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Coincidence point results of multivalued weak C-contractions on metric spaces with a partial order

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Dedicated to the memory of Professor Viorel Radu

Communicated by Professor D. Miheţ

Abstract

In this paper we obtain some coincidence point results of a family of multivalued mappings with a singlevalued mapping in a complete metric space endowed with a partial order. We use δ- distance in this paper. A generalized weak C-contraction inequality for multivalued functions and δ-compatibility for certain pairs of functions are assumed in the theorems. The corresponding singled valued cases are shown to extend a number of existing results. An example is constructed which shows that the extensions are actual improvements.

Keywords: Partially ordered set, multivalued C-contraction, δ - compatible, control function, coincidence point.
2010 MSC: 54 H 10, 54 H 25, 47 H 10.

1. Introduction

The purpose of this paper is to establish some coincidence point results for weak multivalued C-contraction type mappings in partially ordered metric spaces. Weakening of contractive inequalities began with the work of Alber et al. [2] when they established a weak version of the Banach contraction mapping principle in Hilbert spaces. Later it was proved by Rhoades [25] that the weak contraction introduced in [2] has necessarily a unique fixed point in a complete metric space. Following this result many authors have created weak contraction inequalities and obtained fixed point theorems with the help of these inequalities.

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In the fixed point theory of setvalued maps two types of distances are generally used. One is the Hausdorff distance. Nadler [22] had proved a multivalued version of the Banach's contraction mapping principle by using the Hausdorff metric. There are many other fixed point results using this Hausdorff metric, some instances of these works are in [14, 15, 28]. The another distance is the δ - distance. This is not metric like the Hausdorff distance, but shares most of the properties of a metric. It has been used in many problem on fixed point theory like those in [1, 3, 19].

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. References [10, 23, 24] are some recent instances of these works. A speciality of these problems is that they use both analytic and order theoretic methods. It is also one of the main reasons why they are considered interesting.

Khan et al. [20] initiated the use of a control function in metric fixed point theory which they called Altering distance function. Several works on fixed point theory like those noted in [9, 11, 16, 26] have utilized this control function.

C-contractions are contractive mappings which are different from Banach's contraction. This category of contraction was introduced by Chatterjea [7]. Like the weakened of the Banach's contraction inequality, the C-contraction was weakened in [8]. In the same work it has been shown that the weak C-contraction has a unique fixed point in complete metric spaces. Following this work several other works on C-contraction have appeared in [4, 17, 21, 27].

In this paper we utilize a weak C-contraction inequality with δ- distance to establish the existence of a coincidence point of a family of multivalued functions with a singlevalued function in a complete metric space with a partial order. We have also utilized δ-compatible pairs in our theorems. In another theorem we have replaced the continuities of the functions with an order condition. We also give here the corresponding singlevalued versions of the theorems which generalize a number of existing works. An illustrative example for the multivalued case is given.

2. Mathematical Preliminaries

In the following we give some technical definitions which are used in our theorems.

Let $(X,\ d)$ be a metric space. We denote the class of nonempty and bounded subsets of X by $B(X)$. For $A,\ B \in B(X)$, functions $D(A,\ B)$ and $\delta(A,\ B)$ are defined as follows:

$D(A,\ B) = \inf\ \{d(a,\ b) : a \in A,\ b \in B\}$,

$\delta(A,\ B) = \sup\ \{d(a,\ b) : a \in A,\ b \in B\}$.

If $A = \{a\}$, then we write $D(A,\ B) = D(a,\ B)$ and $\delta(A,\ B) = \delta(a,\ B)$. Also in addition, if $B = \{b\}$, then $D(A,\ B) = d(a,\ b)$ and $\delta(A,\ B) = d(a,\ b)$. Obviously, $D(A,\ B) \le \delta(A,\ B)$. For all $A,\ B,\ C \in B(X)$, the definition of $\delta(A,\ B)$ yields the following:

$\delta(A,\ B) = \delta(B,\ A)$,

$\delta(A,\ B) \le \delta(A,\ C) + \delta(C,\ B)$,

$\delta(A,\ B) = 0$ iff $A = B = \{a\}$,

$\delta(A,\ A) = \text{diam } A$. [12]

There are several works which have utilized δ - distance [3, 5, 12, 13, 19].

Lemma 2.1. *([12]) If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$, where (X, d) is a complete metric space and $\{A_n\} \to A$ and $\{B_n\} \to B$ for $A, B \in B(X)$, then $\delta(A_n, B_n) \to \delta(A, B)$ as $n \to \infty$.*

Lemma 2.2. *([13]) If $\{A_n\}$ is a sequence of bounded sets in a complete metric space (X, d) and if $\lim_{n\to\infty} \delta(A_n, \{y\}) = 0$, for some $y \in X$, then $\{A_n\} \to \{y\}$.*

Definition 2.3. ([13]) A set-valued mapping $T : X \to B(X)$, where $(X,\ d)$ is a metric space, is continuous at a point $x \in X$ if $\{x_n\}$ is a sequence in X converging to x, then the sequence $\{Tx_n\}$ in $B(X)$ converges to Tx. T is said to be continuous in X if it is continuous at each point $x \in X$.

Definition 2.4. ([18]) Two self maps g and T of a metric space (X, d) are said to be compatible mappings if $\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

Definition 2.5. ([19]) The mappings $g : X \to X$ and $T : X \to B(X)$, where $(X,\ d)$ is a metric space, are δ- compatible if $\lim_{n\to\infty} \delta(Tgx_n, gTx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $gTx_n \in B(X)$ and $Tx_n \to \{t\}$, $gx_n \to t$, for some t in X.

Definition 2.6. Let $(X,\ d)$ be a metric space and $g : X \to X$ and $T : X \to B(X)$. Then $u \in X$ is called a coincidence point of g and T if $\{gu\} = Tu$.

Definition 2.7. ([5]) Let A and B be two nonempty subsets of a partially ordered set $(X,\ \preceq)$. The relation between A and B is denoted and defined as follows:

$A \prec_1 B$, if for every $a \in A$ there exists $b \in B$ such that $a \preceq b$.

Definition 2.8. ([20]) A function $\psi : [0,\infty) \to [0,\infty)$ is called an Altering distance function if the following properties are satisfied:
(i) ψ is monotone increasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

For $(x, y),\ (u, v) \in \mathbb{R} \times \mathbb{R}$, we say $(x, y) \le (u, v)$ if and only if $x \le u$ and $y \le v$.

Definition 2.9. A function $\phi : [0,\infty)^2 \to [0,\infty)$ is said to be monotone nondecreasing if for $(x, y),\ (u, v) \in [0,\infty)^2$, $(x, y) \le (u, v) \implies \phi(x, y) \le \phi(u, v)$.

As already mentioned, we introduce here the definition of weak multivalued C-contraction type mapping in the following.

Definition 2.10. ([7]) A mapping $T : X \to X$, where $(X,\ d)$ is a metric space, is called a C-contraction if there exists $0 < k < \frac{1}{2}$ such that

$$d(Tx, Ty) \le k\ [d(x, Ty) + d(y, Tx)], \quad \text{for all } x,\ y \in X. \tag{2.1}$$

Definition 2.11. ([8]) A mapping $T : X \to X$, where $(X,\ d)$ is a metric space, is said to be weak C-contractive if for all $x,\ y \in X$,

$$d(Tx, Ty) \le \frac{1}{2}\ [d(x, Ty) + d(y, Tx)] - \phi(d(x, Ty),\ d(y, Tx)), \tag{2.2}$$

where $\phi : [0,\infty)^2 \to [0,\infty)$ is a continuous function with $\phi(x,\ y) = 0$ if and only if $(x,\ y) = (0,\ 0)$.

Definition 2.12. A mapping $T : X \to X$, where $(X,\ d)$ is a metric space, is said to be generalized weak C-contractive if for all $x,\ y \in X$,

$$\psi(d(Tx, Ty)) \le \psi(\frac{1}{2}\ [d(x, Ty) + d(y, Tx)]) - \phi(d(x, Ty),\ d(y, Tx)) \tag{2.3}$$

where ψ is an Altering distance function and $\phi : [0,\infty)^2 \to [0,\infty)$ is a continuous function with $\phi(x,\ y) = 0$ if and only if $(x,\ y) = (0,\ 0)$.

(2.3) reduces to (2.2) if ψ is considered to be the identity mapping. Also, if one takes ψ to be the identity mapping and $\phi(x,\ y) = (\frac{1}{2} - k)(x + y)$, where $0 < k < \frac{1}{2}$, then (2.3) reduces to (2.1). Hence generalized weak C-contraction mappings are generalizations of weak C-contraction and C-contraction mappings.

Definition 2.13. A multivalued mapping $T : X \to B(X)$, where $(X,\ d)$ is a metric space, is said to be weak multivalued C-contractive if for all $x,\ y \in X$,

$$\psi(\delta(Tx, Ty)) \le \psi(\frac{1}{2}\ [D(x, Ty) + D(y, Tx)]) - \phi(\delta(x, Ty),\ \delta(y, Tx)), \tag{2.4}$$

where ψ is an Altering distance function and $\phi : [0,\infty)^2 \to [0,\infty)$ is a continuous function with $\phi(x,\ y) = 0$ if and only if $(x,\ y) = (0,\ 0)$.

3. Main Results

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $\phi : [0, \infty)^2 \to [0, \infty)$ be a monotone nondecreasing and continuous function with $\phi(x, y) = 0$ if and only if $(x, y) = (0, 0)$ and ψ be an altering distance function. Let $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\}$ be a family of multivalued mappings. Let $g : X \to X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that*

(i) *T_{α_0} and g are continuous,*
(ii) *$T_{\alpha_0}x \subseteq g(X)$ and $gT_{\alpha_0}x \in B(X)$, for every $x \in X$,*
(iii) *there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 T_{\alpha_0}x_0$,*
(iv) *for $x, y \in X$, $gx \preceq gy$ implies $T_{\alpha_0}x \prec_1 T_{\alpha_0}y$,*
(v) *the pair (g, T_{α_0}) is δ - compatible,*
(vi) *$\psi(\delta(T_{\alpha_0}x, T_\alpha y)) \le \psi(\frac{1}{2}[D(gx, T_\alpha y) + D(gy, T_{\alpha_0}x)]) - \phi(\delta(gx, T_\alpha y), \delta(gy, T_{\alpha_0}x))$,*

where $x, y \in X$ such that gx and gy are comparable and $\alpha \in \Lambda$.
Then g and $\{T_\alpha : \alpha \in \Lambda\}$ have a coincidence point.

Proof. First we establish that any coincidence point of g and T_{α_0} is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$ and conversely. Suppose that $p \in X$ be a coincidence point of g and T_{α_0}. Then $\{gp\} = T_{\alpha_0}p$. From (vi) and using the monotone property of ψ, we have

$$\begin{aligned}\psi(\delta(gp, T_\alpha p)) &= \psi(\delta(T_{\alpha_0}p, T_\alpha p))\\ &\le \psi(\frac{1}{2}[D(gp, T_\alpha p) + D(gp, T_{\alpha_0}p)]) - \phi(\delta(gp, T_\alpha p), \delta(gp, T_{\alpha_0}p))\\ &\le \psi(\frac{1}{2} D(gp, T_\alpha p)) \ \text{(by a property of } \phi).\end{aligned}$$

Again using the monotone property of ψ, we have

$$\delta(gp, T_\alpha p) \le \frac{1}{2} D(gp, T_\alpha p),$$

which implies that

$$\delta(gp, T_\alpha p) \le \frac{1}{2} D(gp, T_\alpha p) \le \frac{1}{2} \delta(gp, T_\alpha p),$$

which implies that $\delta(gp, T_\alpha p) = 0$, that is, $\{gp\} = T_\alpha p$, for all $\alpha \in \Lambda$. Hence p is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. Converse part is trivial.

Now it is sufficient to prove that g and T_{α_0} have a coincidence point. Let $x_0 \in X$ be such that $\{gx_0\} \prec_1 T_{\alpha_0}x_0$. Then there exists $u \in T_{\alpha_0}x_0$ such that $gx_0 \preceq u$. Since $T_{\alpha_0}x_0 \subseteq g(X)$ and $u \in T_{\alpha_0}x_0$, there exists $x_1 \in X$ such that $gx_1 = u$. So $gx_0 \preceq gx_1$. Then by the assumption (iv), $T_{\alpha_0}x_0 \prec_1 T_{\alpha_0}x_1$. Since $u = gx_1 \in T_{\alpha_0}x_0$, there exists $v \in T_{\alpha_0}x_1$ such that $gx_1 \preceq v$. As $T_{\alpha_0}x_1 \subseteq g(X)$ and $v \in T_{\alpha_0}x_1$, there exists $x_2 \in X$ such that $gx_2 = v$. So $gx_1 \preceq gx_2$. Continuing this process we construct a sequence $\{x_n\}$ in X such that

$$gx_{n+1} \in T_{\alpha_0}x_n, \text{ for all } n \ge 0, \tag{3.1}$$

and

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq \ldots\ldots \preceq gx_n \preceq gx_{n+1} \ldots. \tag{3.2}$$

Since $gx_n \preceq gx_{n+1}$, putting $\alpha = \alpha_0$, $x = x_{n+1}$ and $y = x_n$ in (vi) and using the monotone properties of ψ and ϕ, we have

$$\begin{aligned}\psi(d(gx_{n+2},\ gx_{n+1})) &\le \psi(\delta(T_{\alpha_0}x_{n+1},\ T_{\alpha_0}x_n)) \\ &\le \psi(\frac{1}{2}\ [D(gx_{n+1}, T_{\alpha_0}x_n) + D(gx_n, T_{\alpha_0}x_{n+1})]) \\ &\quad - \phi(\delta(gx_{n+1}, T_{\alpha_0}x_n),\ \delta(gx_n, T_{\alpha_0}x_{n+1})) \\ &\le \psi(\frac{1}{2}[d(gx_{n+1},\ gx_{n+1}) + d(gx_n,\ gx_{n+2})]) \\ &\quad - \phi(d(gx_{n+1},\ gx_{n+1}),\ d(gx_n,\ gx_{n+2})) \\ &= \psi(\frac{1}{2}d(gx_n,\ gx_{n+2})) - \phi(0,\ d(gx_n,\ gx_{n+2})), \end{aligned} \tag{3.3}$$

which by monotone property of ψ and a property of ϕ implies that

$$d(gx_{n+2},\ gx_{n+1}) \le \frac{1}{2}d(gx_n,\ gx_{n+2}) \le \frac{1}{2}[d(gx_n,\ gx_{n+1}) + d(gx_{n+1},\ gx_{n+2})], \tag{3.4}$$

that is,

$$d(gx_{n+2},\ gx_{n+1}) \le d(gx_{n+1},\ gx_n).$$

Therefore, $\{d(gx_{n+1},\ gx_n)\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists an $r \ge 0$ such that

$$\lim_{n\to\infty} d(gx_{n+1}, gx_n) = r. \tag{3.5}$$

Taking the limit as $n \to \infty$ in (3.4) and using (3.5), we have

$$\lim_{n\to\infty} d(gx_n,\ gx_{n+2}) = 2r. \tag{3.6}$$

Letting $n \to \infty$ in (3.3), using (3.5), (3.6) and continuities of ψ and ϕ, we have

$$\psi(r) \le \psi(r) - \phi(0,\ 2r),$$

which is a contradiction unless $r = 0$. Hence

$$\lim_{n\to\infty} d(gx_{n+1},\ gx_n) = 0 \tag{3.7}$$

and

$$\lim_{n\to\infty} d(gx_n,\ gx_{n+2}) = 0. \tag{3.8}$$

Next we show that $\{gx_n\}$ is a Cauchy sequence. If $\{gx_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(gx_{n(k)},\ gx_{m(k)}) \ge \epsilon$. Assuming that $n(k)$ is the smallest such positive integer, we get

$$n(k) > m(k) > k,$$

$$d(gx_{n(k)},\ gx_{m(k)}) \ge \epsilon$$

and

$$d(gx_{n(k)-1},\ gx_{m(k)}) < \epsilon.$$

Now,

$$\epsilon \le d(gx_{n(k)},\ gx_{m(k)}) \le d(gx_{n(k)},\ gx_{n(k)-1}) + d(gx_{n(k)-1},\ gx_{m(k)}),$$

that is,

$$\epsilon \le d(gx_{n(k)},\ gx_{m(k)}) < d(gx_{n(k)},\ gx_{n(k)-1}) + \epsilon.$$

Letting $k \to \infty$ in the above inequality and using (3.7), we have

$$\lim_{k\to\infty} d(gx_{n(k)},\ gx_{m(k)}) = \epsilon. \tag{3.9}$$

Again,

$$d(gx_{n(k)},\ gx_{m(k)}) \leq d(gx_{n(k)},\ gx_{n(k)+1}) + d(gx_{n(k)+1},\ gx_{m(k)+1}) + d(gx_{m(k)+1},\ gx_{m(k)})$$

and

$$d(gx_{n(k)+1},\ gx_{m(k)+1}) \leq d(gx_{n(k)+1},\ gx_{n(k)}) + d(gx_{n(k)},\ gx_{m(k)}) + d(gx_{m(k)},\ gx_{m(k)+1}).$$

Letting $k \to \infty$ in above inequalities, using (3.7) and (3.9), we have

$$\lim_{k\to\infty} d(gx_{n(k)+1},\ gx_{m(k)+1}) = \epsilon. \tag{3.10}$$

Again,

$$d(gx_{n(k)},\ gx_{m(k)}) \leq d(gx_{n(k)},\ gx_{m(k)+1}) + d(gx_{m(k)+1},\ gx_{m(k)})$$

and

$$d(gx_{n(k)},\ gx_{m(k)+1}) \leq d(gx_{n(k)},\ gx_{m(k)}) + d(gx_{m(k)},\ gx_{m(k)+1}).$$

Letting $k \to \infty$ in the above inequalities and using (3.7) and (3.9), we have

$$\lim_{k\to\infty} d(gx_{n(k)},\ gx_{m(k)+1}) = \epsilon. \tag{3.11}$$

Similarly, we have

$$\lim_{k\to\infty} d(gx_{m(k)},\ gx_{n(k)+1}) = \epsilon. \tag{3.12}$$

For each positive integer k, $gx_{m(k)}$ and $gx_{n(k)}$ are comparable. Then putting $\alpha = \alpha_0$, $x = x_{n(k)}$ and $y = x_{m(k)}$ in (vi) and using the monotone properties of ψ and ϕ, we have

$$\begin{aligned}\psi(d(gx_{n(k)+1}, gx_{m(k)+1})) &\leq \psi(\delta(T_{\alpha_0}x_{n(k)},\ T_{\alpha_0}x_{m(k)}))\\ &\leq \psi(\frac{1}{2}[D(gx_{n(k)},\ T_{\alpha_0}x_{m(k)}) + D(gx_{m(k)},\ T_{\alpha_0}x_{n(k)})])\\ &\quad -\phi(\delta(gx_{n(k)},\ T_{\alpha_0}x_{m(k)}),\ \delta(gx_{m(k)},\ T_{\alpha_0}x_{n(k)}))\\ &\leq \psi(\frac{1}{2}[d(gx_{n(k)},\ gx_{m(k)+1}) + d(gx_{m(k)},\ gx_{n(k)+1})])\\ &\quad -\phi(d(gx_{n(k)},\ gx_{m(k)+1}),\ d(gx_{m(k)},\ gx_{n(k)+1})).\end{aligned}$$

Letting $k \to \infty$ in the above inequality, using (3.10), (3.11), (3.12) and the properties of ϕ and ψ, we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon, \epsilon),$$

which is a contradiction by virtue of a property of ϕ. Hence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since X is complete and $g(X)$ is closed in X, there exists $u \in g(X)$ such that

$$gx_n \to u \quad \text{as} \quad n \to \infty.$$

Since $u \in g(X)$, there exists $z \in X$ such that $u = gz$. Then

$$gx_n \to u = gz \quad \text{as} \quad n \to \infty. \tag{3.13}$$

From (3.3), we have

$$\begin{aligned}\psi(d(gx_{n+2},\ gx_{n+1})) &\leq \psi(\delta(T_{\alpha_0}x_{n+1},\ T_{\alpha_0}x_n))\\ &\leq \psi(\frac{1}{2}d(gx_n,\ gx_{n+2})) - \phi(0,\ d(gx_n,\ gx_{n+2})).\end{aligned}$$

Using the properties of ψ and ϕ, we have

$$d(gx_{n+2},\ gx_{n+1}) \leq \delta(T_{\alpha_0}x_{n+1},\ T_{\alpha_0}x_n) \leq \frac{1}{2}d(gx_n,\ gx_{n+2}).$$

Taking $n \to \infty$ in the above inequality, and using (3.7) and (3.8), we have

$$\lim_{n\to\infty} \delta(T_{\alpha_0}x_{n+1},\ T_{\alpha_0}x_n) = 0. \tag{3.14}$$

Now,

$$\delta(T_{\alpha_0}x_n, \{u\}) \leq \delta(T_{\alpha_0}x_n, gx_n) + \delta(gx_n, \{u\}) \leq \delta(T_{\alpha_0}x_n, T_{\alpha_0}x_{n-1}) + d(gx_n, u).$$

Letting $n \to \infty$ in the above inequality using (3.13) and (3.14), we have

$$\lim_{n\to\infty} \delta(T_{\alpha_0}x_n,\ \{u\}) = 0,$$

which by Lemma 2.2 implies that

$$T_{\alpha_0}x_n \to \{u\} \quad \text{as} \quad n \to \infty. \tag{3.15}$$

Since the pair $(g,\ T_{\alpha_0})$ is δ - compatible, from (3.13) and (3.15), we have

$$\lim_{n\to\infty} \delta(T_{\alpha_0}gx_n,\ gT_{\alpha_0}x_n) = 0.$$

As g and T_{α_0} are continuous, it follows that $\delta(T_{\alpha_0}u,\ gu) = 0$, that is, $T_{\alpha_0}u = \{gu\}$. Hence $u \in g(X) \subseteq X$ is a coincidence point of g and T_{α_0}. By what we have already proved, u is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. □

In our next theorem we relax the continuity assumption on T_{α_0} and g by imposing an order condition. We also relax the condition that $gT_{\alpha_0}x \in B(X)$, for every $x \in X$.

Theorem 3.2. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n. Let $\phi : [0,\infty)^2 \to [0,\infty)$ be a monotone nondecreasing and continuous function with $\phi(x,\ y) = 0$ if and only if $(x,\ y) = (0,\ 0)$ and ψ be an altering distance function. Let $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\}$ be a family of multivalued mappings. Let $g : X \to X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that*

(i) *$T_{\alpha_0}x \subseteq g(X)$ for every $x \in X$,*
(ii) *there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 T_{\alpha_0}x_0$,*
(iii) *for $x,\ y \in X$, $gx \preceq gy$ implies $T_{\alpha_0}x \prec_1 T_{\alpha_0}y$,*
(iv) *$\psi(\delta(T_{\alpha_0}x,\ T_\alpha y)) \leq \psi(\frac{1}{2}\ [D(gx, T_\alpha y) + D(gy, T_{\alpha_0}x)]) - \phi(\delta(gx, T_\alpha y),\ \delta(gy, T_{\alpha_0}x))$,*

where $x,\ y \in X$ such that gx and gy are comparable and $\alpha \in \Lambda$.
Then g and $\{T_\alpha : \alpha \in \Lambda\}$ have a coincidence point.

Proof. We take the same sequence $\{gx_n\}$ as in the proof of Theorem 3.1 Then we have $gx_{n+1} \in T_{\alpha_0}x_n$, for all $n \geq 0$, $\{gx_n\}$ is monotonic nondecreasing and $gx_n \to gz$ as $n \to \infty$. Then by the order condition of the metric space, we have $gx_n \preceq gz$, for all n.
Using the monotone properties of ψ and ϕ, and the condition (iv), we have

$$\begin{aligned}
\psi(\delta(gx_{n+1},\ T_\alpha z)) &\leq \psi(\delta(T_{\alpha_0}x_n,\ T_\alpha z)) \\
&\leq \psi(\frac{1}{2}\ [D(gx_n,\ T_\alpha z) + D(gz,\ T_{\alpha_0}x_n)]) \\
&\quad -\phi(\delta(gx_n,\ T_\alpha z),\ \delta(gz,\ T_{\alpha_0}x_n)) \\
&\leq \psi(\frac{1}{2}\ [D(gx_n,\ T_\alpha z) + d(gz,\ gx_{n+1})] \\
&\quad -\phi(\delta(gx_n,\ T_\alpha z),\ d(gz,\ gx_{n+1})).
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality and using the continuities of ϕ and ψ, we have

$$\psi(\delta(gz,\ T_\alpha z)) \leq \psi(\frac{1}{2}\ D(gz,\ T_\alpha z)) - \phi(\delta(gz,\ T_\alpha z),\ 0),$$

which implies that

$$\psi(\delta(gz,\ T_\alpha z)) \leq \psi(\frac{1}{2}\ D(gz,\ T_\alpha z)) \text{ (by a property of } \phi).$$

Using the monotone property of ψ, we have

$$\delta(gz,\ T_\alpha z) \leq \frac{1}{2}\ D(gz,\ T_\alpha z),$$

which implies that

$$\delta(gz,\ T_\alpha z) \leq \frac{1}{2}\ D(gz,\ T_\alpha z) \leq \frac{1}{2}\ \delta(gz,\ T_\alpha z),$$

which implies that $\delta(gz,\ T_\alpha z) = 0$, that is, $\{gz\} = T_\alpha z$, for all $\alpha \in \Lambda$. Hence z is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. □

Considering $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\} = \{T\}$ in theorem 3.1, we have the following corollary.

Corollary 3.3. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $\phi : [0, \infty)^2 \to [0, \infty)$ be a monotone nondecreasing and continuous function with $\phi(x,\ y) = 0$ if and only if $(x,\ y) = (0,\ 0)$ and ψ be an altering distance function. Let $T : X \to B(X)$ be a multivalued mapping and $g : X \to X$ a mapping such that*
(i) T and g are continuous,
(ii) $Tx \subseteq g(X)$ and $gTx \in B(X)$, for every $x \in X$, and $g(X)$ is closed in X,
(iii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 Tx_0$,
(iv) for $x,\ y \in X$, $gx \preceq gy$ implies $Tx \prec_1 Ty$,
(v) the pair $(g,\ T)$ is δ - compatible,
(vi) $\psi(\delta(Tx,\ Ty)) \leq \psi(\frac{1}{2}\ [D(gx, Ty) + D(gy, Tx)]) - \phi(\delta(gx, Ty),\ \delta(gy, Tx))$,
where $x,\ y \in X$ such that gx and gy are comparable.
Then g and T have a coincidence point.

Considering $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\} = \{T\}$ in theorem 3.2, we have the following corollary.

Corollary 3.4. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n. Let $\phi : [0, \infty)^2 \to [0, \infty)$ be a monotone nondecreasing and continuous function with $\phi(x,\ y) = 0$ if and only if $(x,\ y) = (0,\ 0)$ and ψ be an altering distance function. Let $T : X \to B(X)$ be a multivalued mapping and $g : X \to X$ a mapping such that*
(i) $Tx \subseteq g(X)$ for every $x \in X$, and $g(X)$ is closed in X,
(ii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 Tx_0$,
(iii) for $x,\ y \in X$, $gx \preceq gy$ implies $Tx \prec_1 Ty$,
(iv) $\psi(\delta(Tx,\ Ty)) \leq \psi(\frac{1}{2}\ [D(gx, Ty) + D(gy, Tx)]) - \phi(\delta(gx, Ty),\ \delta(gy, Tx))$,
where $x,\ y \in X$ such that gx and gy are comparable.
Then g and T have a coincidence point.

The following theorems are single valued cases of the theorems 3.1 and 3.2 respectively. Here we treat T as a multivalued mapping in which case Tx is a singleton set for every $x \in X$. For the following theorems function ϕ need not to be monotone nondecreasing.

Theorem 3.5. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $\phi : [0, \infty)^2 \to [0, \infty)$ be a continuous function with $\phi(x,\ y) = 0$*

if and only if $(x,\ y) = (0,\ 0)$ *and* ψ *be an altering distance function. Let* $\{T_\alpha : X \to X : \alpha \in \Lambda\}$ *be a family of mappings. Let* $g : X \to X$ *be a mapping such that* $g(X)$ *is closed in* X*. Suppose that there exists* $\alpha_0 \in \Lambda$ *such that*
(i) T_{α_0} *and* g *are continuous,*
(ii) $T_{\alpha_0}(X) \subseteq g(X)$*,*
(iii) there exists $x_0 \in X$ *such that* $gx_0 \preceq T_{\alpha_0}x_0$*,*
(iv) for $x,\ y \in X$*,* $gx \preceq gy$ *implies* $T_{\alpha_0}x \preceq T_{\alpha_0}y$*,*
(v) the pair $(g,\ T_{\alpha_0})$ *is compatible,*
(vi) $\psi(d(T_{\alpha_0}x,\ T_\alpha y)) \leq \psi(\frac{1}{2}\ [d(gx, T_\alpha y) + d(gy, T_{\alpha_0}x)]) - \phi(d(gx, T_\alpha y),\ d(gy, T_{\alpha_0}x))$*,*
where $x,\ y \in X$ *such that* gx *and* gy *are comparable and* $\alpha \in \Lambda$*.*
Then g *and* $\{T_\alpha : \alpha \in \Lambda\}$ *have a coincidence point.*

Theorem 3.6. *Let* $(X,\ \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *on* X *such that* (X, d) *is a complete metric space. Assume that if* $x_n \to x$ *is a nondecreasing sequence in* X*, then* $x_n \preceq x$*, for all* n*. Let* $\phi : [0,\infty)^2 \to [0,\infty)$ *be a continuous function with* $\phi(x,\ y) = 0$ *if and only if* $(x,\ y) = (0,\ 0)$ *and* ψ *be an altering distance function. Let* $\{T_\alpha : X \to X : \alpha \in \Lambda\}$ *be a family of mappings. Let* $g : X \to X$ *be a mapping such that* $g(X)$ *is closed in* X*. Suppose that there exists* $\alpha_0 \in \Lambda$ *such that*
(i) $T_{\alpha_0}(X) \subseteq g(X)$*,*
(ii) there exists $x_0 \in X$ *such that* $gx_0 \preceq T_{\alpha_0}x_0$*,*
(iii) for $x,\ y \in X$*,* $gx \preceq gy$ *implies* $T_{\alpha_0}x \preceq T_{\alpha_0}y$*,*
(iv) $\psi(d(T_{\alpha_0}x,\ T_\alpha y)) \leq \psi(\frac{1}{2}\ [d(gx, T_\alpha y) + d(gy, T_{\alpha_0}x)]) - \phi(d(gx, T_\alpha y),\ d(gy, T_{\alpha_0}x))$*,*
where $x,\ y \in X$ *such that* gx *and* gy *are comparable and* $\alpha \in \Lambda$*.*
Then g *and* $\{T_\alpha : \alpha \in \Lambda\}$ *have a coincidence point.*

Example 3.7. Let $X = [0,\ \infty)$ with usual order $\leq$ be a partially ordered set.
Let $d : X \times X \to \mathbb{R}$ be given as

$$d(x,\ y) = \mid x - y \mid, \quad \text{for} \quad x,\ y \in X.$$

Then (X, d) is a complete metric space with the required properties of theorems 3.1 and 3.2.
Let $g : X \to X$ be defined as follows:

$$gx = 10x, \quad \text{for} \quad x \in X.$$

Then g has the properties mentioned in Theorems 3.1 and 3.2.
Let $\Lambda = \{1,\ 2,\ 3,\ ...\}$. Let the family of mappings $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\}$ be defined as follows:

$$T_1x = \{0\},\ \text{for} \quad x \in X, \quad \text{and for} \quad \alpha \geq 2, \quad T_\alpha x = \begin{cases} \{0\} & \text{if} \quad 0 \leq x \leq 1, \\ \{0, \frac{\alpha}{\alpha+1}\} & \text{if} \quad x > 1. \end{cases}$$

For any sequence $\{x_n\}$ in X such that $gT_1x_n \in B(X)$ and $T_1x_n \to \{t\}$, $gx_n \to t$, for some t in X implies $t = 0$. Then clearly, the pair $(g,\ T_1)$ is δ - compatible. Also, g and T_1 satisfy required conditions mentioned in theorems 3.1 and 3.2.
Let $\psi : [0,\ \infty) \to [0,\ \infty)$ be defined as follows:

$$\psi(t) = 8t^2,\ \text{for}\ t \in [0,\ \infty).$$

Let $\phi : [0,\ \infty)^2 \to [0,\ \infty)$ be defined as follows:
for $(x,\ y) \in [0,\ \infty)^2$ with $z = \ \max\ \{x,\ y\}$,

$$\phi(x,\ y) = \frac{z}{100}.$$

Then ψ and ϕ have the properties mentioned in theorems 3.1 and 3.2.
The condition (vi) of theorem 3.1 and the condition (iv) of theorem 3.2 are satisfied. Hence all the condition of theorems 3.1 and 3.2 are satisfied and it is seen that 0 is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$.

Note In the above example if one takes $g : X \to X$ to be function as follows:

$$gx = \begin{cases} \frac{x}{2} & \text{if } 0 \le x \le 1, \\ 200 & \text{if } x > 1. \end{cases}$$

Then the above example is still applicable to theorem 3.2 but not applicable to theorem 3.1 because g is not continuous and hence does not satisfy required properties mentioned in Theorem 3.1.

Remark 3.8. Considering ψ and g to be the identity mappings and $\{T_\alpha : \alpha \in \Lambda\} = \{T\}$ in theorems 3.5 and 3.6, we have respectively theorems 2.1 and 2.2 in [17]. It may be mentioned that theorems 2.1 and 2.2 of Harjani et al [17] are directly ordered version of the result proved by Choudhury [8] and generalization of ordered version of the result proved by Chatterjea in [7].

Remark 3.9. Considering ψ to be the identity mapping and $\{T_\alpha : \alpha \in \Lambda\} = \{T\}$ in theorems 3.5 and 3.6, we have theorem 2 in [6].

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