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Hyers-Ulam–Rassias stability of Pexiderized Cauchy functional equation in 2-Banach spaces

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Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

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Abstract

In this paper, we investigate stability of the Pexiderized Cauchy functional equation in 2-Banach spaces and pose an open problem.

Keywords: Linear 2-normed space, Generalized Hyers-Ulam stability, Pexiderized Cauchy functional equation .
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1. Introduction and preliminaries

In 1940, S.M. Ulam [34] asked the first question on the stability problem for mappings. In 1941, D. H. Hyers [14] solved the problem of Ulam. This result was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [24] for linear mappings by considering an *unbounded Cauchy difference.* The paper of Th. M. Rassias has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. In 1994, a further generalization was obtained by P. Găvruta [13]. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [5]-[12],[16]-[20],[22]-[23], [25]-[29], [32],[33]). We also refer the readers to the books: P. Czerwik [4] and D.H. Hyers, G. Isac and Th.M. Rassias [15].

In the 1960s, S. Gahler [8, 9] introduced the concept of linear 2-normed spaces.

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Definition 1.1. Let X be a linear space over $\mathbb{R}$ with $dimX > 1$ and let $\|.,.\| : X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:
(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(b) $\|x, y\| = \|y, x\|$,
(c)$\|\lambda x, y\| = |\lambda|\|x, y\|$,
(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$
for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|.,.\|$ is called a 2-norm on X and the pair $(X, \|.,.\|)$ is called a linear 2-normed space. Sometimes the condition (d) called the triangle inequality.

It follows from (d), that $\|x+y, z\| \leq \|x, z\|+\|y, z\|$ and $\big|\|x, z\|-\|y, z\|\big| \leq \|x-y, z\|$. Hence the functions $x \longrightarrow \|x, y\|$ are continuous functions of X into $\mathbb{R}$ for each fixed $y \in X$.

Lemma 1.2. *([21]) Let $(X, \|.,.\|)$ be a linear 2-normed space. If $x \in X$ and $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.*

Definition 1.3. A sequence $\{x_n\}$ in a linear 2-normed space X is called a Cauchy sequence if there are two points $y, z \in X$ such that y and z are linearly independent,
$\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0$ and $\lim_{m,n\to\infty} \|x_m - x_n, z\| = 0$.

Definition 1.4. A sequence $\{x_n\}$ in a linear 2-normed space X is called a convergent sequence if there is an $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0,$$

for all $y \in X$. If $\{x_n\}$ converges to x, write $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.

Definition 1.5. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.

Lemma 1.6. *([21]) For a convergent sequence $\{x_n\}$ in a linear 2-normed space X,*

$$\lim_{n\to\infty} \|x_n, y\| = \| \lim_{n\to\infty} x_n, y\|,$$

for all $y \in X$.

In [21] Won-Gil Park has investigated approximate additive mappings, approximate Jensen mappings and approximate quadratic mappings in 2-Banach spaces. In this paper, we investigate stability of the Pexiderized Cauchy functional equation in 2-Banach spaces and pose an open problem.

2. Stability of the Pexiderized Cauchy functional equation

Throughout this paper, let X be a normed linear space, Y be a 2-Banach space with $dimY > 1$ and k is a fixed integer greater than 1.

Theorem 2.1. *Let $\varphi : X \times X \times X \longrightarrow [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{k^n}\varphi(k^n x, k^n y, z) = 0, \tag{2.1}$$

for all $x, y, z \in X$. Suppose that $f, g, h : X \longrightarrow Y$ be mappings with $f(0) = g(0) = h(0) = 0$ and

$$\|f(x + y) - g(x) - h(y), z\| \leq \varphi(x, y, z), \tag{2.2}$$

and

$$\widetilde{M}_k(x,z) := \sum_{n=0}^{\infty}\sum_{i=1}^{k-1}\frac{M(k^n x, ik^n x, z)}{k^n}, \tag{2.3}$$

exist for all $x, y, z \in X$, where $M(x,y,z) := \varphi(x,y,z)+\varphi(0,y,z)+\varphi(x,0,z)$. Then there is a unique additive mapping $A_k : X \longrightarrow Y$ such that

$$\|f(x) - A_k(x), z\| \le \frac{1}{k}\widetilde{M}_k(x,z) \tag{2.4}$$

for all $x, z \in X$.

Proof. Replacing $y = 0$ in (2.2), we get

$$\|f(x) - g(x), z\| \le \varphi(x, 0, z), \tag{2.5}$$

for all $x, z \in X$. Replacing $x = 0$ in (2.2), we get

$$\|f(y) - h(y), z\| \le \varphi(0, y, z), \tag{2.6}$$

for all $y, z \in X$. By using (2.5) and (2.6), we get

$$\|f(x+y) - f(x) - f(y), z\| \le M(x, y, z), \tag{2.7}$$

for all $x, y, z \in X$, where

$$M(x,y,z) := \varphi(x,y,z) + \varphi(x,0,z) + \varphi(0,y,z).$$

By induction on k, we show that

$$\|f(kx) - kf(x), z\| \le M_k(x,z), \tag{2.8}$$

for all $x, z \in X$, where $M_k(x,z) := \sum_{i=1}^{k-1} M(x, ix, z)$. Letting $y = x$ in (2.7), we get

$$\|f(2x) - 2f(x), z\| \le M(x, x, z), \tag{2.9}$$

for all $x, z \in X$. So we get (2.8) for $k = 2$.
Assume that (2.8) holds for k. Letting $y = kx$ in (2.7), we get

$$\|f\big((k+1)x\big) - f(x) - f(kx), z\| \le M(x, kx, z) \tag{2.10}$$

for all $x, z \in X$. It follows from (2.8) and (2.10) that

$$\begin{aligned}\|f\big((k+1)x\big) &- (k+1)f(x), z\| \\ &\le \|f\big((k+1)x\big) - f(x) - f(kx), z\| + \|f(kx) - kf(x), z\| \\ &\le M_{k+1}(x,z),\end{aligned}$$

This completes the induction argument. Replacing x by $k^n x$ in (2.8) and dividing both sides of (2.8) by k^{n+1}, we get

$$\left\|\frac{1}{k^{n+1}}f(k^{n+1}x) - \frac{1}{k^n}f(k^n x), z\right\| \le \frac{1}{k^{n+1}}M_k(k^n x, z), \tag{2.11}$$

for all $x, y \in X$ and all non-negative integers n. Hence

$$\begin{aligned}\left\|\frac{1}{k^{n+1}}f(k^{n+1}x) - \frac{1}{k^m}f(k^m x), z\right\| &\le \sum_{i=m}^{n}\left\|\frac{1}{k^{i+1}}f(k^{i+1}x) - \frac{1}{k^i}f(k^i x), z\right\| \\ &\le \frac{1}{k}\sum_{i=m}^{n}\frac{1}{k^i}M_k(k^i x, z),\end{aligned} \tag{2.12}$$

for all $x, z \in X$ and all non-negative integers m and n with $n \geq m$. Therefore, we conclude from (2.3) and (2.12) that the sequence $\{\frac{1}{k^n} f(k^n x)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete the sequence $\{\frac{1}{k^n} f(k^n x)\}$ converges in Y for all $x \in X$. So one can define the mapping $A_k : X \to Y$ by:

$$A_k(x) := \lim_{n\to\infty} \frac{1}{k^n} f(k^n x), \tag{2.13}$$

for all $x \in X$. That is

$$\lim_{n\to\infty} \|\frac{1}{k^n} f(k^n x) - A_k(x), y\| = 0,$$

for all $x, y \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (2.12), we get (2.4). Now, we show that $A_k : X \to Y$ is an additive mapping. It follows from (2.1), (2.7), (2.13) and Lemma 1.6 that

$$\begin{aligned}\|A_k(x+y) - A_k(x) - A_k(y), z\| &= \lim_{n\to\infty} \frac{1}{k^n} \|f(k^n x + k^n y) - f(k^n x) - f(k^n y), z\| \\ &\leq \lim_{n\to\infty} \frac{1}{k^n} M(k^n x, k^n y, z) = 0,\end{aligned}$$

for all $x, y, z \in X$. By Lemma 1.2, $A_k(x+y) - A_k(x) - A_k(y) = 0$ for all $x, y \in X$. So the mapping $A_k : X \to Y$ is additive.

To prove the uniqueness of A_k, let $T : X \to Y$ be another additive mapping satisfying (2.4). Then

$$\begin{aligned}\|A_k(x) - T(x), z\| &= \lim_{n\to\infty} \frac{1}{k^n} \big\|A_k(k^n x) - T(k^n x), z\big\| \\ &\leq \lim_{n\to\infty} \frac{1}{k^{n+1}} \widetilde{M}_k(k^n x, z) = 0,\end{aligned}$$

for all $x, z \in X$. By Lemma 1.2, $A_k(x) - f(x) = 0$ for all $x, y \in X$. So $A_k = T$. □

Theorem 2.2. *Let $\psi : [0, \infty) \to [0, \infty)$ be a function such that $\psi(0) = 0$ and*

1. $\psi(ts) \leq \psi(t)\psi(s)$,
2. $\psi(t) < t$ *for all* $t > 1$.

Suppose that $f, g, h : X \longrightarrow Y$ be mappings with $f(0) = g(0) = h(0) = 0$ and

$$\|f(x+y) - g(x) - h(y), z\| \leq \psi(\|x\|_X) + \psi(\|y\|_X) + \psi(\|z\|_X), \tag{2.14}$$

for all $x, y, z \in X$. Then there is a unique additive mapping $A_k : X \to Y$ satisfying

$$\|f(x) - A_k(x), z\| \leq \frac{2\sum_{i=1}^{k-1}(1+\psi(i))}{k - \psi(k)} \psi(\|x\|_X) + 3\psi(\|z\|_X), \tag{2.15}$$

for all $x, z \in X$. Moreover, $A_k = A_2$ for all $k \geq 2$.

Proof. Let

$$\varphi(x, y, z) = \psi(\|x\|_X) + \psi(\|y\|_X) + \psi(\|z\|_X),$$

for all $x, y, z \in X$. It follows from (1) that $\psi(k^n) \leq \big(\psi(k)\big)^n$ and

$$\varphi(k^n x, k^n y, z) \leq (\psi(k))^n \big(\psi(\|x\|_X) + \psi(\|y\|_X)\big) + \psi(\|z\|_X)$$

By using Theorem 2.1, we can get (2.15). Now, we show that $A_k = A_2$. It follows from Theorem 2.1 that $A_k(x) = \lim_{n\to\infty} \frac{1}{k^n} f(k^n x)$. Replacing x by $2^n x$ in (2.15) and dividing both sides of (2.15) by 2^n, we get

$$\begin{aligned} \|\frac{f(2^n x)}{2^n} - A_k(x), z\| &\leq \frac{2\sum_{i=1}^{k-1}(1+\psi(i))}{k-\psi(k)} \frac{\psi(\|2^n x\|_X)}{2^n} + \frac{3\psi(\|z\|_X)}{2^n} \\ &\leq \frac{2\sum_{i=1}^{k-1}(1+\psi(i))}{k-\psi(k)} \psi(\|x\|_X)\frac{\psi(2^n)}{2^n} + \frac{3\psi(\|z\|_X)}{2^n} \end{aligned} \tag{2.16}$$

for all $x, z \in X$. By passing the limit $n \longrightarrow \infty$ in (2.16), we get $A_k = A_2$. □

Theorem 2.3. *Let p, q be non-negative real numbers such that $p > 0, q < 1$ and $H : [0,\infty)\times[0,\infty) \to [0,\infty)$ be a homogeneous function of degree q. Suppose that $f, g, h : X \longrightarrow Y$ be mappings with $f(0) = g(0) = h(0) = 0$ and*

$$\|f(x+y) - g(x) - h(y), z\| \leq H(\|x\|_X, \|y\|_X) + \|z\|_X^p,$$

for all $x, y, z \in X$. Then thereis a unique additive mapping $A_k : X \to Y$ such that

$$\|f(x) - A_k(x), z\| \leq \frac{\sigma_k(H)}{k-k^q}\|x\|_X^q + 3\|z\|_X^p \tag{2.17}$$

for all $x \in X$, where $\sigma_k(H) := (k-1)H(1,0) + \sum_{i=1}^{k-1} H(1,i) + H(0,i)$. Moreover, $A_k = A_2$ for all $k \geq 2$.

Proof. Let

$$\varphi(x, y, z) = H(\|x\|_X, \|y\|_X) + \|z\|_X^p$$

for all $x, y, z \in X$. By using Theorem 2.1, we can get (2.17). Now, we show that $A_k = A_2$. It follows from Theorem 2.1 that $A_k(x) = \lim_{n\to\infty} \frac{1}{k^n} f(k^n x)$. Replacing x by $2^n x$ in (2.17) and dividing both sides of (2.17) by 2^n, we get

$$\|\frac{f(2^n x)}{2^n} - A_k(x), z\| \leq \frac{\sigma_k(H)}{k-k^q}\frac{\|2^n x\|_X^q}{2^n} + \frac{3\|z\|_X^p}{2^n} \tag{2.18}$$

for all $x \in X$. By passing the limit $n \longrightarrow \infty$ in (2.18), we get

$$\lim_{n\to\infty} \|\frac{f(2^n x)}{2^n} - A_k(x), z\| = 0,$$
$$\|\lim_{n\to\infty} \frac{f(2^n x)}{2^n} - A_k(x), z\| = 0,$$

for all $x, z \in X$. By Lemma 1.2, $\lim_{n\to\infty} \frac{f(2^n x)}{2^n} - A_k(x) = 0$, so $A_k = A_2$. □

Theorem 2.4. *Let p, q be non-negative real numbers such that $p > 0, q < 1$ and $H : [0,\infty)\times[0,\infty) \to [0,\infty)$ be a homogeneous function of degree q. Suppose that $f, g, h : X \longrightarrow Y$ be mappings with $f(0) = g(0) = h(0) = 0$ and*

$$\|f(x+y) - g(x) - h(y), z\| \leq H(\|x\|_X, \|y\|_X)\|z\|_X^p,$$

for all $x, y, z \in X$. Then there is a unique additive mapping $A_k : X \to Y$ such that

$$\|f(x) - A_k(x), z\| \leq \frac{\sigma_k(H)}{k-k^q}\|x\|_X^q\|z\|_X^p \tag{2.19}$$

for all $x, z \in X$, where $\sigma_k(H) := (k-1)H(1,0) + \sum_{i=1}^{k-1} H(1,i) + H(0,i)$. Moreover, $A_k = A_2$ for all $k \geq 2$.

Corollary 2.5. *Let p be real number such that $0 < p < 1$. Suppose that $f, g, h : X \longrightarrow Y$ be mappings with $f(0) = g(0) = h(0) = 0$ and*

$$\|f(x+y) - g(x) - h(y), z\| \leq \|x\|_X^p + \|y\|_X^p + \|z\|_X^p,$$

for all $x, y, z \in X$. Then there is a unique additive mapping $A_k : X \to Y$ such that

$$\|f(x) - A_k(x), z\| \leq \frac{2(k-1) + 2(1^p + 2^p + ... + (k-1)^p)}{k - k^p} \|x\|_X^p + 3\|z\|_X^p$$

for all $x, z \in X$, Moreover, $A_k = A_2$ for all $k \geq 2$.

Corollary 2.6. *Let r, s, p be real numbers such that $p > 0, r + s < 1$. Suppose that $f, g, h : X \longrightarrow Y$ be mappings with $f(0) = g(0) = h(0) = 0$ and*

$$\|f(x+y) - g(x) - h(y), z\| \leq \|x\|_X^r \|y\|_X^s \|z\|_X^p$$

for all $x, y, z \in X$. Then there is a unique additive mapping $A_k : X \to Y$ such that

$$\|f(x) - A_k(x), z\| \leq \frac{1^s + 2^s + ... + (k-1)^s}{k - k^{r+s}} \|x\|_X^{r+s} \|z\|_X^p$$

for all $x, z \in X$, Moreover, $A_k = A_2$ for all $k \geq 2$ and

$$f(x) = g(x) = h(x),$$

for all $x \in X$.

Open problem: What is the best possible value of k in Corollaries 2.5 and 2.6?

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