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Fixed point technique for a class of backward stochastic differential equations

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Dedicated to the memory of Professor Viorel Radu

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Abstract

We establish a new theorem on the existence and uniqueness of the adapted solution to backward stochastic differential equations under some weaker conditions than the Lipschitz one. The extension is based on Athanassov's condition for ordinary differential equations. In order to prove the existence of the solutions we use a fixed point technique based on Schauder's fixed point theorem. Also, we study some regularity properties of the solution for this class of stochastic differential equations.

Keywords: Backward stochastic differential equations; non-Lipschitz conditions; adapted solutions; pathwise uniqueness; global solutions; fixed point technique; Schauder's fixed point theorem.
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1. Introduction

A backward differential equation (see for example in *Pardoux and Peng* [25]) which appears in the optimal stochastic control is the following:

$$Y_t = \xi + \int_t^1 f(s, Y_s, Z_s)ds - \int_t^1 Z_s dW(s), \quad 0 \le t \le 1 \tag{1.1}$$

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where $\{W(t),\ 0 \le t \le 1\}$ is a Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$ with the natural filtration $\{\mathcal{F}_t,\ 0 \le t \le 1\}$ and ξ is a given $\mathcal{F}_1$-measurable random variable such that $E|\xi|^2 < \infty$. In the field of control, we usually regard $Z(\cdot)$ as an adapted control and $Y(\cdot)$ as the state of the system. We are allowed to choose an adapted control $Z(\cdot)$ which drives the state $Y(\cdot)$ of the system to the given target X at time $t = 1$. This is the so-called reachability problem. So in fact we are looking for a pair of stochastic processes $\{Y(t), Z(t), 0 \le t \le 1\}$ with values in $\mathbb{R} \times \mathbb{R}$ which is $\mathcal{F}_t$-adapted and satisfies the above equation. Such a pair is called an adapted solution of the equation. *Pardoux and Peng* in [25] showed the existence and uniqueness of the adapted solution under the condition that $f(t, y)$ is uniformly Lipschitz continuous in y. Since then, the interest on BSDEs has increased steadily (see [1], [27] or [19]), due to the connections of this subject with computational finance, stochastic control, and partial differential equations. In particular, many efforts have been made to relax the assumptions on the coefficient functions (for instance [20, 16, 17, 26, 18, 15, 4, 3, 8, 23]). An interesting domain for the applications of this class of stochastic differential equations in computational finance has also been found (see e.g. [13, 11, 10, 14, 12]).

The starting point in the study of our equation is given by *Z. Athanassov* in [2], where a uniqueness theorem of Nagumo type for the Cauchy problem is pointed out, which generalizes several known uniqueness theorems and sufficient conditions to guarantee the convergence of the Picard successive approximations for ordinary differential equations. The importance of Athanassov's result comes from the fact that the coefficients functions can have some singularities at the time moment $t = 1$. That result clarifies the relationship between the modulus of the continuity and growth conditions imposed to the coefficient functions. Stochastic generalizations of the results of Athanassov for SDE's are given by *A. Constantin* ([5, 6, 7]) and others (e.g. [21, 8, 9, 23]). The fixed point method was proposed to show the existence of the solution for a BSDE starting with the first papers on this topic (e.g. [1]). Many other mathematicians have applied different fixed point theorems for various classes of BSDEs. Schauder's fixed point theorem is an interesting extension of Brouwer's fixed theorem in the case of infinite dimensional spaces. To our best knowledge, the first intention to apply this theorem in the context of BSDE was proposed in [22]. Other results are given in [23] and [24].

2. Preliminary results

Let $\{B_t\}_{0 \le t \le 1}$ denote a d-dimensional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, P)$ with the natural filtration $\{\mathcal{F}_t, 0 \le t \le 1\}$ and $\xi(\omega)$ be a given $\mathcal{F}_1$-measurable random variable with $E(|\xi|^2) < \infty$. Let $\mathcal{P}$ be the σ-algebra of $\mathcal{F}_t$-progressively measurable subsets of $[0,1] \times \Omega$ and M^2 be the family of real-valued processes which are $\mathcal{P}$-measurable and square integrable with respect to $\lambda \times P$, λ being the Lebesgue measure.

A solution of a backward stochastic differential equation is a pair of stochastic processes $\{Y(t,\omega), Z(t,\omega) : t \in [0,1]\} \in \times M^2([0,1], \mathbb{R}^m) \times M^2([0,1], \mathbb{R}^{m\times d})$.

For $Y \in M^2([0,1], \mathbb{R}^m)$ we define the norm

$$|Y|^2 = \sum_{j=1}^{m} E[\sup_{0\le t\le 1} |Y_j(t)|^2].$$

Similarly, for $Z \in M^2([0,1], \mathbb{R}^{m\times d})$ we define the norm

$$\|Z\|^2 = \sum_{j=1}^{m} \sum_{i=1}^{d} E[\Big|\int_0^1 Z_j(t) dW_i(t)\Big|^2].$$

Next, we recall the following lemma:

Lemma 2.1. *[23] Let $u(t)$ be a continuous, positive function on $0 < t < 1$ having nonnegative derivative $u'(t) \in L([0,1])$. Let $v(t)$ be a continuous, nonnegative function for $0 \le t \le 1$ such that $v(t) = o(u(t))$ as*

$t \to 1^-$ *and*

$$v(t) \leq \int_t^1 \frac{u'(s)}{u(s)} v(s) ds, \ \ \forall\, 0 \leq t \leq 1.$$

Then $v(t) \equiv 0$ *on* $0 \leq t \leq 1$.

3. Main results

3.1. Assumptions

We consider the following assumptions for the system (1.1):

i) f is a $\mathcal{P} \otimes \mathcal{B}_{\mathbb{R}^m} \otimes \mathcal{B}_{\mathbb{R}^{m\times d}}$ measurable and $\mathcal{F}_t$-adapted function, and it is continuous in the variable (y, z) on $M^2([0,1], \mathbb{R}^m) \times M^2([0,1], \mathbb{R}^{m\times d})$;

ii) $f(\cdot, 0, 0)$ is in $M^2([0,1], \mathbb{R}^m) \times M^2([0,1], \mathbb{R}^{m\times d})$;

iii) there exists a continuous, positive and derivable function $u(t)$ on $0 \leq t \leq 1$, having nonnegative derivative $u'(t) \in L([0,1])$ with $u'(t) \to \infty$ for $t \to 1^-$, such that

$$|f(t,y,z) - f(t,\tilde{y},\tilde{z})|^2 \leq \frac{u'(t)}{A_1 u(t)} (|y-\tilde{y}|^2 + \|z-\tilde{z}\|^2), \tag{3.1}$$

for all $y, \tilde{y} \in \mathbb{R}^m, z, \tilde{z} \in \mathbb{R}^{m\times d}$ $0 \leq t \leq 1$, with A_1 a positive real constant;

iv) with the same function $u(t)$ as above,

$$|f(t,y,z)|^2 \leq \frac{u'(t)}{A_2} (1 + |y|^2 + \|z\|^2), \tag{3.2}$$

for all $y \in \mathbb{R}^m, z \in \mathbb{R}^{m\times d}$, $0 \leq t \leq 1$, with A_2 a positive real constant;

v) ξ is a given $\mathcal{F}_1$-measurable random variable such that $E|\xi|^2 < \infty$ and $\alpha \in \mathbb{R}^n$.

It's not difficult to see that any Lipschitz function f verifies the above assumption. Also, any function with a quadratic growth satisfies the assumption iv). Moreover, the assumption iv) does not require the continuity of the coefficient functions at the final moment $t = 1$ because we suppose just $u'(t) \in L([0,1])$, hence we will prove a Nagumo type theorem for BSDE's frame. We will see, in the last section, an example where the coefficient function f can have a jump in a neighborhood of $t = 1$, a situation which is frequently observed in applications of BSDE's to financial modeling (see e.g. [13, 14, 11, 10, 12]). An extension of this result for a class of forward-backward stochastic differential equations is given in [24].

3.2. Existence and uniqueness

Theorem 3.1. *Let f satisfy the above hypotheses and $\xi \in L^2(\Omega, \mathcal{F}_1, P, \mathbb{R})$. Then there exists a unique pair $(Y, Z) \in M^2([0,1], \mathbb{R}^m) \times M^2([0,1], \mathbb{R}^{m\times d})$ which satisfies the equation* (1.1) *on any compact subsets of interval* $[0,1]$.

Proof. Uniqueness.

Let (Y, Z) and $(\tilde{Y}, \tilde{Z})$ be two solutions in $M^2((0,1), \mathbb{R}^m) \times M^2([0,1], \mathbb{R}^{m\times d})$ of the equation (1.1). Then

$$Y(t) - \tilde{Y}(t) + \int_t^1 Z(s)dW(s) - \int_t^1 \tilde{Z}(s)dW(s) =$$

$$= \int_t^1 f(s, Y(s), Z(s)ds - \int_t^1 f(s, \tilde{Y}(s), \tilde{Z}(s))ds.$$

Now we apply the conditioned expectation, and, using the isometry property for the Ito integral we obtain

$$\begin{aligned}
&|Y(t)-\tilde{Y}(t)|^2+\|Z(t)-\tilde{Z}(t)\|^2 \leq \\
&\qquad \leq \int_t^1 |f(s,Y(s),Z(s))-f(s,\tilde{Y}(s),\tilde{Z}(s))|^2\,ds \\
&\qquad \leq \int_t^1 \frac{u'(s)}{A_1u(s)}(|Y(s)-\tilde{Y}(s)|^2+\|Z(s)-\tilde{Z}(s)\|^2)ds, \quad 0<t<1.
\end{aligned}$$

In a similar way as in [7], we consider

$$\tau=\inf\{t\leq 1:(|Y(t)|^2>n)\vee(|\tilde{Y}(t)\|^2>n)\vee(\|Z(t)\|^2>n)\vee(\|\tilde{Z}(t)\|^2>n)\}\wedge t_0$$

for $0\leq t_0<1$, and

$$v(t)=\sup_{s\geq t}(|Y(s)-\tilde{Y}(s)|^2+\|Z(s)-\tilde{Z}(s)\|^2), \quad 0\leq t_0\leq t\leq s<1.$$

Then

$$v(t)\leq \int_{\tau\wedge t}^1 \frac{u'(s)}{A_1u(s)}(|Y(s)-\tilde{Y}(s)|^2+\|Z(s)-\tilde{Z}(s)\|^2)ds, \quad 0\leq t_0\leq t<1\ .$$

From hypothesis $u'(t)\to\infty$ for $t\to 1^-$ taking $\varepsilon>0$, we can choose $\delta_1>0$ such that $|f(t,Y(t),Z(t))|^2\leq \frac{\varepsilon}{4}u'(1-t)$. Then

$$\begin{aligned}
v(t)&\leq \int_{\tau\wedge t}^1 |f(s,Y(s),Z(s))-f(s,\tilde{Y}(s),\tilde{Z}(s))|^2\,ds \\
&\leq 2\int_{\tau\wedge t}^1 |f(s,Y(s),Z(s))|^2\,ds+\int_{\tau\wedge t}^1 |f(s,\tilde{Y}(s),\tilde{Z}(s))|^2\,ds \\
&\leq 4\varepsilon\int_{\tau\wedge t}^1 u'(1-s)\,ds\leq \varepsilon(u(t)-u(0))\leq \varepsilon u(t),\ 0<\delta_1\leq t\leq s<1.
\end{aligned}$$

Applying Lemma 2.1 (with $A_1\geq 1$) for $v(t)$ we obtain

$$|Y(t)-\tilde{Y}(t)|^2+\|Z(t)-\tilde{Z}(t)\|^2\equiv 0,\ \ 0<\delta_1\leq t<1,$$

and therefore

$$|Y(t)-\tilde{Y}(t)|^2\equiv 0,\quad \|Z(t)-\tilde{Z}(t)\|^2\equiv 0,\ \ 0<\delta_1\leq t<1,$$

which proves the uniqueness of the solution for the system (1.1).

Existence.

We will prove the existence of a solution of system (1.1) on some interval $[\delta_1,\delta_2]$ with $0<\delta_1<\delta_2<1$. We will make a reasoning similar to the one in [9].

Consider the Banach space $B^2=(M^2([0,1],\mathbb{R}^m)\times M^2([0,1],\mathbb{R}^{m\times d}))$ with the norm

$$|||(y,z)|||=\sqrt{|y|^2+\|z\|^2},\ |x|^2=E[\sup_{0\leq t\leq 1}|y(t)|^2],\ \|z\|^2=E[\int_t^1 |z|^2].$$

Let $q = |\xi|$ and $Q = 2q$. We define the set

$$\mathcal{S} = \{(Y, Z) \in B^2 : \ |Y| \leq Q, \ \|Z\| \leq Q, \text{ for } t \in [\delta_1, \delta_2]\}$$

which is a closed bounded and convex subset of the Banach space $(B^2(0,1), |||.|||)$.

We will define a map $T : B^2 \to B^2$ such that $(Y, Z) \in B^2$ is a solution of the BSDE (1.1) if it is a fixed point of T.

Let $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}$, and $(Y, Z) = T(\tilde{Y}, \tilde{Z})$ with $\{Y_t\}$ given by the relation (1.1) and $\{Z_t\}$ obtained by using Ito's martingale representation theorem, applied to the square integrable random variable

$$\xi + \int_t^1 f(s, \tilde{Y}_s, \tilde{Z}_s) ds, \ \ t \in [\delta_1, \delta_2].$$

We will prove that $(Y, Z) \in B^2$ is solution of the BSDE if it is a fixed point of T.

First, using the hypothesis iv), we prove that

$$T(\mathcal{S}) \subseteq \mathcal{S}.$$

Indeed,

$$\begin{aligned}
|Y(t)|^2 + \int_t^1 |Z(s)|^2 ds \leq \\
&\leq 2\Big(|\xi|^2 + \int_t^1 |f(s, \tilde{Y}(s), \tilde{Z}(s))|^2 ds\Big) \\
&\leq 2\Big(|\xi|^2 + \int_t^1 ds \frac{u'(s)}{A_2}(1 + |\tilde{Y}(s)|^2 + \|\tilde{Z}(s)\|^2) ds\Big) \\
&\leq 2\Big(q^2 + \int_t^1 \frac{u'(s)}{A_2}(1 + 2Q^2) ds\Big) \leq 2\Big(\frac{Q^2}{4} + (1 + 2Q^2) \int_t^1 \frac{u'(s)}{A_2} ds\Big) \\
&\leq 2\Big(\frac{Q^2}{4} + (1 + 2Q^2)\frac{u(1) - u(t)}{A_2}\Big) \leq 2\Big(\frac{Q^2}{4} + (1 + 2Q^2)\frac{u(1)}{A_2}\Big) \leq \frac{2Q^2}{2} \ ,
\end{aligned}$$

for $A_2 \geq 2u(1)$.

Hence,

$$|Y(t)|^2 + \|Z(t)\|^2 \leq Q^2 \ \ \Rightarrow T(\mathcal{S}) \subset \mathcal{S} \ .$$

Next, we will prove that the set $T(\mathcal{S})$ is equicontinuous.

For $0 < s < t < 1$ we have

$$\begin{aligned}
|||T(\tilde{Y}(t), \tilde{Z}(t)) - T(\tilde{Y}(s), \tilde{Z}(s))|||^2 &= |Y(t) - Y(s)|^2 + \|Z(t) - Z(s)\|^2 \\
&\leq \int_s^t |f(\tau, \tilde{Y}(\tau), \tilde{Z}(\tau))|^2 d\tau \\
&\leq \int_s^t 2\frac{u'(\tau)}{A_2}(1 + |\tilde{Y}(\tau)|^2 + \|\tilde{Z}(\tau)\|^2) d\tau \\
&\leq 2(1 + 2Q^2)\frac{u(t) - u(s)}{A_2},
\end{aligned}$$

for u a continuous, positive and increasing function on $[0,1]$. Therefore the equicontinuity of the set $T(\mathcal{S})$ is proved.

For $(\tilde{Y},\tilde{Z})$ and $(\hat{Y},\hat{Z}) \in \mathcal{S}$, with $T(\tilde{Y},\tilde{Z}) = (Y,Z)$ and $T(\hat{Y},\hat{Z}) = (\bar{Y},\bar{Z})$, we evaluate

$$|||T(\tilde{Y}(t),\tilde{Z}(t)) - T(\hat{Y}(t),\hat{Z}(t))|||^2 = |Y(t)-\bar{Y}(t)|^2 + \|Z(t)-\bar{Z}(t)\|^2, \ \ t \in [\delta_1,\delta_2] \ .$$

But

$$|Y(t)-\bar{Y}(t)|^2 + \|Z(t)-\bar{Z}(t)\|^2 \le$$

$$\le \int_t^1 |f(s,\tilde{Y}(s),\tilde{Z}(s)) - f(\hat{Y}(s),\hat{Z}(s))|^2 ds,$$

and then from the hypothesis iii) on the continuity of the function f in the variables y, z on B^2 we deduce by the Lebesgue convergence theorem that T is continuous.

Applying Schauder's fixed point theorem we obtain that T has a fixed point $\mathcal{S}$, thus the stochastic differential system (1.1) has a solution on $[\delta_1,\delta_2]$, for any positive real numbers $0<\delta_1<\delta_2<1$. □

3.3. Regularity properties

In the formulation of a mathematical model for a physical, biological or economical problem, we make errors in constructing the initial conditions. For theoretical purposes it is sufficient to know that the change in the solution can be made arbitrary small by making the change in the initial values sufficiently small.

Now, we will give some results on the stability properties of the solution of the equation (1.1). We consider the family of backward stochastic integral equations

$$Y_\lambda(t) = \xi_\lambda + \int_t^1 f_\lambda(s,Y_\lambda(s),Z_\lambda(s))ds - \int_t^1 Z_\lambda(s)dB_s, \quad 0 \le t \le 1, \tag{3.3}$$

with $\lambda \in \Lambda$- a open and bounded set in $\mathbb{R}^n$.

First, under the considered hypothesis, we prove the existence and uniqueness of solutions and the continuity with respect to the final condition ξ in the equation (3.3).

Theorem 3.2. *If, for any $\lambda \in \Lambda$, the coefficient functions f_λ satisfy the hypotheses i)-v), then the family* (3.3) *has a unique solution $(Y_\lambda,Z_\lambda) \in M^2([0,1],\mathbb{R}^m) \times M^2([0,1],\mathbb{R}^{m\times d})$.*

Moreover, if

$$\lim_{k\to\infty} |\xi_{\lambda,k} - \xi_\lambda|^2 = 0,$$

then

$$\lim_{k\to\infty} |||(Y_{\lambda,k},Z_{\lambda,k}) - (Y_\lambda,Z_\lambda)|||^2 = 0, \quad 0<t<1,$$

for every fixed $\lambda \in \Lambda$, where (Y_λ,Z_λ) is the solution of the equation (3.3) *and $(Y_{\lambda,k},Z_{\lambda,k})$ is the solution of* (3.3) *with the terminal condition $\xi_{\lambda,k} \in L^2(\mathbb{R}^m)$.*

Proof. The existence and the uniqueness of the solution processes are proved in a similar way as in Theorem 3.1.

Because

$$\xi_{\lambda,k} \xrightarrow{P} \xi_\lambda, \ k \to \infty$$

yields

$$|\xi_{\lambda,k} - \xi_\lambda|^2 \to 0, \ k \to \infty,$$

we obtain

$$|Y_{\lambda,k}(t) - Y_\lambda(t)|^2 + \|Z_{\lambda,k}(t) - Z_\lambda(t)\|^2 \le$$

$$\leq |\xi_{\lambda,k} - \xi_\lambda|^2 + \int_0^t \frac{u'(s)}{A_1 u(s)}(|Y_{\lambda,k}(s) - Y_\lambda(s)|^2 + \|Z_{\lambda,k}(s) - Z_\lambda(s)\|^2)ds.$$

If we denote

$$v_{\lambda,k}^2(t) = |Y_{\lambda,k}(t) - Y_\lambda(t)|^2 + \|Z_{\lambda,k}(t) - Z_\lambda(t)\|^2$$

and

$$V_{\lambda,k}^2(t) = \int_0^t \frac{u'(s)}{A_1 u(s)} v_{\lambda,k}^2(s)ds,$$

it is easy to see that

$$v_{\lambda,k}^2(t) \leq |\xi_{\lambda,k} - \xi_\lambda| + V_{\lambda,k}^2(t).$$

Hence, if we denote $v_\lambda^2(t) = \lim\limits_{k\to\infty} v_{\lambda,k}^2(t)$, and

$$V_\lambda^2(t) = \int_t^1 \frac{u'(s)}{u(s)} v_\lambda^2(s)ds$$

then, from Lebesgue's theorem of dominated convergence, we obtain

$$v_\lambda^2(t) \leq \int_t^1 \frac{u'(s)}{u(s)} v_\lambda^2(s)ds = V_\lambda^2(t), \quad t \in [\delta, 1],$$

and therefore, by Lemma 2.1,

$$|Y_{\lambda,k}(t) - Y_\lambda(t)|^2 + \|Z_{\lambda,k}(t) - Z_\lambda(t)\|^2 \equiv 0, \quad 0 < t < 1.$$

□

It is known that if

$$\varphi_\lambda(t, Y(t), Z(t)) \xrightarrow{P} \varphi_{\lambda_0}(t, Y(t), Z(t)), \quad \lambda \to \lambda_0 \tag{3.4}$$

then

$$\lim_{\lambda\to\lambda_0} \int_t^1 \|f_\lambda(s, Y(s), Z(s)) - f_{\lambda_0}(s, Y(s)), Z(s)\|^2 ds = 0. \tag{3.5}$$

Theorem 3.3. *In the hypotheses i)-v), if*

$$\lim_{\lambda\to\lambda_0} |\xi_\lambda - \xi_{\lambda_0}|^2 = 0,$$

and

$$\lim_{\lambda\to\lambda_0} \int_t^1 \|f_\lambda(s, Y(s), Z(s)) - f_{\lambda_0}(s, Y(s)), Z(s)\|^2 ds = 0,$$

then

$$\lim_{\lambda\to\lambda_0} |||(Y_\lambda, Z_\lambda) - (Y_{\lambda_0}, Z_{\lambda_0})|||^2 = 0$$

on $(0, 1)$.

Proof. In similar way as in the proof of Theorem 3.1, we have

$$|Y_\lambda(t) - Y_{\lambda_0}(t)|^2 + \|Z_\lambda(t) - Z_{\lambda_0}(t)\|^2 \leq$$

$$\leq 2(|\xi_\lambda - \xi_{\lambda_0}|^2 + 2I_2 + 2\int_t^1 \frac{u'(s)}{A_1 u(s)}[|Y_\lambda(s) - Y_{\lambda_0}(s)|^2 + \|Z_\lambda(s) - Z_{\lambda_0}(s)\|^2]ds),$$

where $I_2 = \int_t^1 |f_\lambda(s, X_{\lambda_0}(s), Y_{\lambda_0}(s), Z_{\lambda_0}(s)) - f_{\lambda_0}(s, X_{\lambda_0}(s), Y_{\lambda_0}(s), Z_{\lambda_0}(s))|^2]ds$. Noting that

$$M_2(\lambda) = |\xi_\lambda - \xi_{\lambda_0}|^2 + 2I_2$$

and using the hypothesis, we obtain

$$\lim_{\lambda \to \lambda_0} M_2(\lambda) = 0.$$

Consequently,

$$|Y_\lambda(t) - Y_{\lambda_0}(t)|^2 + \|Z_\lambda(t) - Z_{\lambda_0}(t)\|^2 \leq$$

$$\leq 2M_2(\lambda) + 4\int_t^1 \frac{u'(s)}{A_1 u(s)}[|Y_\lambda(s) - Y_{\lambda_0}(s)|^2 + \|Z_\lambda(s) - Z_{\lambda_0}(s)\|^2]ds,$$

and then, by Lebesgue's convergence theorem

$$\begin{aligned}
\lim_{\lambda \to \lambda_0} [|Y_\lambda(t) - Y_{\lambda_0}(t)|^2 + \|Z_\lambda(t) - Z_{\lambda_0}(t)\|^2] &\leq \\
&\leq 4\int_t^1 \frac{u'(s)}{A_1 u(s)}\Big(\lim_{\lambda \to \lambda_0}[|Y_\lambda(s) - Y_{\lambda_0}(s)|^2 + \|Z_\lambda(s) - Z_{\lambda_0}(s)\|^2]\Big)ds \\
&\leq \int_t^1 \frac{u'(s)}{u(s)} v(s)ds,
\end{aligned}$$

where $v(s) = \lim_{\lambda \to \lambda_0} [|Y_\lambda(s) - Y_{\lambda_0}(s)|^2 + \|Z_\lambda(s) - Z_{\lambda_0}(s)\|^2]ds$ and $A_1 \geq 4$.

Noting that

$$V(t) = \int_t^1 \frac{u'(s)}{u(s)} v(s)ds,$$

similarly as in the proof of Theorem 3.2, using Lemma 2.1, we obtain that $V(t) \equiv 0$, which concludes the proof. □

4. Comments and examples

For more applications with some unexpected external perturbations, there appears a discontinuity in a time moment (in our paper are two such time moments as to the initial time moment $t = 0$ and to the final time $t = 1$) and before this time moment the stochastic control shows its utility by controlling these perturbations. For example, on several financial markets (especially transition financial markets) the strike price for a derivative financial asset is over quoted or higher quoted, and this yields a discontinuity in the path of this asset.

Example 4.1. The following example shows the consistency of our condition:

$$f(t,y,z) := e^{\sqrt{t}}(\sqrt{y} - \sqrt[3]{z^2} + 1)$$

and

$$u(t) = \arcsin(t)$$

After some complex but not very hard computations, it can be showed that the control function u and the coefficient function f satisfy our assumptions.

Moreover, because the control function u has the property $u'(t) \to \infty,\ t \to 1^-$, we have that the coefficient function f can have a jump in a neighborhood of $t = 1$ and therefore, the assumption iv) does not require the continuity of the coefficient functions at the final moment $t = 1$. This situation is frequently observed in applications of BSDE's to financial modeling. The control function $u(t)$ plays the role of risk-free asset for the optimal hedging theory in stochastic finances.

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References

[1] F. Antonelli, *Backward-Forward stochastic differential equations*, Annals of Applied Prob. 3 (1993), 777–793. 1

[2] Z. S. Athanassov, *Uniqueness and convergence of successive approximations for ordinary differential equations*, Math. Japonica 53 (2) (1990), 351–367. 1

[3] P. Briand and Y. Hu, *BSDE with quadratic growth and unbounded terminal value*, Probability Theory and Related Fields 136 (4) (2006), 604–618. 1

[4] R. Buckdahn, H.-J. Engelbert and A. Rascanu, *On weak solutions of backward stochastic differential equations*, Rossiiskaya Akademiya Nauk. Teoriya Veroyatnostei i ee Primeneniya 49 (1) (2004), 70–108. 1

[5] A. Constantin, *Global Existence of Solutions for Perturbed Differential Equations*, Annali di Mat.Pura ed Appl., Serie IV ,Tom CLXVIII (1995), 237–299. 1

[6] A. Constantin, *On the Existence and Pathwise Uniqueness of Solutions of Stochastic Differential Equations*, Stochastic and Stochastic Reports, 56 (1996), 227–239. 1

[7] A. Constantin, *On the Existence and Uniqueness of Solutions of McShane Type Stochastic Differential Equations*, Stoch. Anal. Appl. 16 (2) (1998), 217–229. 1, 3.2

[8] Gh. Constantin, *The uniqueness of solutions of perturbed backward stochastic differential equations*, J. Math. Anal. Appl. 300 (2004), 12–16. 1

[9] Gh. Constantin and R. Negrea, *An application of Schauder's Fixed Point Theorem in Stochastic McShane Modeling*, J. Fixed Point Theory 5 (4) (2004), 37–52. 1, 3.2

[10] J. Cvitanic and I. Karatzas, *Hedging contingent claims with constrained portfolios*, Ann. Appl. Probab. 3 (1993), 652–681. 1, 3.1

[11] D. Duffie, D. and L. Epstein, *Stochastic differential utility*, Econometrica, 60 (2) (1992), 353–394. 1, 3.1

[12] Y. Hu, P. Imkeller and M. Müller, *Utility maximization in incomplete markets*, The Annals of Applied Probability 15 (3) (2005), 1691–1712. 1, 3.1

[13] I. Karatzas and S. Shreve, *Methods for Mathematical Finance*, Springer, Berlin, Heidelberg, New York, 1997. 1, 3.1

[14] N. El Karoui, S. Peng and M. C. Quenez, *Backward Stochastic Differential Equations in Finance*, Mathematical Finance 7 (1) (1997), 1–71. 1, 3.1

[15] M. Kobylanski, *Backward stochastic differential equations and partial differential equations with quadratic growth*, The Annals of Probability 28 (2) (2000), 558–602. 1

[16] J.-P. Lepeltier and J. San Martin, *Backward stochastic differential equations with continuous generator*, Statist. Probab. Letters 32 (1997), 425–430. 1

[17] J.-P. Lepeltier and J. San Martin, *Existence for BSDE with superlinear-quadratic coefficient*, Stoch. Stoch. Reports 63 (1998), 227–240. 1

[18] J.-P. Lepeltier and J. San Martin, *On the existence or non-existence of solutions for certain backward stochastic differential equations*, Bernoulli 8 (2002), 123–137. 1

[19] J. Ma, P. Protter and J. Yong, *Solving forward-backward stochastic differential equations explicity - a four step scheme*, Probab. Theory and Rel. Fields 98 (1994), 339–359. 1

[20] X. Mao, *Adapted Solutions of Backward Stochastic Differential Equations with non Lipschitz coefficients*, Stochastic Processes and their Applications 58 (1995), 281–292. 1

[21] R. Negrea, *On the uniqueness and convergence of successive approximations for a class of stochastic differential equations*, Analele Univ. Bucuresti Ser. Mat. 52 (2003), 225–234. 1

[22] R. Negrea, *An application of Schauder's Fixed Point Theorem to Backward Stochastic Differential Equations*, J. Fixed Point Theory 9 (1) (2008), 199–206. 1

[23] R. Negrea and C. Preda, *On a class of backward stochastic differential equations*, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 18 (2011), 485–499. 1, 2.1

[24] R. Negrea and C. Preda, *On the existence and uniqueness of solution for a class of forward - backward stochastic differential equations*, to appear. 1, 3.1

[25] E. Pardoux and S. G. Peng, *Adapted solution of a backward stochastic differential equation*, Systems & Control Letters 14 (1990), 55–61. 1, 1

[26] E. Pardoux, A. Rascanu, *Backward SDE's with sub differential operator and related variational inequalities*, Stochastic Processes and their Applications 76 (2) (1998), 191–215. 1

[27] S. Peng, *Backward stochastic differential equation and it's application in optimal control*, Appl. Math. and Optim. 27 (1993), 125–144. 1