Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 5 (2012), 439–447
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Application of the infinite matrix theory to the solvability of a system of differential equations

Ali Fares, Ali Ayad*

Équipe Algèbre et Combinatoire, EDST, Faculté des sciences - Section 1, Université libanaise, Hadath, Liban.

Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

Communicated by Professor Gh. Sadeghi

Abstract

In this paper we deal with the solvability of the infinite system of differential equations $x'(t) = \Delta(\lambda)x(t) + b$ with $x(0) = a$, where $\Delta(\lambda)$ is the triangle defined by the infinite matrix whose the nonzero entries are $[\Delta(\lambda)]_{nn} = \lambda_n$ and $[\Delta(\lambda)]_{n,n-1} = \lambda_{n-1}$ for all $n \in \mathbb{N}$, for a given sequence λ and a, b are two given infinite column matrices. We use a new method based on Laplace transformations to solve this system.

Keywords: Infinite linear systems of differential equations, systems of linear equations, Laplace operator.
2010 MSC: Primary 40C05, 44A10.

1. Introduction

Infinite matrix theory is used in many branches of classical mathematics such as infinite quadratic forms, integral equations, matrix transformations, differential equations, operators between sequence spaces, it is also used to provide approximations of solutions. Infinite-dimensional linear systems appear naturally when studying control problems for systems modelled by linear partial differential equations. Many problems in dynamic systems can be written in the form of differential equations or infinite differential systems and lead to infinite linear systems. In this way, we cite Hill's equation that was studied by L. Brillouin, E. L. Ince [[1]], K.G. Valeev [[10]], H. Hochstadt (1963), S. Winkler (1966) and B. Rossetto [[9]]. This equation is the second order differential equation of the form

$$y''(z) + J(z)\,y(z) = 0, \tag{1.1}$$

where $z \in \Omega$ and Ω is an open subset of $\mathbb{C}$, containing the real axis and $J(z)$ is a special periodic function. It was shown that the solutions of (1.1) are of the form $y(z) = e^{\mu z}\sum_{m=-\infty}^{+\infty} x_m e^{2imz}$ where μ is the Floquet exponent. Replacing $y(z)$ by its expression in equation (1.1), we obtain an infinite linear system represented by the matrix equation

$$A_\mu x = 0, \tag{1.2}$$

*Corresponding author
Email addresses: alikfares@yahoo.fr (Ali Fares), ayadali99100@hotmail.com (Ali Ayad)

Received 2011-3-5

where $x^t = (\dots, x_{-1}, x_0, x_1, \dots)$ and $A_\mu = (a_{nm})_{n,m\in\mathbb{Z}}$ is an infinite matrix depending on μ (cf. [5]) defined by:

$$\begin{cases} a_{nn} = \theta_0 + (\mu + 2ni)^2, & \forall n \in \mathbb{Z} \\ a_{nm} = \theta_{|m-n|}, & \text{for } m \neq n \end{cases}$$

The aim is then to determine the values of μ for which (1.2) has a non trivial solution. Some authors determined such values of μ using an *infinite determinant* [[7]]. B. Rossetto provided an algorithm that allows us to calculate the Floquet exponent from the generalization of the notion of the *characteristic equation and of a truncated determinant.* On the other hand, B. de Malafosse [[4]] dealt with equation (1.1) and studied system (1.2) using special additionnal equations.

More recently, B. de Malafosse [[5]] used the same method for the study of the Mathieu equation that is a differential equation with π-periodic coefficients of the form:

$$y^{(n)}(t) + J_1 y^{(n-1)}(t) + \cdots + J_k y^{(n-k)}(t) + \cdots + J_n y(t) = 0, \text{ for all } t \in \mathbb{R}.$$

in which only one of the coefficients J_k is of the form $p + 2a\cos(2t)$ $(a, p \in \mathbb{R})$, the other being constants. In 2006 K.L. Chiu and P.N. Shivakumar [[2]] studied the differential system

$$-y''(z) + f(z)\, y(z) = \lambda y(z)$$

with $y(0) = 0$ and $y(\infty) = 0$. It is well known that this system is a Sturm Liouville problem and hence there is an infinite number of eigenvalues which are all reals, positive and ordered if $f(z)$ is chosen to be a positive function tending to infinity as z tends to ∞. The authors used finite difference scheme to reduce the linear system to an equivalent infinite linear algebraic eigenvalue problem.

In this paper, we use special well known infinite matrices such as the operator of the first difference Δ. Some properties of this operator were studied by Hausdorff, Leibowitz, Reade [[8]] and Okutoyi [[6]]. Then we deal with the infinite linear system of differential equations defined by

$$x'(t) = \Delta(\lambda) x(t) + b, \tag{1.3}$$

where $\Delta(\lambda)$ is the triangle defined by the infinite matrix $[\Delta(\lambda)]_{nn} = \lambda_n$ and $[\Delta(\lambda)]_{n,n-1} = \lambda_{n-1}$ for all $n \in \mathbb{N}$, $b = (b_n)_{n\geq 1}$ is a given sequence and $x(t) = (x_n(t))_{n\geq 1}$ is the unknown sequence of functions. The matrix $\Delta(\lambda)$ generalizes the well known operator of first difference Δ. Here we use a new method based on Laplace transformations to solve equation (1.3) and we will see that the resolution of these systems leads to solutions with complicated expressions although they are associated with infinite lower triangular matrices.

This paper is organized as follows. In Section 2 we define the triangle matrix and we recall some results on *infinite bidiagonal matrices* and Laplace tranformation operator. We consider in section 3 the equation $x'(t) = \Delta(\lambda)\, x(t) + b$ with $x(0) = a$ where λ and $a = (a_n)_{n\geq 1}$ are two given sequences. We consider two cases where all terms of λ are pairewisely distinct or equals. Finally, in Section 4, we give some examples with particular sequences λ, a and b.

2. Preliminaries

In this paper, we consider infinite lower triangular matrices with nonzero diagonal entries that are called triangles. An infinite matrix $T = (t_{nk})_{n,k\geq 1}$ is a triangle if and only if $t_{nk} = 0$ for all $k > n$ and $t_{nn} \neq 0$ for all $n \geq 1$, that is

$$T = \begin{pmatrix} t_{11} & & & & \\ t_{21} & t_{22} & & \mathbf{0} & \\ \vdots & \vdots & \ddots & & \\ t_{n1} & t_{n2} & \cdots & t_{nn} & \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}.$$

We denote by w the set of all the sequences and by U the set of the sequences $u = (u_n)_{n\geq 1}$ with $u_n \neq 0$ for all $n \geq 1$. The matrix T is considered as an operator from w to itself in the following way, for every sequence $x \in w$ which can be written as a column matrix $x = (x_1, ..., x_n, ...)^t$, we have $Tx = (T_1(x), ..., T_n(x), ...)^t$ with

$$T_n(x) = \sum_{k=1}^{n} t_{nk} x_k \quad \text{for all } n \geq 1.$$

It is known that every triangle T is invertible and if T^{-1} denotes its inverse then we have

$$T\left(T^{-1}x\right) = T^{-1}\left(Tx\right) = x \quad \text{for all } x \in w. \tag{2.1}$$

We are interested in solving infinite linear systems represented by

$$Tx = b \tag{2.2}$$

for a given $b \in w$ where $x \in w$ is the unknown. Equation (2.2) is equivalent to

$$\sum_{k=1}^{n} t_{nk}x_k = b_n \qquad n = 1, 2, ...$$

It can be easily deduced from (2.1) that the unique solution of (2.2) is given by

$$x = T^{-1}b.$$

In this paper, we will solve the equation $\widehat{\Delta}(\alpha,\beta)\,x = b$ where $\widehat{\Delta}(\alpha,\beta)$ is the triangle defined by

$$\widehat{\Delta}(\alpha,\beta) = \begin{pmatrix} \alpha_1 & & & & \mathbf{0} \\ -\beta_1 & \alpha_2 & & & \\ & \ddots & \ddots & & \\ & & -\beta_{n-1} & \alpha_n & \\ \mathbf{0} & & & \ddots & \ddots \end{pmatrix}$$

and $\alpha = (\alpha_n)_{n\geq 1} \in U$ and $\beta = (\beta_n)_{n\geq 1} \in w$.

Lemma 2.1. *Let $\alpha = (\alpha_n)_{n\geq 1} \in U$ and $\beta = (\beta_n)_{n\geq 1} \in \omega$. We have*

$$\left(\widehat{\Delta}(\alpha,\beta)\right)^{-1} = \widehat{C}(\alpha,\beta) = (c_{nk})_{n,k\geq 1},$$

where

$$c_{nk} = \begin{cases} \dfrac{1}{\alpha_n} & \text{if } k = n, \\ \dfrac{1}{\alpha_n}\displaystyle\prod_{i=k}^{n-1}\frac{\beta_i}{\alpha_i} & \text{if } k < n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Delta(\alpha) = \widehat{\Delta}(\alpha,\alpha)$ and $C(\alpha) = \widehat{C}(\alpha,\alpha)$. Then $\Delta(\alpha)$ is the triangle defined by:

$$\Delta(\alpha)_{nk} = \begin{cases} \alpha_n & \text{for } k = n \\ -\alpha_{n-1} & \text{for } k = n-1 \\ 0 & \text{for } k \neq n-1 \text{ and } k \neq n \ (n \geq 1) \end{cases}$$

and $C(\alpha)$ is the triangle defined by:

$$C(\alpha)_{n,k} = \begin{cases} 1/\alpha_n & \text{for } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

Note that $C(\alpha)$ is the inverse of $\Delta(\alpha)$.

Let $e \in U$, defined by $e_n = 1$ for all $n \geq 1$. Then $\Delta = \Delta(e)$ is the well known operator of the first-difference defined by

$$\Delta_n(x) = x_n - x_{n-1} \text{ for all } n \geq 1,$$

with the convention $x_0 = 0$. Recall that the operator Δ is invertible and its inverse is usually written $\Sigma = C(e)$.

Finally, we recall some properties on Laplace transformations that are useful in the sequel. For a function f of one variable t, we define the Laplace transformation of f as follows:

$$F(p) = \int_0^{+\infty} f(t)\; e^{-pt}dt$$

where $p \in \mathbb{C}$ is a new variable. We denote by $\pounds : f \longmapsto F$, the Laplace operator.

Lemma 2.2. *Let $\pounds^{-1}$ be the inverse mapping of $\pounds$. Let $c \in \mathbb{R}$ and f be a function of one variable t. Then the following properties hold:*

1. *$\pounds$ and $\pounds^{-1}$ are linear operators.*
2. *$\pounds(c) = \frac{c}{p}$.*
3. *$\pounds(f'(t)) = p\,\pounds(f(t)) - f(0)$.*
4. *For all $m \in \mathbb{N}$,*

$$\pounds^{-1}\Big(\frac{1}{(p-c)^m}\Big) = \frac{t^{m-1}e^{ct}}{(m-1)!}\ .$$

3. The equation $x'(t) = \Delta(\lambda)\,x(t) + b$

Let $\lambda = (\lambda_n)_{n\geq 1} \in U$ be a sequence. We consider the equation

$$\begin{cases} x'(t) &= \Delta(\lambda)\,x(t) + b \\ x(0) &= a \end{cases} \tag{3.1}$$

where $a = (a_n)_{n\geq 1}$ and $b = (b_n)_{n\geq 1}$ are given sequences.
Equation (3.1) is equivalent to the following infinite linear system:

$$\begin{cases} x_n'(t) = (\Delta(\lambda))_n(x(t)) + b_n \\ \qquad\quad = \lambda_n x_n(t) - \lambda_{n-1}x_{n-1}(t) + b_n \\ x_n(0) = a_n \qquad n = 1, 2, ... \end{cases} \tag{3.2}$$

with the convention $\lambda_0 = 0$ and $x_0(t) = 0$.

3.1. Case when all the entries λ_n of λ are pairwisely distinct

In this case, we need the following lemma whose its proof is immediate:

Lemma 3.1. *Let $c_k, \ldots, c_n$ be pairwisely distinct real numbers where $k \leq n$ is an integer and let*

$$F(z) = \frac{1}{z - c_n}\prod_{i=k}^{n-1}\frac{c_i}{z - c_i}$$

Then the decomposition of $F(z)$ into simple fractions is given by:

$$F(z) = \frac{A_n}{z - c_n} + \sum_{i=k}^{n-1}\frac{A_i}{z - c_i}$$

where

$$A_n = \prod_{i=k}^{n-1}\frac{c_i}{c_n - c_i} \quad \textit{and} \quad A_i = \frac{c_i}{c_i - c_n}\prod_{j=k, j\neq i}^{n-1}\frac{c_j}{c_i - c_j} \quad \textit{for all } i = k, \ldots, n-1.$$

The main result of this section is stated in the following theorem:

Theorem 3.2. *Equation (3.1) has a unique solution which is given for each $n \geq 1$, by*

$$\begin{aligned} x_n(t) \quad &= \quad \frac{-1}{\lambda_n}\sum_{k=1}^{n} b_k + \left[a_n + \frac{b_n}{\lambda_n} + \sum_{k=1}^{n-1}(-1)^{n-k}\left(a_k \ + \frac{b_k}{\lambda_n}\right)A_n\right]e^{\lambda_n t} \\ &\quad + \sum_{k=1}^{n-1}\left[\sum_{i=k}^{n-1}(-1)^{n-k}\left(a_k + \frac{b_k}{\lambda_i}\right)A_i e^{\lambda_i t}\right], \end{aligned} \tag{3.3}$$

where

$$A_n = \prod_{i=k}^{n-1}\frac{\lambda_i}{\lambda_n - \lambda_i} \qquad \textit{and} \qquad A_i = \frac{\lambda_i}{\lambda_i - \lambda_n}\prod_{\substack{j=k \\ j\neq i}}^{n-1}\frac{\lambda_j}{\lambda_i - \lambda_j}$$

for all $k = 1, \ldots, n-1$ and $i = k, \ldots, n-1$.

Proof. Applying Laplace operator $\pounds$ to equations (3.2) and using Lemma 2.2, we obtain the equations:

$$\lambda_{n-1} X_{n-1} + (p - \lambda_n) X_n = a_n + \frac{b_n}{p}, \qquad n = 1, 2, ... \tag{3.4}$$

where for all $n \geq 1$,

$$X_n = X_n(p) = \pounds(x_n(t)) \text{ and } X_0 = 0, \ \lambda_0 = 0$$

Then we obtain the following infinite linear system $\widehat{\Delta}(p - \lambda, -\lambda) X = b'$ where

$$\widehat{\Delta}(p-\lambda, -\lambda) = \begin{pmatrix} p-\lambda_1 & & & & \\ \lambda_1 & p-\lambda_2 & & \mathbf{0} & \\ & \ddots & \ddots & & \\ \mathbf{0} & & \lambda_{n-1} & p-\lambda_n & \\ & & & \ddots & \ddots \end{pmatrix},$$

$b' = (b'_n)_{n \geq 1}$ and

$$b'_n = a_n + \frac{b_n}{p} \quad \text{for all } n \geq 1.$$

By Lemma 2.1, we obtain

$$\left[\widehat{\Delta}(p-\lambda, -\lambda)^{-1}\right]_{nk} = \begin{cases} \frac{1}{p-\lambda_n} & \text{if } k = n, \\ \frac{1}{p-\lambda_n} \prod\limits_{i=k}^{n-1} \frac{-\lambda_i}{p-\lambda_i} & \text{if } k < n, \\ 0 & \text{otherwise} \end{cases}$$

that is

$$\widehat{\Delta}(p-\lambda, -\lambda)^{-1} = \begin{pmatrix} \frac{1}{p-\lambda_1} & & & & & \\ \vdots & \ddots & & \mathbf{0} & & \\ \vdots & \vdots & \frac{1}{p-\lambda_k} & & & \\ \vdots & \vdots & \vdots & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{p-\lambda_n} \prod\limits_{i=1}^{n-1} \frac{-\lambda_i}{p-\lambda_i} & \cdots & \frac{1}{p-\lambda_n} \prod\limits_{i=k}^{n-1} \frac{-\lambda_i}{p-\lambda_i} & \cdots & \frac{1}{p-\lambda_n} & \\ \vdots & \vdots & \vdots & . & . & \ddots \end{pmatrix};$$

Then for all $n \geq 1$, we have

$$\begin{aligned} X_n &= \frac{1}{p-\lambda_n}\left[b'_n + \sum_{k=1}^{n-1}\left(\prod_{i=k}^{n-1}\frac{-\lambda_i}{p-\lambda_i}\right) b'_k\right] \\ &= \frac{a_n}{p-\lambda_n} + \frac{b_n}{p(p-\lambda_n)} + \sum_{k=1}^{n-1}\frac{(-1)^{n-k}}{p-\lambda_n}\left(\prod_{i=k}^{n-1}\frac{\lambda_i}{p-\lambda_i}\right)\left(a_k + \frac{b_k}{p}\right). \end{aligned} \tag{3.5}$$

To simplify this expression, we write

$$X_n = \frac{a_n}{p-\lambda_n} + \frac{b_n}{p(p-\lambda_n)} + \sum_{k=1}^{n-1}(-1)^{n-k}\Big[a_k F_{nk}(p) + b_k G_{nk}(p)\Big] \tag{3.6}$$

where

$$F_{nk}(p) = \frac{1}{p-\lambda_n}\prod_{i=k}^{n-1}\frac{\lambda_i}{p-\lambda_i} \quad \text{and} \quad G_{nk}(p) = \frac{1}{p(p-\lambda_n)}\prod_{i=k}^{n-1}\frac{\lambda_i}{p-\lambda_i} = \frac{F_{nk}(p)}{p}.$$

By Lemma 3.1, we obtain the following decompositions into simple fractions:

$$F_{nk}(p) = \frac{A_n}{p-\lambda_n} + \sum_{i=k}^{n-1} \frac{A_i}{p-\lambda_i}$$

and

$$G_{nk}(p) = \frac{A_0}{p} + \frac{A_n}{\lambda_n}\frac{1}{p-\lambda_n} + \sum_{i=k}^{n-1} \frac{A_i}{\lambda_i}\frac{1}{p-\lambda_i}$$

where

$$A_n = \prod_{i=k}^{n-1} \frac{\lambda_i}{\lambda_n-\lambda_i}, \qquad A_0 = \frac{(-1)^{n-k+1}}{\lambda_n} \quad \text{and} \quad A_i = \frac{\lambda_i}{\lambda_i-\lambda_n} \prod_{\substack{j=k\\ j\neq i}}^{n-1} \frac{\lambda_j}{\lambda_i-\lambda_j}.$$

Applying $\pounds^{-1}$ to (3.6) we obtain

$$\begin{aligned} x_n(t) &= a_n\ \pounds^{-1}\left(\frac{1}{p-\lambda_n}\right) + b_n\ \pounds^{-1}\left[\frac{1}{p(p-\lambda_n)}\right] \\ &\quad + \sum_{k=1}^{n-1} (-1)^{n-k}\Big[a_k\ \pounds^{-1}(F_{nk}(p)) + b_k\ \pounds^{-1}(G_{nk}(p))\Big], \end{aligned} \tag{3.7}$$

but in the other hand, using Lemma 2.2, we obtain:

$$\pounds^{-1}\left[\frac{1}{p(p-\lambda_n)}\right] = \pounds^{-1}\left[-\frac{1}{p\lambda_n} + \frac{1}{\lambda_n(p-\lambda_n)}\right] = -\frac{1}{\lambda_n} + \frac{1}{\lambda_n}e^{\lambda_n t} \tag{3.8}$$

$$\pounds^{-1}(F_{nk}(p)) = A_n e^{\lambda_n t} + \sum_{i=k}^{n-1} A_i e^{\lambda_i t} \tag{3.9}$$

and

$$\pounds^{-1}(G_{nk}(p)) = A_0 + \frac{A_n}{\lambda_n}e^{\lambda_n t} + \sum_{i=k}^{n-1} \frac{A_i}{\lambda_i}e^{\lambda_i t}. \tag{3.10}$$

Replacing Formulas (3.8), (3.9) and (3.10) in (3.7), we obtain:

$$\begin{aligned} x_n(t) &= \frac{-1}{\lambda_n}\sum_{k=1}^{n} b_k + \left[a_n + \frac{b_n}{\lambda_n} + \sum_{k=1}^{n-1}(-1)^{n-k}\left(a_k + \frac{b_k}{\lambda_n}\right)A_n\right]e^{\lambda_n t} \\ &\quad + \sum_{k=1}^{n-1}\left[\sum_{i=k}^{n-1}(-1)^{n-k}\left(a_k + \frac{b_k}{\lambda_i}\right)A_i e^{\lambda_i t}\right]. \end{aligned}$$

□

3.2. Case where all terms λ_n of λ are equals

In this subsection, we suppose that $\lambda_n = c$ for all $n \geq 1$ where c is a constant. Then infinite linear differential system (3.2) becomes:

$$\begin{cases} x_n'(t) = cx_n(t) - cx_{n-1}(t) + b_n, \\ x_n(0) = a_n, \qquad n = 1, 2, ... \end{cases} \tag{3.11}$$

After applying Laplace operator $\pounds$ to equations (3.11) we obtain the equations

$$cX_{n-1} + (p-c)X_n = a_n + \frac{b_n}{p}, \qquad n = 1, 2, \ldots \tag{3.12}$$

The solvability of (3.11) can be obtained reasoning as in Theorem 3.2, with $F_{nk} = \frac{c^{n-k}}{(p-c)^{n-k+1}}$ and $G_{nk} = \frac{F_{nk}}{p}$, but here we explicitly calculate $X_1, X_2, \ldots, X_n$ from (3.12) by mathematical induction, for short. In this way, we have,

Theorem 3.3. *The infinite linear differential system (3.11) has a unique solution which is given by:*

$$x_n(t) = -\frac{S_n}{c} + \sum_{k=1}^{n} (-1)^{n-k} \frac{c^{n-k-1}(ca_k + S_k)}{(n-k)!} t^{n-k} e^{ct},$$

for all $n \geq 1$, where $S_k = \sum_{j=1}^{k} b_j$ for all $k = 1, \ldots, n$.

Proof. Let's show by induction on n that

$$X_n = -\frac{S_n}{cp} + \sum_{k=1}^{n} (-1)^{n-k} \frac{c^{n-k-1}(ca_k + S_k)}{(p-c)^{n-k+1}}. \tag{3.13}$$

For $n = 1$, with the convention that $X_0 = 0$, equation (3.12) gives

$$\begin{aligned} X_1 &= \frac{a_1}{p-c} + \frac{b_1}{p(p-c)} \\ &= \frac{a_1}{p-c} + \frac{b_1}{c}\Big(\frac{1}{p-c} - \frac{1}{p}\Big) \\ &= -\frac{b_1}{cp} + \frac{ca_1 + b_1}{c(p-c)} \\ &= -\frac{S_1}{cp} + \frac{c^{-1}(ca_1 + S_1)}{p-c} \end{aligned}$$

Then the formula is true for $n = 1$. Suppose it is true for $n-1$ and prove it for n. Replacing X_{n-1} in equations (3.12), this leads to:

$$\begin{aligned} (p-c)X_n &= a_n + \frac{b_n}{p} + \frac{S_{n-1}}{p} - c\sum_{k=1}^{n-1} (-1)^{n-1-k+1} \frac{c^{n-1-k-1}(ca_k + S_k)}{(p-c)^{n-1-k+1}} \\ &= a_n + \frac{S_n}{p} + \sum_{k=1}^{n-1} (-1)^{n-k} \frac{c^{n-k-1}(ca_k + S_k)}{(p-c)^{n-k}}. \end{aligned}$$

Then

$$\begin{aligned} X_n &= \frac{a_n}{p-c} + \frac{S_n}{p(p-c)} + \sum_{k=1}^{n-1} (-1)^{n-k} \frac{c^{n-k-1}(ca_k + S_k)}{(p-c)^{n-k+1}} \\ &= \frac{a_n}{p-c} + \frac{S_n}{c}\Big(\frac{1}{p-c} - \frac{1}{p}\Big) + \sum_{k=1}^{n-1} (-1)^{n-k} \frac{c^{n-k-1}(ca_k + S_k)}{(p-c)^{n-k+1}} \\ &= -\frac{S_n}{cp} + \sum_{k=1}^{n} (-1)^{n-k} \frac{c^{n-k-1}(ca_k + S_k)}{(p-c)^{n-k+1}}. \end{aligned}$$

Applying $\pounds^{-1}$ to equation (3.13) and using part (4.) of Lemma 2.2, we obtain:

$$\begin{aligned} x_n(t) &= \pounds^{-1}(X_n(p)) \\ &= -\frac{S_n}{c} + \sum_{k=1}^{n} (-1)^{n-k} \frac{c^{n-k-1}(ca_k + S_k)}{(n-k)!} t^{n-k} e^{ct}. \end{aligned}$$

□

4. Examples

Example 4.1. If $\lambda_n = n$, $b_n = 1$ and $a_n = 0$ for all $n \geq 1$, then the infinite linear differential system (3.2) becomes:

$$\begin{cases} x_n'(t) = nx_n(t) - (n-1)x_{n-1}(t) + 1 \\ x_n(0) = a_n, \qquad n = 1, 2, \ldots \end{cases} \tag{4.1}$$

Then the unique solution of the system (4.1) is given by Theorem 3.2:

$$x_n(t) = -1 + \frac{1}{n}\Big[1 + \sum_{k=1}^{n-1}(-1)^{n-k}A_n\Big]e^{nt} + \sum_{k=1}^{n-1}\left[\sum_{s=k}^{n-1}\frac{(-1)^{n-k}}{s}A_s e^{st}\right]$$

for all $n \geq 1$, where

$$A_n = \frac{(n-1)!}{(k-1)!(n-k)!} = C_{n-1}^{k-1}$$

and

$$\begin{aligned}
A_s &= \frac{s}{s-n}\prod_{\substack{j=k\\ j\neq s}}^{n-1}\frac{j}{s-j}\\
&= \frac{s}{s-n}\prod_{j=k}^{s-1}\frac{j}{s-j}\prod_{j=s+1}^{n-1}\frac{j}{s-j}\\
&= (-1)^{n-s-1}\times\frac{s}{s-n}\times\frac{(s-1)!}{(k-1)!\,(s-k)!}\times\frac{(n-1)!}{s!\,(n-s-1)!}\\
&= (-1)^{n-s-1}\times\frac{s}{s-n}\times C_{s-1}^{k-1}\times C_{n-1}^{s}.
\end{aligned}$$

But

$$\begin{aligned}
\sum_{k=1}^{n-1}(-1)^{n-k}A_n &= \sum_{k=1}^{n-1}(-1)^{n-k}\times C_{n-1}^{k-1}\\
&= \sum_{j=0}^{n-2}(-1)^{n-1-j}\times C_{n-1}^{j} \qquad (j = k-1)\\
&= -1 + \sum_{j=0}^{n-1}C_{n-1}^{j}\times 1^j\times(-1)^{n-1-j}\\
&= -1 + (1-1)^{n-1} \qquad \text{(using Newton binomial formula)}\\
&= -1.
\end{aligned}$$

Thus

$$x_n(t) = -1 + \sum_{k=1}^{n-1}\left[\sum_{s=k}^{n-1}\frac{(-1)^{k+s+1}}{s-n}\;C_{s-1}^{k-1}\,C_{n-1}^{s}\,e^{st}\right],$$

for all $n \geq 1$.

Example 4.2. If $\lambda = b = e$, i.e., $\lambda_n = b_n = 1$ for all $n \geq 1$ and $a_n = 0$ for all $n \geq 1$, then $\Delta(\lambda) = \Delta$ is the operator of the first-difference and the infinite linear differential system (3.11) becomes:

$$\begin{cases} x_n'(t) = x_n(t) - x_{n-1}(t) + 1\\ x_n(0) = a_n, \qquad n = 1, 2, \ldots \end{cases} \tag{4.2}$$

Then $S_k = k$ for all $k = 1, \ldots, n$ and the unique solution of the system (4.2) is given by Theorem 3.3:

$$x_n(t) = -n + \sum_{k=1}^{n}(-1)^{n-k}\frac{k}{(n-k)!}\;t^{n-k}e^t,$$

for all $n \geq 1$.

Conclusion

In this paper, we have proposed a new method based on Laplace transformation for solving particular infinite linear systems of differential equations. This leads to solve infinite linear systems. A future work is to consider other particular systems of differential equations, like systems defined by Césaro's operator.

References

[1] E.L. Ince, *Periodic solutions of a linear differential equation of the second order*. Proc. Cam. Philo. Soc., **23** (1926), 44-46. 1

[2] K.C-L. Issic, P.N. Shivakumar, *On the eigenvalue problem $-y'' + f(x)y = \lambda y$ on a semi infinite interval,* Mathematical and Computer Modelling **46** (2007), 316-330. 1

[3] A. Farés, *Contribution à l'étude des opérateurs dans des espaces de suites et applications à l'optimisation et aux systèmes différentiels*, PhD thesis, University of Le Havre, France (2009).

[4] B. de Malafosse, *Recent results in the infinite matrix theory and application to Hill equation,* Demonstratio Matematica, **35**, N°1, (2002), 11-26. 1

[5] B. de Malafosse, *An application of the infinite matrix theory to Mathieu equation,* Comput. Math. Appl. **52** (2006), 1439-1452. 1

[6] J.T. Okutoyi, *On the spectrum of C_1 as operator on bv*, Commun. Fac. Sci. Univ. Ank. Series A_1, **41**, (1992), 197-207. 1

[7] H. Poincaré, *Sur les déterminants d'ordre infini.*, Bull. Soc. Math. Fr., **14,87**, (1886). 1

[8] J.B. Reade, *On the spectrum of the Cesàro operator*, Bull. London Math., Soc. **17**, (1985), 263-267. 1

[9] B. Rosseto, *Détermination des exposants de Floquet de l'équation de Hill d'ordre n*. Thèse de Doctorat d'Etat Es-Sciences, Univ. Toulon, 1983. 1

[10] K.G. Valeev, *On Hill's method in the theory of linear differential equations with periodic coefficients*. J. Appl. Math. Mech., transl. of Prikl. MAT., **24** (1960), 1493-1505.

1