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Common fixed point of mappings satisfying implicit contractive conditions in TVS-valued ordered cone metric spaces

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Abstract

Using the setting of TVS-valued ordered cone metric spaces (order is induced by a non normal cone), common fixed point results for four mappings satisfying implicit contractive conditions are obtained. These results extend, unify and generalize several well known comparable results in the literature.

Keywords: Implicit contraction; fixed point; coincidence point; common fixed point; weakly compatible mappings; metric space; dominating maps; dominated maps; ordered metric space.
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1. Introduction

Metric fixed point theory has primary applications in functional analysis. Extension of fixed point theory to generalized structures as cone metrics, partial metric spaces and quasi-metric spaces has received a lot of attention (see, for instance, [1 - 33] and references mentioned therein). The study of fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity.

Fixed point theory in K-metric and K-normed spaces was developed by Perov et al. [18, 28, 29], Mukhamadijev and Stetsenko [19], and Vandergraft [41]. For more details on this subject, we refer to Zabrejko [42].

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Huang and Zhang [14] reintroduced such spaces under the name of cone metric spaces and reintroduced definition of convergent and Cauchy sequences in the terms of interior points of the underlying cone. They also proved some fixed point results in framework of cone metric spaces. Subsequently, several interesting and valuable results have appeared about existence of fixed point in K- metric spaces (see, e.g., [1, 4, 14, 15, 19, 30, 35]). Recently, Abbas et al. [2] (see also [12]) obtained common fixed point results in topological vector space valued cone metric space which is a larger class than that considered in [14]. The main motivation behind such research is an observation, that the domain of existence of a solution to a system of first-order differential equations may be enhanced by considering generalized ([5]).

Recently, Wei-Shih Du [13] used the scalarization function and investigated the equivalence of vectorial versions of fixed point theorems in K-metric spaces and scalar versions of fixed point theorems in metric spaces. He showed that many of the fixed point theorems for mappings satisfying contractive conditions of a linear type in K-metric spaces can be considered as the corollaries of corresponding theorems in metric spaces. Nevertheless, the fixed point theory in K-metric spaces proceeds to be actual, since the method of scalarization cannot be applied for a wide class of mappings satisfying contractive conditions more general than contractive conditions of a linear type.

Existence of fixed point in ordered metric spaces was first investigated by Ran and Reurings [31, Theorem 2.1]. They also studied applications of their results to matrix equations. Subsequently, Nieto and Rodríguez-López [27] extended the results in [31, Theorem 2.1] for nondecreasing mappings and then applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Since then, a problem of existence and uniqueness of fixed point of mappings on metric spaces endowed with a partial ordering has received much of attention of several mathematicians, see for example [6, 7, 8, 9, 10, 11, 20, 21, 22, 23, 24, 25, 26, 27, 32, 34, 36, 37, 38, 39, 40] and references cited therein.

In this paper, common fixed point theorems involving two pairs of weakly compatible mappings satisfying implicit contractions in TVS-valued ordered cone metric spaces are obtained. Our results generalize, extend and improve some recent fixed point results in K-metric spaces including the results of Abbas and Jungck [1], Olaleru [28], Huang and Zhang [14] and Rezapour and Hamlbarani [33]. It is worth mentioning that our results do not require the assumption that the cone is normal.

2. Preliminaries

We shall recall some definitions and mathematical preliminaries.

Let E be always a topological vector space (in shortly, TVS).

Definition 2.1. (See Zabrejko [42]). A non-empty subset K of E is called a cone if and only if
(i) $\overline{K} = K$, $K \neq 0_E$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in K \Rightarrow ax + by \in K$,
(iii) $K \cap (-K) = \{0_E\}$, where K is the closure of K.

A cone K defines a partial ordering $\leq_E$ in E by $x \leq_E y$ if and only if $y - x \in K$. We shall write $x <_E y$ to indicate that $x \leq_E y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(K)$, where $int(K)$ denotes the interior of K. A cone K is said to be normal if there exists a constant $M \geq 1$ such that $0_E \leq_E x \leq_E y$ implies $\|x\|_E \leq M\|y\|_E$. The least positive number M satisfying this inequality is called the normal constant of cone K. For further details on cone theory, one can refer to [33].

Definition 2.2. Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies
(d1) $0_E \leq_E d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq_E d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a TVS-valued cone metric on X and (X, d) is called a TVS-valued cone metric space.

Example 2.3. Let $X = \{0, 1, 2\}$, $E = \mathbb{R}^2_+$ and $P = \{(x, y) : x \geq 0, y \geq 0\}$. Define $d : X \times X \to E$ by

(x, y)	$d(x, y)$
$(0, 0)$	$(0, 0)$
$(0, 1), (1, 0)$	$(1, 1)$
$(0, 2), (2, 0)$	$(2, 2).$
$(1, 2), (2, 1)$	$(3, 3)$

It is straightforward to check that d satisfies all axioms of TVS-valued cone metric.

Definition 2.4. Let (X, d) be a TVS-valued cone metric space and $\{x_n\}$ is a sequence in X. We say that $\{x_n\}$ is Cauchy if for every $c \in E$ with $0_E \ll c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n > m > N$. We say that $\{x_n\}$ converges to $x \in X$ if for every $c \in E$ with $0_E \ll c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > N$. In this case, we denote $x_n \to x$ as $n \to \infty$.

A TVS-valued cone metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

Remark 2.5. Let E is a TVS-valued cone metric space with a cone P then,

(a) if with a cone P and if $a \leq ha$ where $a \in P$ and $h \in (0, 1)$, then $a = 0$.

(b) if $0 \leq u \ll c$ for each $0 \ll c$, then $u = 0$.

(c) if $a \leq b + c$ for each $0 \ll c$, then $a \leq b$.

For more on the properties of cone, we refer to [17].

Definition 2.6. Let $f : E \to E$ be a given mapping. We say that f is a non-decreasing mapping with respect to $\leq_E$ if for every $x, y \in E$, $x \leq_E y$ implies $fx \leq_E fy$.

Definition 2.7. Let f and g be self-maps on a set X. If $w = fx = gx$, for some x in X, then x is called coincidence point of f and g, where w is called a point of coincidence of f and g.

Definition 2.8. [3]. Let f and g be two self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.

Also f and g are said to be compatible if $\lim\limits_{n\to\infty} gfx_n = \lim\limits_{n\to\infty} fgx_n$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$ for some t in X.

Definition 2.9. Let X be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric space if and only if d is a metric on X having partial order $\preceq$.

Definition 2.10. Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 2.11. [3]. Let $(X, \preceq)$ be a partially ordered set. A mapping f is called dominating if $x \preceq fx$ for each x in X.

Example 2.12. [3]. Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = \sqrt[n]{x}$. Since $x \leq x^{\frac{1}{3}} = fx$ for all $x \in X$. Therefore f is a dominating map.

Definition 2.13. Let $(X, \preceq)$ be a partially ordered set. A mapping f is called dominated if $fx \preceq x$ for each x in X.

Example 2.14. Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$. Therefore f is a dominated map.

Definition 2.15. A subset $\mathcal{K}$ of a partially ordered set X is said to be well ordered if every two elements of $\mathcal{K}$ are comparable.

3. Main Results

To complete the results, we need following setting of implicit contraction.

We consider the set L of functions $\varphi : K^5 \to K$ satisfying the following properties:

(i) φ is continuous;
(ii) φ is non-decreasing with respect to $\leq_E$ in the 4th and 5th variable;
(iii) there are $h_1 > 0$ and $h_2 > 0$ such that $h = h_1 h_2 < 1$ and if $u, v \in K$ satisfy $u \leq_E \varphi(v, v, u, u+v, 0_E)$, then $u \leq_E h_1 v$ and if $u, v \in K$ satisfy $u \leq_E \varphi(v, u, v, 0_E, u+v)$, then $u \leq_E h_2 v$;
(iv) if $u \in K$ is such that $u \leq_E \varphi(u, 0_E, 0_E, u, u)$ or $u \leq_E \varphi(0_E, u, 0_E, 0_E, u)$ or $u \leq_E \varphi(0_E, 0_E, u, u, 0_E)$, then $u = 0_E$.

Our main result of this paper is as follows:

Theorem 3.1. *Let $(X, d, \preceq)$ be an ordered TVS-valued cone metric space with cone K over a solid cone P. Let S, T, I and J be self-maps on X such that*

$$d(Sx, Ty) \leq_E \varphi(d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Sx, Jy)), \tag{3.1}$$

for all comparable $x, y \in X$, where $\varphi \in L$. Suppose that

(i) $TX \subseteq IX$ and $SX \subseteq JX$;
(ii) I and J are dominating maps and S and T are dominated maps.

If for a nonincreasing sequence $\{x_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ implies that $u \preceq x_n$ and either

(a) $\{S, I\}$ are compatible, S or I is continuous and $\{T, J\}$ are weakly compatible or
(b) $\{T, J\}$ are compatible, T or J is continuous and $\{S, I\}$ are weakly compatible,

then S, T, I and J have a common fixed point. Moreover, the set of common fixed points of S, T, I and J is well ordered if and only if S, T, I and J have one and only one common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $TX \subseteq IX$ and $SX \subseteq JX$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n+1} = Sx_{2n} = Jx_{2n+1}, y_{2n+2} = Tx_{2n+1} = Ix_{2n+2}, n = 1, 2, \cdots .$$

By given assumptions $x_{2n+1} \preceq Jx_{2n+1} = Sx_{2n} \preceq x_{2n}$ and $x_{2n} \preceq Ix_{2n} = Tx_{2n-1} \preceq x_{2n-1}$. Thus, for all $n \geq 1$, we have $x_{n+1} \preceq x_n$. We suppose that $d(y_{2n}, y_{2n+1}) > 0$, for every n. If not then $y_{2n} = y_{2n+1}$, for some n. From (3.1), we obtain

$$\begin{aligned}
& d(y_{2n+1}, y_{2n+2}) \\
= \; & d(Sx_{2n}, Tx_{2n+1}) \\
\leq_E \; & \varphi(d(Ix_{2n}, Jx_{2n+1}), d(Ix_{2n}, Sx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), d(Ix_{2n}, Tx_{2n+1}), d(Sx_{2n}, Jx_{2n+1})) \\
= \; & \varphi(d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+2}), d(y_{2n+1}, y_{2n+1})) \\
= \; & \varphi(0_E, 0_E, d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}), 0_E).
\end{aligned}$$

From (iv), this implies that $d(y_{2n+1}, y_{2n+2}) = 0_E$, that is,

$$y_{2n+1} = y_{2n+2}.$$

Following the similar arguments, we obtain $y_{2n+2} = y_{2n+3}$ and so on. Thus $\{y_n\}$ becomes a constant sequence and y_{2n} is the common fixed point of S, T, I and J.

Take, $d(y_{2n}, y_{2n+1}) > 0$ for each n. Since x_{2n} and x_{2n+1} are comparable, from (3.1), we have

$$\begin{aligned}
& d(y_{2n+1}, y_{2n+2}) \\
= \quad & d(Sx_{2n}, Tx_{2n+1}) \\
\leq_E \quad & \varphi(d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+2}), 0_E) \\
\leq_E \quad & \varphi(d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}), 0_E).
\end{aligned}$$

By (iii), we have

$$d(y_{2n+1}, y_{2n+2}) \leq_E h_1 d(y_{2n}, y_{2n+1}). \tag{3.2}$$

Again, using (3.1), we have

$$\begin{aligned}
& d(y_{2n+1}, y_{2n}) \\
= \quad & d(Sx_{2n}, Tx_{2n-1}) \\
\leq_E \quad & \varphi(d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), 0_E, d(y_{2n+1}, y_{2n-1})) \\
\leq_E \quad & \varphi(d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), 0_E, d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})).
\end{aligned}$$

By (iii), we obtain

$$d(y_{2n+1}, y_{2n}) \leq_E h_2 d(y_{2n}, y_{2n-1}). \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$d(y_{2n+1}, y_{2n+2}) \leq_E h d(y_{2n-1}, y_{2n}).$$

Continuing this process, we get

$$d(y_{2n+1}, y_{2n+2}) \leq_E h^n d(y_1, y_2). \tag{3.4}$$

Again, using (3.1), we have

$$\begin{aligned}
& d(y_{2n+3}, y_{2n+2}) \\
= \quad & d(Sx_{2n+2}, Tx_{2n+1}) \\
\leq_E \quad & \varphi(d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2}), 0_E, d(y_{2n+3}, y_{2n+1})) \\
\leq_E \quad & \varphi(d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2}), 0_E, d(y_{2n+3}, y_{2n+2}) + d(y_{2n+2}, y_{2n+1})).
\end{aligned}$$

From (iii), we get

$$d(y_{2n+2}, y_{2n+3}) \leq_E h_2 d(y_{2n+1}, y_{2n+2}).$$

Using (3.4), we obtain

$$d(y_{2n+2}, y_{2n+3}) \leq_E h_2 h^n d(y_1, y_2). \tag{3.5}$$

From (3.4) and (3.5), we get

$$d(y_n, y_{n+1}) \leq_E \frac{\max\{1, h_2\}}{\sqrt{h}} (\sqrt{h})^n d(y_1, y_2), \text{ for all } n = 2, 3, \ldots. \tag{3.6}$$

From (3.6) and using the triangular inequality, for all $n, m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
d(y_n, y_{n+m}) \quad \leq_E \quad & d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m) \\
\leq_E \quad & \frac{\max\{1, h_2\}}{\sqrt{h}} ((\sqrt{h})^n + (\sqrt{h})^{n+1} + \ldots + (\sqrt{h})^{m-1}) d(y_1, y_2) \\
\leq_E \quad & \frac{\max\{1, h_2\}}{\sqrt{h}} \frac{(\sqrt{h})^n}{1 - \sqrt{h}} d(y_1, y_2).
\end{aligned}$$

Let c be an arbitrary element in E with $0_E \ll c$. Since $0 < h < 1$, there exists $N \in \mathbb{N}$ such that

$$\frac{\max\{1, h_2\}}{\sqrt{h}} \frac{(\sqrt{h})^n}{1 - \sqrt{h}} d(y_1, y_2) \ll c, \quad \text{for all } n > N. \tag{3.7}$$

Thus, for all $m, n \in \mathbb{N}$,

$$d(y_n, y_m) \preceq_E \frac{\max\{1, h_2\}}{\sqrt{h}} \frac{(\sqrt{h})^n}{1-\sqrt{h}} d(y_1, y_2) \ll c$$

and so the sequence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point z in X, such that y_{2n} converges to z. Therefore,

$$y_{2n+1} = Jx_{2n+1} = Sx_{2n} \to z \text{ as } n \to \infty \tag{3.8}$$

and

$$y_{2n+2} = Ix_{2n+2} = Tx_{2n+1} \to z \text{ as } n \to \infty. \tag{3.9}$$

Assume that I is continuous. Since $\{S, I\}$ are compatible, we have

$$\lim_{n\to\infty} SIx_{2n+2} = \lim_{n\to\infty} ISx_{2n+2} = Iz.$$

Also, $Ix_{2n+2} = Tx_{2n+1} \preceq x_{2n+1}$. Now

$$\begin{aligned} & d(SIx_{2n+2}, Tx_{2n+1}) \\ \preceq_E \;& \varphi(d(IIx_{2n+2}, Jx_{2n+1}), d(IIx_{2n+2}, SIx_{2n+2}), d(Jx_{2n+1}, Tx_{2n+1}), d(IIx_{2n+2}, Tx_{2n+1}), d(SIx_{2n+2}, Jx_{2n+1}) \end{aligned}$$

On taking limit as $n \to \infty$, we obtain

$$d(Iz, z) \preceq_E \varphi(d(Iz, z), 0_E, 0_E, d(Iz, z), d(Iz, z)).$$

From (iv), this implies that $d(Iz, z) = 0_E$, that is,

$$Iz = z. \tag{3.10}$$

Now, $Tx_{2n+1} \preceq x_{2n+1}$ and $Tx_{2n+1} \to z$ as $n \to \infty$, $z \preceq x_{2n+1}$ and (3.1) becomes

$$d(Sz, Tx_{2n+1}) \preceq_E \varphi(d(Iz, Jx_{2n+1}), d(Sz, Iz), d(Jx_{2n+1}, Tx_{2n+1}), d(Iz, Tx_{2n+1}), d(Sz, Jx_{2n+1})).$$

Taking limit as $n \to \infty$ in the above inequality,

$$d(Sz, z) \preceq_E \varphi(0_E, d(Sz, z), 0_E, 0_E, d(Sz, z)).$$

From (iv), this implies that $d(Sz, z) = 0_E$, that is,

$$Sz = z. \tag{3.11}$$

Since $S(X) \subseteq J(X)$, there exists a point $w \in X$ such that $Sz = Jw$. Suppose that $Tw \neq Jw$. Since $w \preceq Jw = Sz \preceq z$ implies $w \preceq z$. From (3.1), we obtain

$$\begin{aligned} d(Jw, Tw) &= d(Sz, Tw) \preceq_E \varphi(d(Iz, Jw), d(Iz, Sz), d(Jw, Tw), d(Iz, Tw), d(Sz, Jw)) \\ &= \varphi(d(z, z), d(z, z), d(Jw, Tw), d(Jw, Tw), d(Jw, Jw)) \\ &= \varphi(0_E, 0_E, d(Jw, Tw), d(Jw, Tw), 0_E). \end{aligned}$$

From (iv), this implies that $d(Jw, Tw) = 0_E$, that is,

$$Jw = Tw. \tag{3.12}$$

Since T and J are weakly compatible, $Tz = TSz = TJw = JTw = JSz = Jz$. Thus z is a coincidence point of T and J.

Now, since $Sx_{2n} \preceq x_{2n}$ and $Sx_{2n} \to z$ as $n \to \infty$, implies that $z \preceq x_{2n}$, from (3.1)

$$d(Sx_{2n}, Tz) \preceq_E \varphi(d(Ix_{2n}, Jz), d(Ix_{2n}, Sx_{2n}), d(Jz, Tz), d(Ix_{2n}, Tz), d(Sx_{2n}, Jz)).$$

On taking limit as $n \to \infty$, we have

$$\begin{aligned} d(z,Tz) &\leq_E \varphi(d(z,Tz), d(z,z), d(Tz,Tz), d(z,Tz), d(z,Tz)) \\ &= \varphi(d(z,Tz), 0_E, 0_E, d(z,Tz), d(z,Tz)). \end{aligned}$$

From (iv), this implies that $d(z,Tz) = 0_E$, that is,

$$z = Tz. \tag{3.13}$$

Therefore $Sz = Tz = Iz = Jz = z$. The proof is similar when S is continuous.

Similarly, the result follows when (b) holds.

Now suppose that set of common fixed points of S, T, I and J is well ordered. We claim that common fixed point of S, T, I and J is unique. Assume on contrary that, $Su = Tu = Iu = Ju = u$ and $Sv = Tv = Iv = Jv = v$ but $u \neq v$. By supposition, we can replace x by u and y by v in (3.1) to obtain

$$\begin{aligned} d(u,v) = d(Su,Tv) &\leq_E \varphi(d(Iu,Jv), d(Iu,Su), d(Jv,Tv), d(Iu,Tv), d(Su,Jv)) \\ &= \varphi(d(u,v), 0_E, 0_E, d(u,v), d(u,v)). \end{aligned}$$

From (iv), we get $u = v$, that is,

$$Su = Iu = Tu = Ju = u. \tag{3.14}$$

Conversely, if S, T, I and J have only one common fixed point then the set of common fixed point of S, T, I and J being singleton is well ordered. □

Using the obtained result given by Theorem 3.1, we will prove the following theorem.

Theorem 3.2. *Let $(X, d, \preceq)$ be an ordered TVS-valued cone metric space with cone K over a solid cone P. Let S, T, I and J be self-maps on X such that*

$$d(Sx,Ty) \leq_E Ad(Ix,Jy) + B[d(Ix,Sx) + d(Jy,Ty)] + C[d(Ix,Ty) + d(Sx,Jy)], \tag{3.15}$$

for all comparable $x, y \in X$, where $A, B, C > 0$ with $A + 2B + 2C < 1$. Suppose that

(i) $TX \subseteq IX$ and $SX \subseteq JX$;
(ii) I and J are dominating maps and S and T are dominated maps.

If for a nonincreasing sequence $\{x_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ implies that $u \preceq x_n$ and either

(a) $\{S, I\}$ are compatible, S or I is continuous and $\{T, J\}$ are weakly compatible or
(b) $\{T, J\}$ are compatible, T or J is continuous and $\{S, I\}$ are weakly compatible,

then S, T, I and J have a common fixed point. Moreover, the set of common fixed points of S, T, I and J is well ordered if and only if S, T, I and J have one and only one common fixed point.

Proof. Define $\varphi : K^5 \to K$ by

$$\varphi(u_1, u_2, u_3, u_4, u_5) = Au_1 + B(u_2 + u_3) + C(u_4 + u_5), \quad \text{for all } u_i \in K.$$

Denote

$$h_1 = h_2 = \frac{A + B + C}{1 - (B + C)}.$$

Since $A + 2B + 2C < 1$, we have $h_1 > 0, h_2 > 0$. If $u \leq_E \varphi(v, v, u, u + v, 0_E)$, we have

$$u \leq_E Av + Bv + Bu + Cu + Cv,$$

which implies that $u \leq_E h_1 v$. Now, if $u \leq_E \varphi(v, u, v, 0_E, u + v)$, we have

$$u \leq_E Av + Bu + Bv + Cu + Cv,$$

which implies that $u \leq_E h_2 v$. Suppose now that $u \leq_E \varphi(u, 0_E, 0_E, u, u)$. We get $u \leq_E Au + 2Cu$, which implies that $-[1-(A+2C)]u \in K$. Since $A + 2C < 1$, we have also $[1-(A+2C)]u \in K$. Then $u = 0_E$. The same result holds if $u \leq_E \varphi(0_E, u, 0_E, 0_E, u)$ or $u \leq_E \varphi(0_E, 0_E, u, u, 0_E)$. Therefore, $\varphi \in L$. Moreover, inequality (3.15) is equivalent to inequality (3.1). Then, to obtain the desired result, we have only to apply Theorem 3.1 for the considered function. □

Example 3. Let $X = [0, 1)$, $E = C^1_R$ and let $P = \{x \in E : x(t) \geq 0\}$. Mapping $d : X \times X \to E$ is defined by

$$d(x, y) = |x - y|\,\psi(t),$$

where $\psi \in P$ is a fixed function, for example, (i) $\psi(t) = e^t$, (ii) $\psi(t) = 2^t$, (ii) $\psi(t) = \lambda t$, $\lambda \in [0, 1)$, $t \in P$. Clearly, the metric given above is TVS-valued ordered cone metric on X. Define the self maps I, J, S and T on X by

$$S(x) = \begin{cases} 0, \text{ if } x \leq \frac{1}{3} \\ \frac{1}{2}(x - \frac{1}{3}), \text{ if } x \in (\frac{1}{3}, 1] \end{cases}, \quad Tx = \begin{cases} 0, \text{ if } x \leq \frac{1}{3} \\ \frac{1}{3}, \text{ if } x \in (\frac{1}{3}, 1] \end{cases},$$

$$J(x) = \begin{cases} 0, \text{ if } x = 0 \\ x, \text{ if } x \in (0, \frac{1}{3}] \\ 1, \text{ if } x \in (\frac{1}{3}, 1] \end{cases}, \quad Ix = \begin{cases} 0, \text{ if } x = 0 \\ \frac{1}{3}, \text{ if } x \in (0, \frac{1}{3}] \\ 1, \text{ if } x \in (\frac{1}{3}, 1] \end{cases}.$$

Then I and J are dominating maps and S and T are dominated maps with $S(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$,i.e.

	S is dominated map	*T is dominated map*	*I is dominating map*	*J is dominating map*
for each x in X	$Sx \leq x$	$Tx \leq x$	$x \leq Ix$	$x \leq Jx$
$x = 0$	$S(0) = 0$	$T(0) = 0$	$0 = I(0)$	$0 = J(0)$
$x \in (0, \frac{1}{3}]$	$Sx = 0 < x$	$Tx = 0 < x$	$x \leq \frac{1}{3} = I(x)$	$x = J(x)$
$x \in (\frac{1}{3}, 1]$	$Sx = \frac{1}{2}(x - \frac{1}{3}) < x$	$Tx = \frac{1}{3} < x$	$x \leq 1 = I(x)$	$x \leq 1 = J(x)$

Also, $\{S, I\}$ are compatible, S is continuous and $\{T, J\}$ are weakly compatible.

Define $\varphi : K^5 \to K$ by

$$\varphi((u_1, u_2, u_3, u_4, u_5)\psi(t)) = \frac{1}{30}(u_1 + 7(u_2 + u_3) + 7(u_4 + u_5))\psi(t), \quad \textit{for all } u_i \in K.$$

It is easy to see that φ satisfies axioms (i) to (iv) of Theorem 3.1.
Now we shall show that S, T, I and J satisfy (3.1).

We consider the following cases:

(i) If $x = y = 0$, then $d(S0, T0) = 0$ and (3.1) is satisfied.

(ii) For $x = 0$ and $y \in (0, \frac{1}{3}]$, then again $d(Sx, Ty) = 0$ and (3.1) is satisfied.

(iii) For $x = 0$ and $y \in (\frac{1}{3}, 1]$,

$$\begin{aligned} d(Sx, Ty) &= \frac{1}{3}\psi(t) \\ &<_E \frac{1}{2}\psi(t) \\ &= \varphi((1, 0, \frac{2}{3}, \frac{1}{3}, 1)\psi(t)) \\ &= \varphi((d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Sx, Jy))\psi(t)). \end{aligned}$$

(iv) For $x \in (0, \frac{1}{3}]$ and $y = 0$, then $d(Sx, T0) = 0$ and (3.1) is satisfied.

(v) For $x, y \in (0, \frac{1}{3}]$, then $d(Sx, Ty) = 0$ and hence (3.1) is satisfied.

(vi) For $x = (0, \frac{1}{3}]$ and $y \in (\frac{1}{3}, 1]$,

$$\begin{aligned} d(Sx,Ty) &= \frac{1}{3}\psi(t) \\ &<_E \frac{22}{45}\psi(t) \\ &= \varphi((\frac{2}{3},\frac{1}{3},\frac{2}{3},0,1)\psi(t)) \\ &= \varphi((d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),d(Ix,Ty),d(Sx,Jy))\psi(t)). \end{aligned}$$

(vii) For $x \in (\frac{1}{3}, 1]$ and $y = 0$,

$$\begin{aligned} d(Sx,Ty) &= \frac{1}{2}(x-\frac{1}{3})\psi(t) \\ &<_E \frac{1}{2}\psi(t) \\ &= \varphi((1,1-\frac{1}{2}(x-\frac{1}{3}),0,1,\frac{1}{2}(x-\frac{1}{3}))\psi(t)) \\ &= \varphi((d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),d(Ix,Ty),d(Sx,Jy))\psi(t)). \end{aligned}$$

(viii) For $x \in (\frac{1}{3}, 1], y \in (0, \frac{1}{3}]$,

$$\begin{aligned} d(Sx,Ty) &= \frac{1}{2}(x-\frac{1}{3})\psi(t) \le \frac{1}{3}\psi(t) \\ &\le_E \frac{38}{90}\psi(t) \\ &\le_E \varphi((1-y,1-\frac{1}{2}(x-\frac{1}{3}),y,1,\left|\frac{1}{2}(x-\frac{1}{3})-y\right|)\psi(t)) \\ &= \varphi((d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),d(Ix,Ty),d(Sx,Jy))\psi(t)). \end{aligned}$$

(ix) For $x, y \in (\frac{1}{3}, 1]$,

$$\begin{aligned} d(Sx,Ty) &= \frac{1}{2}(1-x)\psi(t) \le \frac{1}{3}\psi(t) \\ &\le_E \frac{56}{90}\psi(t) \\ &\le_E \varphi((0,\frac{7-3x}{6},\frac{2}{3},\frac{2}{3},\frac{7-3x}{6})\psi(t)) \\ &= \varphi((d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),d(Ix,Ty),d(Sx,Jy))\psi(t)). \end{aligned}$$

Thus (3.1) is satisfied for all $x, y \in X$. Therefore, all conditions of *Theorem 1* are satisfied. Moreover, 0 is the unique common fixed point.

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