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Convergence results for solutions of a first-order differential equation

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To the memory of Viorel Radu, a friend and companion on the probabilistic topological ways

Communicated by Professor D. Miheţ

Abstract

We consider the first order differential problem:

$$(P_n) \qquad \begin{cases} u'(t) = f_n(t, u(t)), \text{ for almost every } t \in [0,1], \\ u(0) = 0. \end{cases}$$

Under certain conditions on the functions f_n, the problem (P_n) admits a unique solution $u_n \in W^{1,1}([0,1], E)$. In this paper, we propose to study the limit behavior of sequences $(u_n)_{n\in\mathbb{N}}$ and $(u'_n)_{n\in\mathbb{N}}$, when the sequence $(f_n)_{n\in\mathbb{N}}$ is subject to different growing conditions.

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1. Introduction

The subject treated below is inspired by the paper [4], a work that studies the results to limit for the sequence $(u_n)_{n\in\mathbb{N}}$ of solutions of second order differential equations:

$$\begin{cases} u''(t) = f_n(t, u(t), u'(t)), \\ u(0) = u(1) = 0. \end{cases}$$

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In the quoted work the basic tools used are Helly's compactness theorem for the sequence of derivatives $(u'_n)_{n\in\mathbb{N}}$ and Prokhorov's compactness theorem for the tight sequence of derivatives of second order, $(u''_n)_{n\in\mathbb{N}}$. The application of these compactness results was possible due to the assumption that the sequence $(f_n(\cdot,0,0))_{n\in\mathbb{N}}$ is bounded in $L^1([0,1],E)$.

The question was whether such limit results remain valid for some unbounded sequences of $L^1([0,1],E)$.

The boundedness of a sequence in $L^1([0,1],E)$ provides, in addition to its tightness (permitting Prokhorov's compactness theorem), the using of a weak compactness result: Biting lemma.

As we have seen (see [7]), these results continue to function acceptably for a class of unbounded sequences in $L^1([0,1],E)$ - the Jordan finite-tight sequences. In this context, the theorem of Fréchet of compactness in measure will replace the more restrictive theorem of Helly.

In a first variant, we introduced Jordan finite-tight sets in the case of real functions of real variable in [6], where we presented an alternative to the paper [4] for the particular problem

$$\begin{cases} u''(t) = f_n(t), \\ u(0) = u(1) = 0. \end{cases}$$

The results were then extended to the general case of functions defined on a space with a finite measure $(\Omega,\mathcal{A},\mu)$ taking values in a separable Banach space ([7]). Biting lemma can be extended for Jordan finite-tight sequences. In the particular case when Ω is an open convex set in $\mathbb{R}^d$, we obtained a compactness in measure result for sequences of Sobolev space $W^{1,1}(\Omega,\mathbb{R}^p)$. Thus, if a sequence $(u_n)_{n\in\mathbb{N}}\subseteq W^{1,1}(\Omega,\mathbb{R}^p)$ is tight and the sequence of its gradients $(\nabla u_n)_{n\in\mathbb{N}}$ is Jordan finite-tight, then $(u_n)_{n\in\mathbb{N}}$ admits a subsequence convergent in measure. This result essentially intervened to get relaxed solutions to the classic problem of variational calculus in [8].

In this paper, we treat the unbounded case for the general problem of [4]. Since the study of the second order differential problem can be reduced to that of a first order problem, we will deal with a differential equation of order 1.

2. The differential problem

Let μ be the Lebesgue measure on $[0,1]$, let $E=\mathbb{R}^p$ be the p-euclidean space and let $C([0,1],E)$ $(L^1([0,1],E))$ be the space of all E-valued continuous (integrable) functions on $[0,1]$. We consider on $C([0,1],E)$ the norm $\|\cdot\|_\infty$, where $\|u\|_\infty=\sup_{t\in[0,1]}\|u(t)\|_E$ and on $L^1([0,1],E)$ the norm $\|\cdot\|_1$, where $\|v\|_1=\int_0^1\|v(t)\|_E d\mu(t)$.

A mapping $v=(v_1,\cdots,v_p)\in L^1([0,1],E)$ is a weak derivative of the mapping $u=(u_1,\cdots,u_p)\in L^1([0,1],E)$ if, for every $i\in\{1,\dots,p\}$ and every application ∞-times differentiable $\phi:[0,1]\to\mathbb{R}$ with $supp\phi\subseteq(0,1)$ we have

$$\int_0^1 u_i(t)\phi'(t)d\mu(t) = -\int_0^1 v_i(t)\phi(t)d\mu(t).$$

If v,w are two derivatives of u, then $v=w$ almost everywhere. We will note the weak derivative of u with u'.

The Sobolev space $W^{1,1}([0,1],E)$ consists of all mappings $u\in C([0,1],E)$ with $u'\in L^1([0,1],E)$. If $W^{1,1}([0,1],E)$ is equipped with the norm $\|\cdot\|_W$, where $\|u\|_W=\|u\|_\infty+\|u'\|_1$, then it becomes a Banach space. We remark that the norm $\|\cdot\|_W$ is stronger that the usual norm defined by $\|u\|_{W^{1,1}}=\|u\|_1+\|u'\|_1$.

Definition 2.1. A map $f:[0,1]\times E\to E$ is a Lipschitz integrand if:

L1) $f(\cdot,x)$ is Lebesgue measurable, for every $x\in E$.

L2) $f(\cdot,0)\in L^1([0,1],E)$.

L3) There exists $\beta\in L^1_+([0,1])$ with $\|\beta\|_1<\frac{1}{2}$ such that, a.e. on $[0,1]$,

$$\|f(t,x)-f(t,y)\|_E\le\beta(t)\cdot\|x-y\|_E, \forall x,y\in E.$$

We denote by $\mathbb{L}([0,1] \times E, E)$, or simply $\mathbb{L}$, the family of all Lipschitz integrands.

Theorem 2.2. *For every $f \in \mathbb{L}([0,1], E)$ there exists exactly one function $u \in W^{1,1}([0,1], E)$ such that*

$$(P) \qquad \begin{cases} u'(t) = f(t, u(t)), \text{ for almost every } t \in [0,1], \\ u(0) = 0. \end{cases}$$

Moreover,

$$\|u\|_\infty \leq 2 \int_0^1 \|f(t,0)\|_E d\mu(t),$$

and

$$\|u'\|_1 \leq 2 \int_0^1 \|f(t,0\|_E d\mu(t).$$

Proof. From the hypotheses L1)-L3), it follows that, for almost every $t \in [0,1]$ and for every $x \in E$,

$$\|f(t,x)\|_E \leq \beta(t) \cdot \|x\|_E + \|f(t,0)\|_E, \tag{2.1}$$

If we note $c(t) = \max\{\beta(t), \|f(t,0)\|_E\}, c \in L^1_+([0,1])$, then

$$\|f(t,x)\|_E \leq c(t)(1 + \|x\|_E), \forall t \in [0,1], \text{ for every } x \in E.$$

From the inequality (2.1), it follows that, for every $u \in C([0,1], E)$ and every $t \in [0,1]$,

$$\|f(t,u(t))\|_E \leq \beta(t) \cdot \|u(t)\|_E + \|f(t,0)\|_E \leq c(t)(1 + \|u\|_\infty).$$

Therefore $f(\cdot, u(\cdot)) \in L^1([0,1], E)$, for every $u \in C([0,1], E)$.

It is easy to note that the differential problem (P) is equivalent to the integral equation

$$(I) \qquad u(t) = \int_0^t f(s, u(s)) d\mu(s).$$

For every $u \in W^{1,1}([0,1], E)$, we define $T(u) : [0,1] \to E$ letting $T(u)(t) = \int\limits_0^t f(s, u(s)) d\mu(s)$. $T(u)$ is continuous on $[0,1]$ and

$$\|T(u)\|_E \leq \int_0^1 \|f(s,u(s))\|_E d\mu(s) = \|f(\cdot, u(\cdot))\|_1.$$

Moreover, almost everywhere on $[0,1]$, $(T(u))' = f(\cdot, u(\cdot)) \in L^1([0,1], E)$ and

$$\|(T(u))'\|_1 = \int_0^1 \|f(t,u(t))\|_E d\mu(t) = \|f(\cdot, u(\cdot))\|_1.$$

Hence $T : W^{1,1}([0,1], E) \to W^{1,1}([0,1], E)$.

A simple calculation leads to:

$$\|T(u) - T(v)\|_W \leq 2\|\beta\|_1 \cdot \|u - v\|_W, \text{ for every } u, v \in W^{1,1}([0,1], E),$$

which shows that T is a contraction. From Banach's fixed-point theorem, there exists only one function $u \in W^{1,1}([0,1], E)$ such that $T(u) = u$.

Using relation (2.1), we obtain:

$$\begin{aligned}
\|u\|_\infty = \|T(u)\|_\infty &\leq \int_0^1 \|f(t, u(t))\|_E d\mu(t) \\
&\leq \int_0^1 \beta(t) \cdot \|u(t)\|_E d\mu(t) + \int_0^1 \|f(t, 0)\|_E d\mu(t) \\
&\leq \|\beta\|_1 \cdot \|u\|_\infty + \int_0^1 \|f(t, 0)\|_E d\mu(t) \\
&< \frac{1}{2}\|u\|_\infty + \int_0^1 \|f(t, 0)\|_E \mu(t),
\end{aligned}$$

whence $\|u\|_\infty \leq 2 \int_0^1 \|f(t, 0)\|_E d\mu(t)$.

Because $\|u'\|_1 = \int_0^1 \|f(t, u(t))\|_E d\mu(t)$, as above, it follows that

$$\|u'\|_1 \leq 2 \int_0^1 \|f(t, 0\|_E d\mu(t).$$

□

3. Young measures

In this section we recall the necessary notions and results from the theory of Young.

The Young measures were introduced in order to obtain relaxed solutions for variational problems. The theory begins with the works of L. C. Young (1937); the extensions to Polish and Suslin spaces were made by E. J. Balder (1984) and M. Valadier (1990). A general presentation of theory can be found in [9] (see also [3]).

The Young measures generalize measurable functions. Thus, a Young measure on the euclidean space $F = \mathbb{R}^q$ is itself a measurable application that, to every point $t \in [0, 1]$, associates a probability τ_t on F; for every Borel set $B \in \mathcal{B}_F$, $\tau_t(B)$ may be interpreted as the probability that the value in t of generalized function $\tau_.$ belongs to B.

Let $\mathcal{P}_F \subseteq ca^+(\mathcal{B}_F)$ be the set of all probabilities on F endowed with the narrow topology $\mathfrak{T}$ and let $\mathcal{C}$ be the Borel sets of $(\mathcal{P}_F, \mathfrak{T})$.

Definition 3.1. A Young measure on F is an $(\mathcal{A} - \mathcal{C})$-measurable map $\tau_. : [0, 1] \to \mathcal{P}_F$; here $\mathcal{A}$ is the σ-algebra of Lebesgue measurable sets on $[0, 1]$. Let $\mathcal{Y}([0, 1], F)$ be the space of Young measures on F.

For every measurable function $u : [0, 1,] \to F$, let $\tau^u : [0, 1] \to \mathcal{P}_F, \tau^u_t = \delta_{u(t)}$, for every $t \in [0, 1]$ ($\delta_.$ is the Dirac measure). τ^u is the Young measure associated to measurable application u. The mapping $u \mapsto \tau^u = \delta_{u(\cdot)}$ is an injection of all F-valued measurable functions on $[0, 1]$, $\mathcal{M}([0, 1], F)$, in $\mathcal{Y}([0, 1], F)$. Therefore we will consider that $\mathcal{M}([0, 1], F) \subseteq \mathcal{Y}([0, 1], F)$.

The stable topology on $\mathcal{Y}([0, 1], F)$ is the projective limit topology generated by the family of mappings $\{I_{A,f} : A \in \mathcal{A}, f \in C_b([0, 1], F)\}$, where $I_{A,f} : \mathcal{Y}([0, 1], F) \to \mathbb{R}$ is defined by $I_{A,f}(\tau) = \int_A \tau_t(f) d\mu(t)$ and $C_b([0, 1], F)$ is the set of all F-valued, bounded continuous functions on $[0, 1]$. This topology is denoted by $\mathcal{S}$.

A sequence $(\tau^n)_{n\in\mathbb{N}} \subseteq \mathcal{Y}([0,1],F)$ is $\mathcal{S}$-convergent to $\tau \in \mathcal{Y}([0,1],F)$ iff

$$\int_A \tau_t^n(f)d\mu(t) \to \int_A \tau_t(f)d\mu(t), \forall A \in \mathcal{A}, \forall f \in C_b([0,1],F).$$

If $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}([0,1],F)$, then $(u_n)_{n\in\mathbb{N}}$ is $\mathcal{S}$-convergent to $\tau \in \mathcal{Y}([0,1],F)$ iff

$$\int_A f(u_n)d\mu(t) \to \int_A \tau_t(f)d\mu(t), \forall A \in \mathcal{A}, \forall f \in C_b([0,1],F).$$

We denote this by $u_n \xrightarrow{\mathcal{S}} \tau$. If $u \in \mathcal{M}([0,1],F)$, then we write $u_n \xrightarrow{\mathcal{S}} u$ instead of $u_n \xrightarrow{\mathcal{S}} \tau^u$, i.e. $\int_A f(u_n)d\mu \to \int_A f(u)d\mu$, for every measurable set $A \subseteq [0,1]$ and for every $f \in C_b([0,1],F)$. Hoffmann-Jørgensen showed that this is equivalent with the convergence in measure of $(u_n)_{n\in\mathbb{N}}$ to u, $u_n \xrightarrow{\mu} u$ (see [5, Corollary 4.6]).

The following result will be very useful in the following (for a proof see [9, Corollary 3.36]):

Theorem 3.2. *Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}([0,1],F)$ and let $\tau \in \mathcal{Y}([0,1],F)$ such that $u_n \xrightarrow{\mathcal{S}} \tau$. Then, for every Carathéodory integrand $\Psi : [0,1] \times F \to \mathbb{R}$ for which $\{\Psi(\cdot, u_n(\cdot)) : n \in \mathbb{N}\}$ is an uniformly integrable subset of $L^1([0,1],\mathbb{R})$ and such that there exists $\int_0^1 \left(\int_F \Psi(t,x)d\tau_t(x)\right) d\mu(t)$, we have:*

$$\int_0^1 \left(\int_F \Psi(t,x)d\tau_t(x)\right) d\mu(t) = \lim_{n\to\infty} \int_0^1 \Psi(t,u_n(t))d\mu(t).$$

In the previous theorem, Ψ is a Carathéodory integrand on F if, for every $x \in F$, $\Psi(\cdot,x)$ is measurable on $[0,1]$ and, for every $t \in [0,1]$, $\Psi(t,\cdot)$ is continuous on F.

A proof for the following theorem can be found in [9, Theorem 3.50], in a more general setting.

Theorem 3.3. *$\{\tau^u : u \in \mathcal{M}([0,1],F)\}$ is dense in $(\mathcal{Y}([0,1],F),\mathcal{S})$.*

Definition 3.4. A subset $\mathcal{H} \subseteq \mathcal{Y}([0,1],F)$ is tight if, for every $\varepsilon > 0$, there exists a compact set $K \subseteq F$ such that

$$(T) \qquad \int_0^1 \tau_t(F \setminus K)d\mu(t) < \varepsilon, \text{ for every } \tau \in \mathcal{H}.$$

A set $H \subseteq \mathcal{M}([0,1],F)$ is tight if $\mathcal{H} = \{\tau^u : u \in H\} \subseteq \mathcal{Y}([0,1],F)$ is tight, i.e., for every $\varepsilon > 0$, there exists $k > 0$ such that $\mu(\{t \in [0,1] : \|u(t)\|_F \geq k\}) < \varepsilon$.

We can note that, for every bounded set $H \subseteq L^1([0,1],F)$, the set $\mathcal{H} = \{\tau^u : u \in H\} \subseteq \mathcal{Y}([0,1],F)$ is tight (see [9, Proposition 3.56]).

The interest for tight sets is given by Prohorov's compactness theorem ([9, Theorem 3.64 and Proposition 3.65]).

Theorem 3.5. *A set $\mathcal{H} \subseteq \mathcal{Y}([0,1],F)$ is sequentially $\mathcal{S}$-compact if and only if $\mathcal{H}$ is tight.*

As a corollary of the previous theorem we obtain:

Corollary 3.6. *Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}([0,1],F)$ be a sequence; if $\{u_n : n\in\mathbb{N}\}$ is tight, then $(u_n)_{n\in\mathbb{N}}$ has a subsequence $(u_{k_n})_{n\in\mathbb{N}}$ $\mathcal{S}$-convergent to a Young measure $\tau\in\mathcal{Y}([0,1],F)$.*

Moreover, if $(u_n)_n$ is uniformly integrable in $L^1([0,1],F)$, then

$$\mathrm{bar}\tau_{\cdot} \equiv \int_{\mathbb{R}^m} x d\tau_{\cdot}(x) \in L^1([0,1],F)$$

and $(u_{k_n})_n$ is weakly convergent to $\mathrm{bar}\tau_{\cdot}$.

For a proof see [9, Proposition 3.37].

We conclude this section with the fiber product lemma (see [3, Theorem 3.3.1]).

Definition 3.7. Let $E=\mathbb{R}^p$. For every $\tau\in\mathcal{Y}([0,1],E)$ and every $\sigma\in\mathcal{Y}([0,1],E)$ the mapping $t\mapsto\tau_t\otimes\sigma_t$ is a Young measure $\tau\otimes\sigma\in\mathcal{Y}([0,1],E\times E)$; $\tau\otimes\sigma$ is called the fiber product of Young measures τ and σ.

In the case where $u,v\in\mathcal{M}([0,1],E)\subseteq\mathcal{Y}([0,1],E)$, then the Young measure $\tau^u\otimes\tau^v$ is the mapping $t\mapsto\delta_{(u(t),v(t))}$.

We can find a proof of the following result in a more general setting in [9, Theorem 3.87 and Corollary 3.89] (see also [2]).

Theorem 3.8 (Fiber product lemma). *Let $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}([0,1],E)$, $u\in\mathcal{M}([0,1],E)$ and $\tau\in\mathcal{Y}([0,1],E)$.*

Then $(u_n,v_n)\xrightarrow[E\times E]{\mathcal{S}}\tau^u\otimes\tau$ if and only if $u_n\xrightarrow{\mu} u$ and $v_n\xrightarrow{\mathcal{S}}\tau$.

4. The bounded case

In this section we treat a case similar to that studied in [4].

We recall that by $\mathbb{L}([0,1]\times E,E)=\mathbb{L}$ we denote the family of all Lipschitz integrands (see Definition 2.1).

Theorem 4.1. *For every $n\in\mathbb{N}$, let $f_n\in\mathbb{L}$ and let $u_n\in W^{1,1}([0,1],E)$ be the unique solution of problem*

$$(P_n)\qquad \begin{cases} u'(t)=f_n(t,u(t)), \text{ for almost every } t\in[0,1],\\ u(0)=0.\end{cases}$$

(see Theorem 2.2).

If $(f_n(\cdot,0))_{n\in\mathbb{N}}$ is a bounded sequence in $L^1([0,1],E)$, then there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ (still noted $(u_n)_{n\in\mathbb{N}}$), and there exist $u\in BV([0,1],E)$ (the set of all E-valued mappings of bounded variation on $[0,1]$) and $\tau\in\mathcal{Y}_E([0,1])$ such that:

(i) *$u_n\xrightarrow{\|\cdot\|_1} u$ and $u(0)=0$.*

(ii) *$u'_n\xrightarrow{\mathcal{S}}\tau$.*

(iii) *The mapping $v:[0,1]\to E$, defined by $v(t)=\mathrm{bar}\tau_t=\int_E y d\tau_t(y)$ is integrable on $[0,1]$.*

(iv) *Moreover, if the sequence $(\langle u_n,u'_n\rangle)_{n\in\mathbb{N}}\subseteq L^1([0,1],\mathbb{R})$ is uniformly integrable, then*

$$\int_0^1 \langle u(t),v(t)\rangle\, d\mu(t) = \lim_{n\to\infty}\int_0^1 \langle u_n(t),u'_n(t)\rangle\, d\mu(t).$$

(here $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{R}^p$).

Proof. Let $M > 0$ such that $\int_0^1 \|f_n(t,0)\|_E d\mu(t) \le M$, for every $n \in \mathbb{N}$ and let $\Delta = \{t_0, \dots, t_q\}$ be a partition of the interval $[0,1]$. Then, for every $n \in \mathbb{N}$,

$$\begin{aligned}
V_\Delta(u_n) &= \sum_{k=0}^{q-1} \|u_n(t_{k+1}) - u_n(t_k)\|_E \\
&= \sum_0^{q-1} \| \int_{t_k}^{t_{k+1}} f_n(s, u_n(s)) d\mu(s) \|_E \\
&\le \sum_0^{q-1} \int_{t_k}^{t_{k+1}} \|f_n(s, u_n(s))\|_E d\mu(s) \\
&\le \int_0^1 \beta_n(t) \cdot \|u_n(t)\|_E d\mu(t) + \int_0^1 \|f_n(t,0)\|_E d\mu(t) \\
&\le \|u_n\|_\infty \cdot \|\beta_n\|_1 + \int_0^1 \|f_n(t,0)\|_E d\mu(t) \\
&\le 2 \int_0^1 \|f_n(t,0)\|_E d\mu(t) \le 2M.
\end{aligned}$$

It follows that $(u_n)_{n\in\mathbb{N}}$ is a sequence of uniformly bounded variation and $\|u_n\|_\infty \le 2M$; from Helly's selection theorem it has a subsequence, still noted $(u_n)_{n\in\mathbb{N}}$, pointwise convergent to a function $u \in BV([0,1], E)$ of bounded variation. Moreover, $u(0) = 0$.

Using the bounded convergence theorem, $u_n \xrightarrow{\|\cdot\|_1} u$.

From Theorem 2.2, $\|u_n'\|_1 \le 2M$, for every $n \in \mathbb{N}$ and so, according to the remark from Definition 3.4, $(u_n')_{n\in\mathbb{N}}$ is tight. We then call Prokhorov's theorem (Theorem 3.5); therefore there exist a Young measure $\tau \in \mathcal{Y}([0,1], E)$ and a subsequence of $(u_n')_{n\in\mathbb{N}}$ (still noted with $(u_n')_{n\in\mathbb{N}}$) such that $u_n' \xrightarrow{\mathcal{S}} \tau$.

(iii) According to Theorem 2.2, $\|u_n\|_\infty \le 2M$, for every $n \in \mathbb{N}$; therefore $(u_n)_{n\in\mathbb{N}}$ is uniformly integrable in $L^1([0,1], E)$. From Corollary 3.6, the mapping $v : [0,1] \to E$ defined by $v(t) = \text{bar}\tau_t = \int_E y d\tau_t(y)$ is integrable on $[0,1]$.

(iv) Now, let us suppose that $(\langle u_n, u_n'\rangle)_{n\in\mathbb{N}} \subseteq L^1([0,1], \mathbb{R})$ is uniformly integrable. Since $u_n \xrightarrow{\mu} u$ and $u_n' \xrightarrow{\mathcal{S}} \tau$, we can apply the fiber product lemma (see Theorem 3.8); therefore the sequence $((u_n, u_n'))_{n\in\mathbb{N}} \subseteq \mathcal{Y}([0,1], E \times E)$ is stable convergent to $\tau^u \otimes \tau$.

Let $\Psi : [0,1] \times E \times E \to \mathbb{R}$ defined by $\Psi(t,x,y) = \langle x, y\rangle$. Ψ is an integrand Carathéodory on $F = E \times E$. For every $n \in \mathbb{N}$, $\Psi(\cdot, u_n, u_n') = \langle u_n, u_n'\rangle$, such that the sequence $(\Psi(\cdot, u_n, u_n'))_{n\in\mathbb{N}}$ is uniformly integrable in

$L^1([0,1],\mathbb{R})$. Moreover,

$$\begin{aligned}
&\left|\int_0^1 \left(\int_F \Psi(t,x,y)d(\tau_t^u \otimes \tau_t)(x,y)\right) d\mu(t)\right| = \\
&= \left|\int_0^1 \left(\int_E \left(\int_E \langle x,y\rangle \, d\tau_t^u(x)\right) d\tau_t(y)\right) d\mu(t)\right| \\
&= \left|\int_0^1 \left(\int_E \langle u(t),y\rangle \, d\tau_t(y)\right) d\mu(t)\right| \\
&= \left|\int_0^1 \left\langle u(t), \int_E y d\tau_t(y)\right\rangle d\mu(t)\right| \\
&= \left|\int_0^1 \langle u(t),v(t)\rangle \, d\mu(t)\right| \leq \|u\|_\infty \cdot \|v\|_1 < +\infty.
\end{aligned}$$

It follows that the conditions of Theorem 3.2 are satisfied, therefore

$$\int_0^1 \langle u(t),v(t)\rangle \, d\mu(t) = \lim_{n\to\infty} \int_0^1 \langle u_n(t),u_n'(t)\rangle \, d\mu(t).$$

$\square$

5. Jordan finite-tight sets

In this section we recall the extension of Biting Lemma to the unbounded case. The proof of this result can be found in [9, Theorem 3.84 and 3.85], (see also [7]). We recall also a result of compactness in measure proved in [9, Theorem 3.102], (see also [7, Theorem 3.12]).

Definition 5.1. A set of measurable mappings $H \subseteq \mathcal{M}([0,1],E)$ is called Jordan finite-tight if, for every $\varepsilon > 0$, there exist $k > 0$ and a finite family $\mathcal{I}$ of sub-intervals of $[0,1]$ such that, for every $u \in H$, there exists a sub-family $\mathcal{I}_u \subseteq \mathcal{I}$ with $\mu\left(\bigcup \mathcal{I}_u\right) < \varepsilon$ and

$$\{t \in [0,1] : \|u(t)\|_E \geq k\} \subseteq \bigcup \mathcal{I}_u$$

($\bigcup \mathcal{I}_u$ is the union of intervals of family $\mathcal{I}_u$).

A sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}([0,1],E)$ is Jordan finite-tight if the set $H = \{u_n : n \in \mathbb{N}\}$ is Jordan finite-tight.

Remark 5.2. Every Jordan finite-tight set is tight. The converse is not true.

Indeed, let $\mathbb{Q} \cap [0,1] = \{q_0, q_1, \ldots, q_n, \ldots\}$ be the set of all rational numbers of $[0,1]$ and let $u : [0,1] \to \mathbb{R}$, $u = \sum\limits_{n=0}^{\infty} n \cdot \chi_{\{q_n\}}$; the set $H = \{u\} \subseteq W^{1,1}([0,1],\mathbb{R})$ is tight but H is not a Jordan finite-tight set.

On the other hand, if $u_n = n^2 \cdot \chi_{]q_n, q_n + \frac{1}{n}[}$, then $H = \{u_n : n \in \mathbb{N}^*\}$ is a tight set but it is not bounded in $\mathcal{L}^1(]0,1[,\mathbb{R})$. For every $k > 0$ and every $n \in \mathbb{N}^*$,

$$\{t \in]0,1[: |u_n(t)| > k\} = \begin{cases} \emptyset & , n^2 \leq k, \\ ]q_n, q_n + \frac{1}{n}[& , n^2 > k. \end{cases}$$

$H = \{u_n : n \in \mathbb{N}^*\}$ is a Jordan finite-tight set.

The following theorem gives a justification for the denomination *Jordan finite-tight set* (see [9, Theorem 3.94] and [7, Theorem 3.4]).

For every $B \subseteq [0,1]$ let $\mu_J^*(B)$ be the Jordan outer measure of B:

$$\mu_J^*(B) = \inf\{\mu(\cup\mathcal{I}) : \mathcal{I} \text{ a finite cover of } B \text{ with intervals}\}.$$

Obviously, $\mu_J^*(B) = 0$ if and only if B is a Jordan-negligible set.

Theorem 5.3. *For every $H \subseteq \mathcal{M}([0,1],E)$, let*

$$I_H = \left\{ t \in [0,1] : \limsup_{s\to t} \left(\sup_{u\in H} \|u(s)\|_E \right) = +\infty \right\}.$$

A set $H \subseteq \mathcal{M}([0,1],E)$ is Jordan finite-tight if and only if, for every $\varepsilon > 0$, there exists a finite cover of H, $\{H_1,\ldots,H_p\}$, such that

$$\mu_J^*(I_{H_i}) < \varepsilon, \text{ for any } i = 1,\ldots,p.$$

In the following we present versions of Biting lemma for the case of unbounded sequences in L^1.

Theorem 5.4. *[9, Theorems 3.84, 3.85], [7, Theorems 2.10, 2.11] For every Jordan finite-tight sequence $(u_n)_{n\in\mathbb{N}} \subseteq L^1([0,1],E)$, there exist a subsequence (still noted $(u_n)_{n\in\mathbb{N}}$), a decreasing sequence $(B_p)_{p\in\mathbb{N}} \subseteq \mathcal{A}$ with $\mu(\cap_{p=0}^{\infty} B_p) = 0$ and a Young measure $\tau \in \mathcal{Y}([0,1],E)$ such that:*

(i) $u_n \xrightarrow{\mathcal{S}} \tau$.

(ii) *τ_t has a barycenter $u(t)$, for almost every $t \in [0,1]$,*

$$u(t) = \mathrm{bar}(\tau_t) = \int_E x d\tau_t(x).$$

(iii) *$u \in L^1([0,1]\setminus B_p, E)$, for every $p \in \mathbb{N}$.*

(iv) *$u_n \xrightarrow[{[0,1]\setminus B_p}]{w} u$, for every $p \in \mathbb{N}$.*

This result helps us obtain a result of compactness in measure.

Theorem 5.5. *[9, Theorem 3.102], [7, Theorem 3.12] Let $H \subseteq W^{1,1}([0,1],E)$ be a tight set such that $H' = \{u' : u \in H\}$ is Jordan finite-tight; then H is relatively compact in the topology of convergence in measure on $\mathcal{M}([0,1],E)$.*

6. The Jordan finite-tight case

The result obtained in Theorem 4.1 is based on the assumption that the sequence $(f_n(\cdot,0))_{n\in\mathbb{N}}$ is bounded in $L^1([0,1],E)$. Now, we replace this condition by a domination of $(f_n(\cdot,x))_{n\in\mathbb{N}}$ with a Jordan finite-tight sequence.

Theorem 6.1. *For every $n \in \mathbb{N}$, let $f_n \in \mathbb{L}$ and let $u_n \in W^{1,1}([0,1],E)$ be the unique solution of problem*

$$(P_n) \qquad \begin{cases} u'(t) = f_n(t,u(t)), \text{ for almost every } t \in [0,1], \\ u(0) = 0. \end{cases}$$

(see Theorem 2.2).

We suppose that $(u_n)_{n\in\mathbb{N}}$ is tight and that there exists a Jordan finite-tight sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_+([0,1],\mathbb{R})$ such that $\|f_n(t,x)\|_E \le \varphi_n(t)$, for almost every $t \in [0,1]$ and for all $x \in E$.

Then, there exist a subsequence of $(u_n)_{n\in\mathbb{N}}$ (still denoted by $(u_n)_{n\in\mathbb{N}}$), a mapping $u \in \mathcal{M}([0,1],E)$ and a Young measure $\tau \in \mathcal{Y}([0,1],E)$ such that:

(i) $u_n \xrightarrow{\mu} u$, *and* $u(0) = 0$.

(ii) $u'_n \xrightarrow{\mathcal{S}} \tau$.

(iii) *The mapping* $v : [0,1] \to E$ *defined by* $v(t) = \mathrm{bar}\tau_t = \int\limits_E y d\tau_t(y)$ *is measurable.*

(iv) *If* $(\langle u_n, u'_n \rangle)_{n\in\mathbb{N}} \subseteq L^1([0,1],\mathbb{R})$ *is uniformly integrable, then*

$$\int_0^1 \langle u(t), v(t)\rangle \, d\mu(t) = \lim_{n\to\infty} \int_0^1 \left\langle u_n(t), u'_n(t)\right\rangle d\mu(t).$$

(v) *If* $(u_n)_{n\in\mathbb{N}}$ *is bounded in* $L^1([0,1],E)$*, then* $u \in L^1([0,1],E)$ *and*

$$\|u_n - u\|_1 \longrightarrow \eta((u_n)) = \lim_k \sup_n \int\limits_{(\|u_n\|_E \geq k)} \|u_n(t)\|_E d\mu(t)$$

($\eta((u_n))$ *is the modulus of uniform integrability of* $(u_n)_{n\in\mathbb{N}}$*).*

(vi) *If* $(u_n)_{n\in\mathbb{N}}$ *is uniformly integrable, then* $u_n \xrightarrow{\|\cdot\|_1} u$.

Proof. For every $n \in \mathbb{N}$ and every $t \in [0,1]$,

$$\|u'_n(t)\|_E = \|f_n(t, u_n(t))\|_E \leq \varphi_n(t).$$

It follows that $(u'_n)_{n\in\mathbb{N}} \subseteq L^1([0,1],E)$ is Jordan finite-tight. Since $(u_n)_{n\in\mathbb{N}}$ is tight we can apply Theorem 5.5, therefore $(u_n)_{n\in\mathbb{N}}$ admits a subsequence (still denoted by $(u_n)_{n\in\mathbb{N}}$) convergent in measure to a measurable function $u \in \mathcal{M}([0,1],E)$.

Moreover, from Theorem 5.4, the subsequence can be chosen so that $(u'_n)_{n\in\mathbb{N}}$ to be stable convergent to a Young measure $\tau \in \mathcal{Y}([0,1],E)$.

Almost for every $t \in [0,1]$, τ_t has a barycenter $v(t) = \mathrm{bar}\tau_t = \int\limits_E y d\tau_t(y)$ and there exists a decreasing sequence $(B_p)_{p\in\mathbb{N}} \subseteq \mathcal{A}$ with $\mu(\cap_1^\infty B_p) = 0$ such that, for every $p \in \mathbb{N}$, $v \in L^1([0,1]\setminus B_p, E)$ and $u'_n \xrightarrow[{[0,1]\setminus B_p}]{w} v$. Obviously, $v \in \mathcal{M}([0,1],E)$.

Let us now suppose that $(\langle u_n, u'_n\rangle)_{n\in\mathbb{N}}$ is uniformly integrable. Following a similar argument to that of the proof of Theorem 4.1, we obtain

$$\int_0^1 \langle u(t), v(t)\rangle \, d\mu(t) = \lim_{n\to\infty} \int_0^1 \left\langle u_n(t), u'_n(t)\right\rangle d\mu(t).$$

If $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1([0,1],E)$, then we can apply Theorem 3.70 and (2) of Remark 3.71 from [9]. It follows that we can extract the above subsequence so that $\|u_n - u\|_1 \to \eta((u_n))$.

Moreover, if $(u_n)_{n\in\mathbb{N}} \subseteq L^1([0,1],E)$ is uniformly integrable, then $\eta((u_n)) = 0$ and then $u_n \xrightarrow{\|\cdot\|_1} u$. $\square$

Remark 6.2. By a similar procedure to that used in the proof of (iv) of above theorem we can show that, for every $t \in [0,1]$,

$$\int_0^t \langle u(s), \mathrm{bar}\tau_t\rangle \, d\mu(s) = \lim_{n\to\infty} \int_0^t \left\langle u_n(s), u'_n(s)\right\rangle d\mu(s).$$

If we note that, for every $t \in [0,1]$,

$$\int_0^t \left\langle u_n(s), u'_n(s)\right\rangle d\mu(s) = \frac{1}{2}\cdot \|u_n(t)\|_E^2 \xrightarrow[n\to\infty]{} \frac{1}{2}\cdot \|u(t)\|_E^2,$$

then $\frac{1}{2} \cdot \|u(t)\|_E^2 = \int\limits_0^t \langle u(s), \mathrm{bar}\tau_s \rangle \, d\mu(s)$.

In the additional assumption that u is a differentiable function, we obtain that $u'(t) = \mathrm{bar}\tau_t$, for every $t \in [0, 1]$.

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