p\| &\le \|x_n - T^n y_n\| + \|T^n y_n - p\| \\ &\le \|x_n - T^n y_n\| + \lambda_n \mu_n \|y_n - p\|, \end{aligned} \tag{3.26}$$

which implies

$$\|x_n - p\| - \|x_n - T^n y_n\| \le \lambda_n \mu_n \|y_n - p\|. \tag{3.27}$$

The last inequality with (3.3) yields that

$$\|x_n - p\| - \|x_n - T^n y_n\| \le \lambda_n \mu_n \|y_n - p\| \le \lambda_n^2 \mu_n^2 \|x_n - p\|. \tag{3.28}$$

Then (3.22) and (3.23) with the Squeeze Theorem imply that

$$\lim_{n\to\infty} \|y_n - p\| = d. \tag{3.29}$$

Again from (1.1) we can see that

$$\|y_n - p\| = \|(1-\beta_n)(x_n - p) + \beta_n (I^n x_n - p)\| \to \infty, \quad n \to \infty. \tag{3.30}$$

From (3.7) one finds

$$\lim_{n\to\infty} \sup \|I^n x_n - p\| \le \lim_{n\to\infty} \sup \mu_n \|x_n - p\| = d. \tag{3.31}$$

Now applying Lemma 2.2 to (3.29) we obtain

$$\lim_{n\to\infty} \|x_n - I^n x_n\| = 0. \tag{3.32}$$

From (3.24) and (3.32) we have

$$\lim_{n\to\infty}\|x_{n+1}-I^nx_n\|\le\lim_{n\to\infty}\|x_{n+1}-x_n\|+\lim_{n\to\infty}\|x_n-I^nx_n\|=0. \tag{3.33}$$

It follows from (1.1) that

$$\|y_n-x_n\|=\beta_n\|x_n-I^nx_n\|. \tag{3.34}$$

Hence, from (3.32) and (3.34) we obtain

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.35}$$

Consider

$$\|x_n-T^nx_n\|\le\|x_n-T^ny_n\|+L_1\|y_n-x_n\|. \tag{3.36}$$

Then from (3.23) and (3.35) we obtain

$$\lim_{n\to\infty}\|x_n-T^nx_n\|=0. \tag{3.37}$$

From (3.24) and (3.35) we have

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|\le\lim_{n\to\infty}\|x_{n+1}-x_n\|+\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.38}$$

Finally, from

$$\|x_n-Tx_n\|\le\|x_n-T^nx_n\|+L_1\|x_n-y_{n-1}\|+L_1\left\|T^{n-1}y_{n-1}-x_n\right\|, \tag{3.39}$$

which with (3.25), (3.37) and (3.38) we get

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.40}$$

Similarly, one has

$$\|x_n-Ix_n\|\le\|x_n-I^nx_n\|+L_2\|x_n-x_{n-1}\|+L_2\left\|I^{n-1}x_{n-1}-x_n\right\|, \tag{3.41}$$

which with (3.24), (3.32) and (3.33) implies

$$\lim_{n\to\infty}\|x_n-Ix_n\|=0. \tag{3.42}$$

This completes the proof. □

Theorem 3.4. *Let X be a real uniformly convex Banach space satisfying Opial condition and let K be a nonempty closed convex subset of X. Let $E:X\to X$ be an identity mapping, let $T:K\to K$ be a unifornly L_1-Lipschitzian asymptotically quasi-I-nonexpansive mapping with a sequence $\{\lambda_n\}\subset[1,\infty)$,and $I:K\to K$ be a unifornly L_2-Lipschitzian asymptotically quasi-nonexpansive mapping with a sequence $\{\mu_n\}\subset[1,\infty)$ such that $\mathcal{F}=F(T)\cap F(I)\neq\emptyset$. Suppose $N=\lim_n\lambda_n\ge1$, $M=\lim_n\mu_n\ge1$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[t,1-t]$ for some $t\in(0,1)$ such that $\sum_{n=1}^{\infty}(\lambda_n\mu_n-1)\alpha_n<\infty$. If the mappings $E-T$ and $E-I$ are semiclosed at zero, then an explicit iterative sequence $\{x_n\}$ defined by (1.1) converges weakly to a common fixed point of T and I.*

Proof. Let $p\in F$, then according to Lemma 3.1 the sequence $\{\|x_n-p\|\}$ converges. This provides that $\{x_n\}$ is a bounded sequence. Since X is uniformly convex, then every bounded subset of X is weakly compact. Since $\{x_n\}$ is a bounded sequence in K, then there exists a subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q\in K$. Hence, from (3.40) and (3.42) it follows that

$$\lim_{n_k\to\infty}\|x_{n_k}-Tx_{n_k}\|=0,\qquad\lim_{n_k\to\infty}\|x_{n_k}-Ix_{n_k}\|=0. \tag{3.43}$$

Since the mappings $E - T$ and $E - I$ are semiclosed at zero, therefore, we find $Tq = q$ and $Iq = q$, which means $q \in \mathcal{F} = F(T) \cap F(I)$.

Finally, let us prove that $\{x_n\}$ converges weakly to q. In fact, suppose the contrary, that is, there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_1 \in K$ and $q_1 \neq q$. Then by the same method as given above, we can also prove that $q_1 \in \mathcal{F} = F(T) \cap F(I)$.

Taking $p = q$ and $p = q_1$ and using the same argument given in the proof of (3.7), we can prove that the limits $\lim_{n\to\infty} \|x_n - q\|$ and $\lim_{n\to\infty} \|x_n - q_1\|$ exist, and we have

$$\lim_{n\to\infty} \|x_n - q\| = d, \quad \lim_{n\to\infty} \|x_n - q_1\| = d_1, \tag{3.44}$$

where d and d_1 are two nonnegative numbers. By virtue of the Opial condition of X, we obtain

$$\begin{aligned} d &= \lim_{n_k\to\infty} \sup \|x_{n_j} - q\| < \lim_{n_k\to\infty} \sup \|x_{n_k} - q_1\| = d_1 \\ &= \lim_{n_j\to\infty} \sup \|x_{n_j} - q_1\| < \lim_{n_j\to\infty} \sup \|x_{n_j} - q\|. \end{aligned} \tag{3.45}$$

This is a contradiction. Hence $q_1 = q$. This implies that $\{x_n\}$ converges weakly to q. This completes the proof. □

Theorem 3.5. *Let X be a real uniformly convex Banach space and let K be a nonempty closed convex subset of X. Let $T : K \to K$ be a unifornly L_1-Lipschitzian asymptotically quasi-I-nonexpansive mapping with a sequence $\{\lambda_n\} \subset [1,\infty)$,and $I : K \to K$ be a unifornly L_2-Lipschitzian asymptotically quasi-nonexpansive mapping with a sequence $\{\mu_n\} \subset [1,\infty)$ such that $\mathcal{F} = F(T) \cap F(I) \neq \emptyset$. Suppose $N = \lim_n \lambda_n \geq 1$, $M = \lim_n \mu_n \geq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[t, 1-t]$ for some $t \in (0,1)$ such that $\sum_{n=1}^{\infty} (\lambda_n\mu_n - 1)\alpha_n < \infty$. If at least one mapping of the mappings T and I is semicompact, then an explicit iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to a common fixed point of T and I.*

Proof. Without any loss of generality, we may assume taht T is semicompact. This with (3.40) means that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^*$ strongly and $x^* \in K$. Since T, I are continuous, then from (3.40) and (3.42) we find

$$\|x^* - Tx^*\| = \lim_{n_k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0, \quad \|x^* - Ix^*\| = \lim_{n_k\to\infty} \|x_{n_k} - Ix_{n_k}\| = 0. \tag{3.46}$$

This shows taht $x^* \in \mathcal{F} = F(T) \cap F(I)$. According to Lemma 3.1 the limit $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Then

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n_k\to\infty} \|x_{n_k} - x^*\| = 0,$$

which means taht $\{x_n\}$ converges to $x^* \in F$. This completes the proof. □

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