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On the generalized stability of d'Alembert functional equation

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Abstract

In this article, we study the super stability problem for the functional equation:

$$\sum_{\psi\in K_{n-1}} f(\psi(x_1,...,x_n)) = 2^{n-1}\prod_{i=1}^{n} f(x_i)$$

on an Abelian group and the unknown function f is (a complex or a semi simple Banach algebra valued).

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1. Introduction and preliminaries

Questions concerning the stability of functional equation seem to have been first raised by Ulam in [12]. Hyers in [6] showed that if $\delta \succ 0$ an $f : E \to F$, where E and F are Banach spaces, such that $\|f(x+y) - f(x) - f(y)\| \leq \delta$ for all $x, y \in E$ then there exists a unique $S : E \to F$ such that $S(x+y) = S(x) + S(y)$ and $\|f(x) - S(x)\| \leq \delta$ for all $x, y \in E$. In 1979, Baker et. al. [2] postulated that if f satisfies the inequality $|E_1(f) - E_2(f)| \leq \epsilon$, then either f is bounded or $E_1(f) = E_2(f)$.
There after it is called the super stability. Baker in [3] proved the super stability of cosine functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{1.1}$$

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Which is also called the d'Alembert functional equation. The stability of the generalized cosine functional equation has been investigated in many papers ([4, 5, 7, 8, 9, 10, 11]). In 2002, Badora and Ger [1] proved the super stability of d'Alembert functional equation concerning complex-valued mappings, as in the following theorem.

Theorem 1.1. *Let $(G,+)$ be an Abelien group. Let $f : G \to C$ and $\varphi : G \to [0,\infty)$ satisfy the inequality*

$$\|f(x+y)+f(x-y)-2f(x)f(y)\| \leq \varphi(x) \, or \, \varphi(y) \, \forall x,y \in G$$

Then, either f is bounded or f satisfies Eq. (1.1).

In this paper, let $(G,+)$ be an Abelian group, $\mathbb{C}$ the field of complex numbers, and $K_{n-1} = K^{n-1}$ with $K = \{+I, -I\}$. We consider the functional equation :

$$\sum_{\psi \in K_{n-1}} f(\psi(x_1,...,x_n)) = 2^{n-1} \prod_{i=1}^{n} f(x_i) \tag{1.2}$$

and the difference operator $Df : G^n \to \mathbb{C}$ as

$$Df(x_1,...,x_n) = \sum_{\psi \in K_{n-1}} f(\psi(x_1,...,x_n)) - 2^{n-1} \prod_{i=1}^{n} f(x_i)$$

with $\psi(x_1,...,x_n) = x_1 + \sum_{i=2}^{n} \alpha_i(x_i)$ and $\alpha_i \in K$.

The object of Theorem 1.1 is to show that the equation can be viewed as a generalization of the cosine functional equation Eq.(1.1). The aim of this paper is to investigate the improved super stability for functional equation Eq.(1.2) as follows $|Df(x_1,...,x_n)| \leq \varphi(x_i)$ for $i = 2...n$.

Moreover, we extend all super stability result for Eq.(1.2) to the super stability on the commutative semi simple Banach algebra.

2. Super Stability of Eq.(1.2)

In this section we will investigate the supers stability of the functional equation Eq.(1.2). The functional equation Eq. (1.2) is connected with the d'Alembert functional equation Eq.(1.1) as follows:

Lemma 2.1. *A complex valued function f on an Abelian group satisfies the functional equation:*

$$\sum_{\psi \in 2^{n-1}} f(\psi(x_1,...,x_n)) = 2^{n-1} \prod_{i=1}^{n} f(x_i)$$

for all $x_1,...,x_n \in G$ and $f(0) \geq 0$ if and only f satisfies the d'Alembert functional equation:

$$f(x+y)+f(x-y) = 2f(x)f(y) \forall \; x,y \in G$$

Proof. If f is a solution of Eq. (1.2), then by substituting $x_1,...,x_n$ by 0 we have:

$$f(0)^n = f(0)$$

since $f(0) \geq 0$ so $f(0) = 0$ or 1. If $f(0) = 0$ then $f = 0$, and if $f(0) = 1$ then by taking $x_3 = ... = x_n = 0$ we get

$$2^{n-2}(f(x_1+x_2)+f(x_1-x_2)) = 2^{n-1} f(x_1)f(x_2)$$

so that

$$f(x_1+x_2)+f(x_1-x_2) = 2f(x_1)f(x_2).$$

Conversely, let f solution of Eq. (1.1), the assertion is true for $n=2$. Assuming that the assertion is true for $n-1$, hence we have

$$\begin{aligned}\sum_{\psi\in K_{n-1}} f(\psi(x_1,...,x_n)) &= \sum_{\psi\in K_{n-1}} f(\psi(x_1+x_n,x_2,...,x_n))+f(\psi(x_1-x_n,x_2,...,x_n))\\ &= 2^{n-2}f(x_1+x_n).\prod_{i=2}^{n-1}f(x_i)+2^{n-2}f(x_1-x_n).\prod_{i=2}^{n-1}f(x_i)\\ &= 2^{n-1}\prod_{i=1}^{n}f(x_i).\end{aligned}$$

□

Lemma 2.2. *Let $(G,+)$ be an Abelian group, $f:G\to\mathbb{C}$ and $\psi:G\to\mathbb{R}_+$ satisfy the inequality:*

$$|Df(x_1,...,x_n)|\le\varphi(x_n) \tag{2.1}$$

for all $x_1,...,x_n\in G$. If there exist a sequence y_k such that

$$|f(y_k)|\to+\infty$$

for $k\to+\infty$ then

$$|f(2^iy_k)|\to+\infty$$

for $i\in\{0,...,n-3\}$.

Proof. We use induction on i, the assertion of Lemma (2.2) is true for $i=0$. Assuming that is true for i, hence by taking $x_1=x_2=2^iy_k$ and $x_3=...=x_n=0$ in the inequality (2.1). We have

$$|2^{n-2}f(2^{i+1}y_k)+2^{n-2}f(0)-2^{n-1}(f(2^iy_k))^2f(0)^{n-2}|\le\varphi(0)$$

so that

$$|f(2^{i+1}y_k)|\to+\infty.$$

□

Theorem 2.3. *Let $(G,+)$ be an Abelian group, $f:G\to\mathbb{C}$ and $\varphi:G\to\mathbb{R}_+$ satisfies the inequality (2.1) then either f is bounded or satisfies the functional equation Eq.(1.2) for all $x_1,...,x_n\in G$ and $n\ge3$.*

Proof. Let f be unbounded, then by Lemma (2.2) we can choose a sequence $\{y_k\}$ in G such that $|f(2^iy_k|\to\infty$ as $k\to\infty$, for $i=0,...,n-3$. Firstly we use induction on n to prove that:

$$\sum_{\psi\in K_{n-1}} f(\psi(2^{n-3}y_k,...,2y_k,y_k,y_k,x_n)) \tag{2.2}$$

$$\begin{aligned}&= f(2^{n-2}y_k+x_n)+f(2^{n-2}y_k-x_n)+2\sum_{i=1}^{2^{n-3}-1}f((2^{n-2}-2i)y_k+x_n)\\ &+ 2\sum_{i=1}^{2^{n-3}-1}f((2^{n-2}-2i)y_k-x_n)+f(x_n)+f(-x_n)\end{aligned}$$

The assertion (2.4) is true for $n = 3$. Assuming that is true for $n-1$, hence we have:

$$\begin{aligned}
\sum_{\psi\in K_{n-1}} f(\psi(2^{n-3}y_k, ..., 2y_k, y_k, y_k, x_n)) &= \sum_{\psi\in K_{n-2}} f(\psi(2^{n-4}y_k, ..., 2y_k, y_k, y_k, x_n) + 2^{n-3}y_k) \\
&+ \sum_{\psi\in K_{n-2}} f(\psi(-2^{n-4}y_k, 2^{n-5}y_k, ..., 2y_k, y_k, y_k, x_n) - 2^{n-3}y_k) \\
&= \sum_{\psi\in K_{n-2}} f(\psi(2^{n-4}y_k, ..., 2y_k, y_k, y_k, x_n) + 2^{n-3}y_k) \\
&+ \sum_{\psi\in K_{n-2}} f(\psi(-2^{n-4}y_k, ..., -2y_k, -y_k, -y_k, x_n) + 2^{n-3}y_k) \\
&= f((2^{n-3}y_k + x_n) + 2^{n-3}y_k) + f((2^{n-3}y_k - x_n) + 2^{n-3}y_k) \\
&+ 2\sum_{i=1}^{2^{n-4}-1} f((2^{n-3} - 2i)y_k + x_n + 2^{n-3}y_k) \\
&+ 2\sum_{i=1}^{2^{n-4}-1} f((2^{n-3} - 2i)y_k - x_n + 2^{n-3}y_k) \\
&+ f(2^{n-3}y_k + x_n) + f(2^{n-3}y_k - x_n) \\
&+ f(-2^{n-3}y_k + x_n + 2^{n-3}y_k) + f((-2^{n-3}y_k - x_n + 2^{n-3}y_k) \\
&+ 2\sum_{i=1}^{2^{n-4}-1} f(-(2^{n-3} - 2i)y_k + x_n + 2^{n-3}y_k) \\
&+ 2\sum_{i=1}^{2^{n-4}-1} f(-(2^{n-3} - 2i)y_k - x_n + 2^{n-3}y_k) \\
&+ f(2^{n-3}y_k + x_n) + f(2^{n-3}y_k - x_n). \\
&= f(2^{n-2}y_k + x_n) + f(2^{n-2}y_k - x_n) \\
&+ 2\sum_{i=1}^{2^{n-4}-1} f((2^{n-2} - 2i)y_k + x_n) \\
&+ 2\sum_{i=1}^{2^{n-4}-1} f((2^{n-2} - 2i)y_k - x_n) \\
&+ 2\sum_{i=1}^{2^{n-4}-1} f(2iy_k + x_n) + f(2iy_k - x_n) \\
&+ 2(f(2^{n-3}y_k + x_n) + f(2^{n-3}y_k - x_n)) \\
&+ f(x_n) + f(-x_n).
\end{aligned}$$

And using the fact that:

$$\begin{aligned}
2\sum_{i=1}^{2^{n-4}-1} f(2iy_k + x_n) + f(2iy_k - x_n) &= 2\sum_{i=2^{n-4}+1}^{2^{n-3}-1} f((2^{n-2} - 2i)y_k + x_n) \\
&+ 2\sum_{i=2^{n-4}+1}^{2^{n-3}-1} f((2^{n-2} - 2i)y_k - x_n)
\end{aligned}$$

the induction proof is completed.

Now putting

$$\begin{aligned} A(x_n) &= f(2^{n-2}y_k + x_n) + f(2^{n-2}y_k - x_n) \\ &+ 2\sum_{i=1}^{2^{n-3}-1} f((2^{n-2}-2i)y_k + x_n) + f((2^{n-2}-2i)y_k - x_n). \end{aligned}$$

And taking $x_1 = 2^{n-3}y_k ... x_{n-3} = 2y_k$ and $x_{n-2} = x_{n-1} = y_k$ in (2.1) we obtain

$$\left| \frac{A(x_n) + f(x_n) + f(-x_n)}{2^{n-1} f(y_k) \prod_{i=0}^{n-3} f(2^i y_k)} - f(x_n) \right| \leq \frac{\varphi(x_n)}{\left| 2^{n-1} f(y_k) \prod_{i=0}^{n-3} f(2^i y_k) \right|}$$

then

$$\lim_{k \to +\infty} \frac{A(x_n)}{2^{n-1} f(y_k) \prod_{i=0}^{n-3} f(2^i y_k)} = f(x_n), x_n \in G \tag{2.3}$$

Note that, the result is invariant under the inversion of x_n, we deduce that f is even.

In the next we will show that f satisfies the functional equation Eq.(1.2), putting

$$\begin{aligned} B_\psi &= f(\psi(2^{n-2}y_k + x_1, x_2, ..., x_n)) + f(\psi(2^{n-2}y_k - x_1, x_2, ..., x_n)) \\ &+ 2\sum_{i=1}^{2^{n-3}-1} f(\psi((2^{n-2}-2i)y_k + x_1, x_2, ..., x_n)) \\ &+ 2\sum_{i=1}^{2^{n-3}-1} f(\psi((2^{n-2}-2i)y_k - x_1, x_2, ..., x_n)) \end{aligned}$$

$$\left| \sum_{\psi \in K_{n-1}} f(\psi((2^{n-2}-2i)y_k + x_1, x_2, ..., x_n)) - 2^{n-1} f((2^{n-2}-2i)y_k + x_1) \prod_{i=2}^{n} f(x_i) \right| \leq \varphi(x_n). \tag{2.4}$$

And letting: $x_1 = (2^{n-2}-2i)y_k - x_1$ for $i = 0, ..., 2^{n-3}-1$ in (2.1) we have

$$\left| \sum_{\psi \in K_{n-1}} f(\psi((2^{n-2}-2i)y_k - x_1, x_2, ..., x_n)) - 2^{n-1} f((2^{n-2}-2i)y_k - x_1) \prod_{i=2}^{n} f(x_i) \right| \leq \varphi(x_n) \tag{2.5}$$

For all $x_1, ..., x_n \in G$:

Combining (2.4) and (2.5), and using the evenness of f, we see that

$$\left| \sum_{\psi \in K_{n-1}} B_\psi - 2^{n-1} \prod_{i=2}^{n} f(x_i) A(x_1) \right| \leq (2^{n-1}-2)\varphi(x_n) \tag{2.6}$$

for all $x_1, ..., x_n \in G$. Now, we fix $\psi_j \in K_{n-1}$ for $j = 1, 2, 3, ..., 2^{n-1}$, then we get

$$\begin{aligned} B_{\psi_j} &= f(2^{n-2}y_k+\psi_j(x_1,x_2,...,x_n))+f(2^{n-2}y_k-\psi_j(x_1,x_2,...,x_n)) \\ &+ 2\sum_{i=1}^{2^{n-3}-1} f((2^{n-2}-2i)y_k+\psi_j(x_1,x_2,...,x_n)) \\ &+ 2\sum_{i=1}^{2^{n-3}-1} f((2^{n-2}-2i)y_k-\psi_j(x_1,x_2,...,x_n)) \end{aligned}$$

Using the fact (2.3) of f, we see that

$$\lim_{k\to+\infty}\frac{B_{\psi_j}}{2^{n-1}f(y_k)\prod_{i=0}^{n-3}f(2^iy_k)}=f(\psi_j(x_1,x_2,...,x_n))$$

Therefore, dividing each side of the inequality (2.6) by

$$2^{n-1}f(y_k)\prod_{i=0}^{n-3}f(2^iy_k)$$

and taking the limit as $k\to+\infty$, we get

$$\sum_{\psi\in K_{n-1}} f(\psi(x_1,x_2,...,x_n))=2^{n-1}\prod_{i=1}^{n}f(x_i)$$

□

Corollary 2.4. *Let δ be positive real number and let $f: G\to\mathbb{C}$ be a function satisfying the inequality $|Df(x_1,...,x_n)|\leq\delta$ for all $x_1,...,x_n\in G$ and $n\geq 3$, then either f is bounded or f satisfies the functional equation:*

$$\sum_{\psi\in K_{n-1}} f(\psi(x_1,x_2,...,x_n))=2^{n-1}\prod_{i=1}^{n}f(x_i)$$

3. Extension to Banach algebra

All results in section 2 can be extended to the super stability on the commutative semi simple Banach algebra. In this section, let $(G,+)$ be an Abelian group, and $(E,\|.\|)$ be a commutative semi simple algebra.

Theorem 3.1. *Assume that $f: G\to E$ and $\varphi: G\to\mathbb{R}^+$ satisfy the inequality*

$$\|Df(x_1,...,x_n)\|\leq\varphi(x_n) \tag{3.1}$$

for all $x_1,...,x_n\in G$ and $n\geq 3$. If the superposition $x^*\circ f$ is unbounded for each linear multiplicative functional $x^*\in E^*$, then f satisfies the functional equation Eq.(1.2).

Proof. Assume that (3.1) holds and fixe arbitrarily linear multiplicative functional $x^*\in E$. As is well known we have $\|x^*\|=1$, whence, for every $x_1,...,x_n\in G$, we have

$$\begin{aligned} \varphi(x_n) &\geq \|Df(x_1,...,x_n)\| \\ &= Sup_{\|y^*\|=1}|D(y^*\circ f)(x_1,...,x_n)| \\ &\geq |D(x^*\circ f)(x_1,...,x_n)| \end{aligned}$$

which states that the superposition $x^* \circ f$ yield a solution of the inequality (2.1) of section 2 in the Theorem (2.3). Since by assumption the superposition $x^* \circ f$ is unbounded, an appeal to Theorem (2.3) shows that $x^* \circ f$ solve functional equation Eq.(1.2). In other words, bearing the linear multi pliability of x^*, for all $x_1, ..., x_n \in G$ the difference $Df(x_1, ..., x_n)$ falls into the kernel of x^*.
Therefore, in view of the unrestricted choice of x^*, we infer that

$$Df(x_1, ..., x_n) \in \{kerx^*; x^* \ is \ multiplicative \ of \ E^*\}$$

for all $x_1, ..., x_n \in G$. Since the algebra E has been assumed to be semi simple, the last term of the above formula coincides with the singleton $\{0\}$, that is

$$Df(x_1, ..., x_n) = 0 \ \ \forall x_1, ..., x_n \in G$$

as claimed. This completes the proof. □

Corollary 3.2. *Let δ be positive real number and let $f : G \to \mathbb{E}$ be a function satisfying the inequality: $\|Df(x_1, ..., x_n)\| \leq \delta$ for all $x_1, ..., x_n \in G$ and $n \geq 3$. If the superposition $x^* \circ f$ is unbounded for each linear multiplicative functional $x^* \in E^*$, then f satisfies the functional equation*

$$\sum_{\psi \in K_{n-1}} f(\psi(x_1, x_2, ..., x_n)) = 2^{n-1} \prod_{i=1}^{n} f(x_i).$$

Similarly, one can prove that if the difference $Df(x_1, ..., x_n)$ is bounded by $\varphi(x_2)$ or $\varphi(x_3)$, ..., or $\varphi(x_{n-1})$ one obtains the same result as in the Theorem (2.3).

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