Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 6 (2013), 227–233
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

A characterization of completeness in cone metric spaces

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Communicated by C. Alaca

Abstract

In this paper, we introduce the concept of d-point in cone metric spaces and characterize cone completeness in terms of this notion.

Keywords: Cone metric space, strongly minihedral cone, d-point, lower semicontinuous function.
2010 MSC: 54H25, 47H10.

1. Introduction

In 2007, Huang and Zhang [13] introduced the concept of cone metric spaces by replacing the set of real numbers with an ordered Banach space. Afterwards, a series of articles in this field have been dedicated to the improvement of fixed point theory. Some of those articles dealt with the structure of the spaces. Recently, some authors used cone valued lower semicontinuous functions to establish some results in cone metric spaces. In 1977, Weston[25] had shown that completeness criterion of metric spaces has got some relation with the family of real valued semicontinuous functions carried over the space. In fact, he had proved a necessary and sufficient condition for the metric space (X, d) to be complete in terms of the notion of d-point for lower semicontinuous functions. In this paper, our main purpose is to introduce the concept of d-point in cone metric spaces and obtain a result by using this notion. The cone under consideration is assumed to be strongly minihedral and normal. Our result extend the result of Weston [25] to cone metric spaces.

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Received 2012-6-20

2. Preliminaries

Let E be a real Banach space and P be a subset of E. Then P is called a cone if and only if

(i) P *is closed, nonempty and* $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \ \Rightarrow\ ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by $x \leq y$ (equivalently, $y \geq x$) if and only if $y - x \in P$. We shall write $x < y$ (equivalently, $y > x$) if $x \leq y$ and $x \neq y$, while $x \ll y$(equivalently, $y \gg x$) will stand for $y - x \in Int(P)$, where $Int(P)$ denotes the interior of P. The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq k\, \|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if (x_n) is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0\ (n \to \infty)$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. Razapour and Hamlbarani [19] proved that every regular cone is normal and there are no normal cones with normal constant $k < 1$.

Definition 2.1. *[13] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*

(i) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 2.2. *[13] Let (X, d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$, then (x_n) is said to be convergent and (x_n) converges to x, and x is the limit of (x_n). We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x\ (n \to \infty)$.*

Definition 2.3. *[13] Let (X, d) be a cone metric space, (x_n) be a sequence in X. If for any $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then (x_n) is called a Cauchy sequence in X.*

Definition 2.4. *[13] Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.*

Lemma 2.5. *[20] Let E be a real Banach space with a cone P. Then*
(i) If $a \ll b$ and $b \ll c$, then $a \ll c$.
(ii) If $a \leq b$ and $b \ll c$, then $a \ll c$.

Lemma 2.6. *[13] Let (X, d) be a cone metric space, P a normal cone with normal constant k, $x \in X$ and (x_n) a sequence in X. Then*
(i) (x_n) converges to x if and only if $d(x_n, x) \to \theta$ (or equivalently, $\| d(x_n, x) \| \to 0$);
(ii) Limit point of every sequence is unique;
(iii) Every convergent sequence is Cauchy;
(iv) (x_n) is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$ (or equivalently, $\| d(x_n, x_m) \| \to 0$);
(v) If $x_n \to x$ and $y_n \to y$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

Lemma 2.7. *[13] Let (X, d) be a cone metric space over a cone P in E. Then one has the following.*
(i) $Int(P) + Int(P) \subseteq Int(P)$ and $\lambda\, Int(P) \subseteq Int(P)$, $\lambda > 0$.
(ii) If $c \gg \theta$, then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.
(iii) For any given $c \gg \theta$ and $c_0 \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $\frac{c_0}{n_0} \ll c$.
(iv) If a_n, b_n are sequences in E such that $a_n \to a$, $b_n \to b$ and $a_n \leq b_n$ for all $n \geq 1$, then $a \leq b$.

Proposition 2.8. *[14] If E is a real Banach space with cone P and if $a \leq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$ then $a = \theta$.*

Definition 2.9. *[12] P is called minihedral cone if $sup\{x, y\}$ exists for all $x, y \in E$, and strongly minihedral if every subset of E which is bounded from above has a supremum or equivalently, every subset of E which is bounded from below has an infimum.*

It is easy to see that every strongly minihedral normal cone is regular.

Definition 2.10. *[1] Let (X, d) be a cone metric space and $\varphi : X \to E$ a function on X. Then, the function φ is called a lower semicontinuous on X whenever*

$$\lim_{n\to\infty} x_n = x \Longrightarrow \varphi(x) \leq \lim_{n\to\infty} inf\, \varphi(x_n) := \sup_{n\geq 1} \inf_{m\geq n} \varphi(x_m).$$

Definition 2.11. *Let (X, d) be a cone metric space and $f : X \to E$. Then, the function f is called uniformly continuous on X if for any $\epsilon > 0$ there is a $c \in E$ with $\theta \ll c$ such that*

$$d(x, y) \ll c \Longrightarrow \|f(x) - f(y)\| < \epsilon.$$

3. Main Results

In this section we always suppose that E is a real Banach space, P is a cone in E with $Int(P) \neq \emptyset$ and $\leq$ is the partial ordering on E with respect to P. Throughout the paper we denote by $\mathbb{N}$ the set of all natural numbers.

We begin with a definition.

Definition 3.1. *Let (X, d) be a cone metric space and $h : X \to E$. A point $x_0 \in X$ is called a d-point for h if for every other point $x \in X$ with $h(x_0) - h(x) \in Int(P)$,*

$$h(x_0) - h(x) < d(x_0, x)$$

Example 3.2. *Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$, $X = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\} \cup \{(0, x) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$ and $d : X \times X \to E$ be such that*

$$d\left((x, 0), (y, 0)\right) = \left(\frac{4}{3} \mid x - y \mid, \mid x - y \mid\right),$$

$$d\left((0, x), (0, y)\right) = \left(\mid x - y \mid, \frac{2}{3} \mid x - y \mid\right),$$

$$d\left((x, 0), (0, y)\right) = d\left((0, y), (x, 0)\right) = \left(\frac{4}{3}x + y, x + \frac{2}{3}y\right).$$

Then (X, d) is a cone metric space.
We define $h : X \to E$ by

$$h((x,0)) = (0,x),\ h((0,y)) = \left(-\frac{y}{2}, 0\right).$$

Then,

$$h\left((\frac{1}{2},0)\right) - h((0,y)) = \left(0,\frac{1}{2}\right) - \left(-\frac{y}{2},0\right) = \left(\frac{y}{2},\frac{1}{2}\right) \in Int(P) \ for\ y > 0,$$

$$h\left((\frac{1}{2},0)\right) - h((y,0)) = \left(0,\frac{1}{2}\right) - (0,y) = \left(0,\frac{1}{2}-y\right) \notin Int(P).$$

Also, for $y > 0$

$$h\left((\frac{1}{2},0)\right) - h((0,y)) = \left(\frac{y}{2},\frac{1}{2}\right) < d\left((\frac{1}{2},0),(0,y)\right).$$

Thus, $(\frac{1}{2}, 0)$ is a d-point for h.

Theorem 3.3. *Let (X, d) be a complete cone metric space and let P be a strongly minihedral normal cone of comparable elements. Then any lower semicontinuous function $h : X \to E$ which is bounded below has a d-point. If (X, d) is not complete, then there is a uniformly continuous function $g : X \to E$ which is bounded below but has no d-point.*

Proof. For any point $x_1 \in X$, we construct a sequence (x_n) in the following way:
For each $n \in \mathbb{N}$, let

$$\alpha(x_n) = inf\{h(x) : h(x_n) - h(x) \geq d(x_n, x) > \theta\}.$$

h being bounded below, $\alpha(x_n)$ exists by strong minihedrality of P.
Let x_{n+1} be a point such that

$$h(x_n) - h(x_{n+1}) \geq d(x_n, x_{n+1}) \tag{3.1}$$

and

$$h(x_{n+1}) < \alpha(x_n) + \frac{c_0}{n},\ where\ c_0 \in Int(P). \tag{3.2}$$

It follows from (3.1) that the sequence $(h(x_n))$ is nonincreasing in E. Also, it is bounded below. Since P is regular, the sequence $(h(x_n))$ is convergent.
For $m \geq n$, the triangle inequality implies that

$$\begin{aligned} h(x_n) - h(x_m) &= h(x_n) - h(x_{n+1}) + h(x_{n+1}) - h(x_{n+2}) \\ &\quad + \cdots + h(x_{m-1}) - h(x_m) \\ &\geq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\geq d(x_n, x_m). \end{aligned} \tag{3.3}$$

Hence,

$$\| d(x_n, x_m)\| \leq k\ \| h(x_n) - h(x_m)\| \to 0\ as\ m, n \to \infty.$$

By Lemma 2.6, $\| d(x_n, x_m) \| \to 0$ yields that the sequence (x_n) is Cauchy in X. Completeness of X implies that the sequence (x_n) is convergent to some point in X, say x_0. From (3.3), it follows that

$$h(x_m) \leq h(x_n) - d(x_n, x_m) \tag{3.4}$$

for all $m \geq n$. By regarding (3.4), Lemma 2.6 and lower semicontinuity of the function h, one can obtain that

$$\begin{aligned} h(x_0) &\leq \lim_{m\to\infty} inf\, h(x_m) \\ &\leq \lim_{m\to\infty} inf\ [h(x_n) - d(x_n, x_m)] \\ &= h(x_n) - d(x_n, x_0) \end{aligned}$$

for all $n \geq 1$. Thus,

$$h(x_n) - h(x_0) \geq d(x_n, x_0) \tag{3.5}$$

for all $n \geq 1$.
If x_0 is not a d-point for h, then for some $x(\neq x_0) \in X$ with $h(x_0) - h(x) \in Int(P)$,

$$h(x_0) - h(x) \geq d(x_0, x) > \theta. \tag{3.6}$$

Using (3.5) and (3.2), we obtain

$$h(x) \leq h(x_{n+1}) + h(x) - h(x_0) < \alpha(x_n) + \frac{c_0}{n} + h(x) - h(x_0). \tag{3.7}$$

Since $\theta \ll c_0$ and $\theta \ll h(x_0) - h(x)$, by Lemma 2.7, there exists $n \in \mathbb{N}$ such that

$$\frac{c_0}{n} \ll h(x_0) - h(x)$$

which implies that $h(x_0) - h(x) - \frac{c_0}{n} \geq \theta$. Thus, (3.7) gives that $h(x) < \alpha(x_n)$.
From (3.5) and (3.6), it follows that

$$\begin{aligned} h(x_n) - h(x) &= h(x_n) - h(x_0) + h(x_0) - h(x) \\ &\geq d(x_n, x_0) + d(x_0, x) \\ &> \theta \end{aligned}$$

which implies that $h(x_n) > h(x)$. So, $x_n \neq x$ and therefore $d(x_n, x) > \theta$.
Moreover,

$$h(x_n) - h(x) \geq d(x_n, x_0) + d(x_0, x) \geq d(x_n, x) > \theta.$$

It now follows from the definition of $\alpha(x_n)$ that $h(x) \geq \alpha(x_n)$ which contradicts the fact that $h(x) < \alpha(x_n)$.
Thus, x_0 is a d-point for h.

Now suppose that (X, d) is not complete. So there exists a Cauchy sequence (x_n) in X which is not convergent. We show that for any $x \in X$, the sequence $(2d(x, x_n))$ is Cauchy in E.
For $x \in X$, we have

$$d(x, x_n) \leq d(x, x_m) + d(x_m, x_n)$$

and so

$$d(x, x_n) - d(x, x_m) \leq d(x_m, x_n). \tag{3.8}$$

Interchanging n and m, we obtain

$$d(x, x_m) - d(x, x_n) \leq d(x_m, x_n).$$

By hypothesis, the elements of P are comparable. Then either $d(x, x_n) \leq d(x, x_m)$ or $d(x, x_m) \leq d(x, x_n)$.
Without loss of generality, we may assume that $d(x, x_m) \leq d(x, x_n)$ and so from (3.8), we have

$$\theta \leq d(x, x_n) - d(x, x_m) \leq d(x_m, x_n).$$

If k is the normal constant of P, then

$$\|2d(x, x_n) - 2d(x, x_m)\| \leq k \, \|2d(x_m, x_n)\| \to 0 \; as \; m, n \to \infty$$

which implies that $(2d(x, x_n))$ is Cauchy in E. By completeness of E, let $g(x)$ be its limit. Clearly, $g(x) > \theta$.
Because $g(x) = \theta$ implies that the sequence (x_n) is convergent, a contradiction.

Thus, the function g is bounded below.
If $x_0 \in X$, then

$$\begin{aligned} g(x_0) - g(x) &= \lim_{n\to\infty} 2d(x_0, x_n) - \lim_{n\to\infty} 2d(x, x_n) \\ &= \lim_{n\to\infty} [2d(x_0, x_n) - 2d(x, x_n)] \\ &\leq \lim_{n\to\infty} 2d(x_0, x) \\ &= 2d(x_0, x). \end{aligned} \tag{3.9}$$

Interchanging x_0 and x, we obtain

$$g(x) - g(x_0) \leq 2d(x_0, x).$$

Since the elements of P are comparable, either $g(x_0) \leq g(x)$ or $g(x) \leq g(x_0)$. Suppose that $g(x) \leq g(x_0)$. Then using (3.9), we have

$$\theta \leq g(x_0) - g(x) \leq 2d(x_0, x)$$

which implies that

$$\|g(x_0) - g(x)\| \leq k \,\|2d(x_0, x)\|.$$

Let $\epsilon > 0$ be a given real number. We choose $c \in E$ with $\theta \ll c$ and $k^2\|c\| < \epsilon$. Then,

$$\begin{aligned} \|g(x_0) - g(x)\| &\leq 2k^2 \|\frac{c}{2}\| \textit{ whenever } d(x_0, x) \ll \frac{c}{2} \\ &< \epsilon \textit{ whenever } d(x_0, x) \ll \frac{c}{2}. \end{aligned}$$

So, g is uniformly continuous. Also,

$$\begin{aligned} \frac{1}{2}[g(x_0) + g(x)] &= \frac{1}{2}\left[\lim_{n\to\infty} 2d(x_0, x_n) + \lim_{n\to\infty} 2d(x, x_n)\right] \\ &\geq d(x_0, x). \end{aligned}$$

Now,

$$\begin{aligned} g(x_0) - g(x) &= \frac{1}{2}[g(x_0) + g(x)] + \frac{1}{2}[g(x_0) - 3g(x)] \\ &\geq d(x_0, x) + \frac{1}{2}[g(x_0) - 3g(x)]. \end{aligned} \tag{3.10}$$

If $g(x_0) - g(x) \in Int(P)$, then $g(x) \ll g(x_0)$. Again, $g(x) > \theta$. So, it must be the case that $\theta \ll g(x_0)$. By the definition of g, $3g(x_m) \to \theta$ as $m \to \infty$.
By Lemma 2.7, for $g(x_0) \in E$ with $\theta \ll g(x_0)$, there is $\delta > 0$ such that $\|b\| < \delta$ implies $g(x_0) - b \in Int(P)$. Since $3g(x_m) \to \theta$ as $m \to \infty$, for this δ there is $n_0 \in \mathbb{N}$ such that $\|3g(x_m)\| < \delta$ for all $m > n_0$. So, $g(x_0) - 3g(x_m) \in Int(P)$ for all $m > n_0$ i.e., $3g(x_m) \ll g(x_0)$ for all $m > n_0$. Thus, $3g(x) \ll g(x_0)$ if $x = x_m$ and m is large. It now follows from (3.10) that $g(x_0) - g(x) \geq d(x_0, x)$ if $x = x_m$ and m is large. So, x_0 is not a d-point for g.

□

The following Corollary is the result [[25], Theorem].

Corollary 3.4. *If the metric space (X, d) is complete then any lower semicontinuous function $X \to \mathbb{R}$ which is bounded below has a d-point. If (X, d) is not comlete there is a uniformly continuous function $X \to \mathbb{R}$ which is bounded below but has no d-point.*

Proof. Taking $E = \mathbb{R},\ P = [0, \infty)$ in Theorem 3.3, the conclusion of the Corollary follows. □

Remark 3.5. *Theorem 3.3 is an extension of the result [[25], Theorem] in metric spaces to cone metric spaces.*

Acknowledgements:

The authors are very grateful to the referees for their valuable suggestions.

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