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Existence results for impulsive neutral functional integrodifferential equation with infinite delay

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Abstract

In this paper, we study the existence of mild solutions for a impulsive semilinear neutral functional integrodifferential equations with infinite delay in Banach spaces. The results are obtained by using the Hausdorff measure of noncompactness. Examples are provided to illustrate the theory.

Keywords: Impulsive differential equation, Neutral functional differential equation, Mild solution, Hausdorff measures of noncompactness.
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1. Introduction

Measures of noncompactness are a very useful tool in many branches of mathematics. They are used in the fixed point theory, linear operators theory, theory of differential and integral equations and others [5]. There are two measures which are the most important ones. The Kuratowski measure of noncompactness $\sigma(X)$ of a bounded set X in a metric space is defined as infimum of numbers $r > 0$ such that X can be covered with a finite number of sets of diameter smaller than r. The Hausdorff measure of noncompactness $\chi(X)$ defined as infimum of numbers $r > 0$ such that X can be covered with a finite number of balls of radii smaller than r. The Hausdorff measure is convenient in applications.

The notion of a measure of noncompactness turns out to be a very important and useful tool in many branches of mathematical analysis. The notion of a measure of weak compactness was introduced by De Blasi

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[12] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations. Several authors have studied the measures of noncompactness in Banach spaces [3, 5, 6, 7, 8, 9].

The study of the impulsive differential equations has attracted a great deal of attention. The theory of impulsive differential and integrodifferential equations become an important area of invetigation in recent years. In [13, 18, 19], the authors studied the existence of solutions for first-order impulsive partial neutral functional differential equations with infinite delay in Banach space with the compactness assumption on associated semigroups. Now impulsive partial neutral functional differential equations have become an important object of investigation in recent years stimulated by their numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc. With regard to this matter, we refer the reader to [10, 13, 17, 18, 19, 25, 26, 28, 29, 32, 33].

On the other hand, study of the existence and stability of the differential equations with delay was initiated by Travis and Webb [30] and Webb [31]. Since such equations are often more realistic to describe natural phenomena than those without delay, they have been investigated in variant aspects by many authors. Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the past decades; see, for example, [4, 10, 11, 14, 15, 19, 20, 21, 22, 23] and the references therein.

In this paper, we study the following impulsive neutral functional integrodifferential equations with infinite delay

$$\frac{d}{dt}(x(t) - g(t, x_t)) = A(x(t) - g(t, x_t)) + f(t, x_t, \int_0^t e(t, s, x_s)ds), t \in J = [0, b] \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

$$\Delta x(t_i) = I_i(x_{t_i}), \quad i = 1, 2, ..., n, 0 < t_1 < t_2 < ... < t_n < b, \tag{1.3}$$

where A is the infinitesimal generator of an analytic semigroup of linear operators defined on a Banach space X. The history $x_t : (-\infty, 0] \to X, x_t(\theta) = x(t + \theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $g : J \times \mathcal{B} \to X$, $f : J \times \mathcal{B} \times X \to X$, $e : J \times J \times \mathcal{B} \to X$, $I_i : X \to X, i = 1, 2, ..., p$ are appropriate functions; $0 < t_1 < t_2 < ... < t_n < b$ are fixed numbers and the symbol $\Delta\xi(t)$ represent the jump of the function ξ at t, which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$. We give the existence of mild solution of the initial value problem (1.1)-(1.3) under the condition in respect of Hausdorff's measure of non-compactness. The results obtained in this paper are generlizations of the results given by Arjunan [24, 27], Banas and Goebel [5] and Hernandez [18, 19, 21, 22] .

2. Preliminaries

Let X be a Banach space and $A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup of linear operators $(T(t))_{t\geq 0}$ on $X, 0 \in \rho(A)$. M is a constant such that $\|T(t)\| \leq M$ for every $t \in J = [0, b]$. The notation $(-A)^\alpha, \alpha \in (0, 1)$ is a closed linear operator on its domain $D((-A)^\alpha)$. Furthermore, the subspace $D((-A)^\alpha)$ is dense in X and the expression

$\|x\|_\alpha = \|(-A)^\alpha x\|, \quad x \in D((-A)^\alpha)$

defines a norm on $D((-A)^\alpha)$. Hereafter we denote by X_α the Banach space $D((-A)^\alpha)$ normed with $\|X\|_\alpha$. Then for each $0 < \alpha \leq 1, X_\alpha$ is a Banach space. For semigroup $\{T(t) : t \geq 0\}$, the following properties will be used.

Lemma 2.1 ([19]). *Let $0 < \beta < \alpha \leq 1$, then the following properties hold :*

(1) *X_β is a Banach space and $X_\beta \hookrightarrow X_\alpha$ and the imbedding is compact.*

(2) *there exists $M \geq 1$ such that $\|T(t)\| \leq M$, for all $0 \leq t \leq b$;*

(3) *for any* $0 \leq \alpha \leq 1$, *there exists a positive constant* C_α *such that*
$\|(-A)^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}, \qquad 0 < t \leq b.$

To describe appropriately our problems we say that a function $u : [\sigma, \tau] \to X$ is a normalized piecewise continuous function on $[\sigma, \tau]$ if u is piecewise continuous and left continuous on $(\sigma, \tau]$. We denothe by $\mathcal{PC}([\sigma, \tau]; X)$ the space formed by the normalized piecewise continuous fuction from $[\sigma, \tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ fromed by the all functions $u : [0, b] \to X$ such that u is continuous at $t \neq t_i, u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists, for all $i = 1, ..., n$. It is clear that $\mathcal{PC}$ endowed with the norm of the uniform convergence is a Banach space.

In this work, we will employ an axiomatic definition of the phase space $\mathcal{B}$ which is similar to that introduced by Hale and Kato [16] and it is appropiate to treat retarded impulsive differential equations.

Definition 2.1 ([16]). *Let* $\mathcal{B}$ *be a linear space of functions mapping* $(-\infty, 0]$ *into* X *endowed with a seminorm* $\|\cdot\|_\mathcal{B}$ *and we will assume that* $\mathcal{B}$ *satisfies the following axioms:*

(**A**) *If* $x : (-\infty, \sigma + b] \to X, b > 0$, *such that* $x_\sigma \in \mathcal{B}$ *and* $x|_{[\sigma,\sigma+b]} \in \mathcal{PC}([\sigma, \sigma + b] : X)$, *then for every* $t \in [\sigma, \sigma + b)$ *the following conditions hold:*

- (**i**) x_t *is in* $\mathcal{B}$,
- (**ii**) $\|x(t)\| \leq H\|x_t\|_\mathcal{B}$,
- (**iii**) $\|x_t\|_\mathcal{B} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t + \sigma)\|x_\sigma\|_\mathcal{B}$,

where $H > 0$ *is a constant;* $K, M : [0, \infty) \to [1, \infty)$, K *is continuous,* M *is locally bounded and* H, K, M *are independent of* $x(\cdot)$.

(**B**) *The space* $\mathcal{B}$ *is complete.*

Definition 2.2. *The Hausdorff's measures of noncompactness* χ_Y *is defined by*
$\chi_Y(B) = \inf\ \{r > 0, B\ \ can\ be\ covered\ by\ finite\ number\ of\ balls\ with\ radii\ \ \ r\}$ *for bounded set* B *in any Banach space* Y.

Lemma 2.2 ([5]). *Let* Y *be a real Banach space and* $B, C \subseteq Y$ *be bounded, the following properties are satisfied:*

(1) B *is pre-compact if and only if* $\chi_Y(B) = 0$;

(2) $\chi_Y(B) = \chi_Y(\bar{B}) = \chi_Y(convB)$ *where* $\bar{B}$ *and* $convB$ *mean the closure and convex hull of* B *respectively;*

(3) $\chi_Y(B) \leq \chi_Y(C)$ *when* $B \subseteq C$;

(4) $\chi_Y(B + C) \leq \chi_Y(B) + \chi_Y(C)$, *where* $B + C = \{x + y; x \in B, y \in C\}$;

(5) $\chi_Y(B \cup C) \leq \max\{\chi_Y(B), \chi_Y(C)\}$;

(6) $\chi_Y(\lambda B) = |\lambda|\, \chi_Y(B)$ *for any* $\lambda \in R$;

(7) *If the map* $Q : D(Q) \subseteq Y \to Z$ *is Lipschitsz continuous with constant* k *then*
$\chi_Z(QB) \leq k\chi_Y(B)$ *for any bounded subset* $B \subseteq D(Q)$, *where* Z *be a Banach space;*

(8) *If* $\{W_n\}_{n=1}^{+\infty}$ *is a decreasing sequence of bounded closed nonempty subsets of* Y
and $\lim\limits_{n\to+\infty} \chi_Y(W_n) = 0$, *then* $\cap_{n=1}^{+\infty} W_n$ *is nonempty and compact in* Y.

Definition 2.3 ([33]). *The map* $Q : W \subseteq Y \to Y$ *is said to be a* $\chi_Y -$ *contraction if there exists a positive constant* $k < 1$ *such that* $\chi_Y(Q(C)) \leq k\chi_Y(C)$ *for any bounded closed subset* $C \subseteq W$ *where* Y *is a Banach space.*

Lemma 2.3 ([1] Darbo). *If $Q : W \subseteq Y$ is closed and convex and $0 \in W$, the continuous map $Q : W \to W$ is a χ_Y-contraction, if the set $\{x \in W : x = \lambda \Gamma x\}$ is bounded for $0 < \lambda < 1$, then the map Q has atleast one fixed point in W.*

Lemma 2.4 ([5] Darbo-Sadovskii). *If $W \subseteq Y$ is bounded closed and convex, the continuous map $Q : W \to W$ is a $\chi_Y -$ contraction, then the map Q has at least one fixed point in W.*

In this paper, we denote χ by the Hausdorff's measure of noncompactness of X and denote χ_c by the Hausdorff's measure of noncompactness of $C([0; b]; X)$. To discuss the existence we need the following lemmas in this paper.

Lemma 2.5 ([5]). (1) *If $W \subseteq C([a, b]; X)$ is bounded, then $\chi(W(t)) \leq \chi_c(W)$, for any $t \in [a, b]$, where $W(t) = \{u(t); u \in W\} \subseteq X$.*

(2) *If W is equicontinuous on $[a, b]$, then $\chi(W(t))$ is continuous for $t \in [a, b]$ and $\chi_c(W) = sup\{\chi(W(t)), t \in [a, b]\}$;*

(3) *If $W \subseteq C([a, b]; X)$ is bounded and equicontinuous, then $\chi(W(t))$ is continuous for $t \in [a, b]$ and $\chi\big(\int_a^t W(s)ds\big) \leq \int_a^t \chi W(s)ds$, for all $t \in [a, b]$, where $\int_a^t W(s)ds = \{\int_a^t x(s)ds : x \in W\}$.*

Lemma 2.6 ([29]). (1) *If $W \subseteq \mathcal{PC}([a, b]; X)$ is bounded, then $\chi(W(t)) \leq \chi_{\mathcal{PC}}(W)$, for any $t \in [a, b]$, where $W(t) = \{u(t); u \in W\} \subseteq X$.*

(2) *If W is piecewise equicontinuous on $[a, b]$, then $\chi(W(t))$ is piecesise continuous for $t \in [a, b]$ and $\chi_{\mathcal{PC}}(W) = sup\{\chi(W(t)), t \in [a, b]\}$;*

(3) *If $W \subseteq C([a, b]; X)$ is bounded and piecewise equicontinuous, then $\chi(W(t))$ is piecewise continuous for $t \in [a, b]$, and $\chi\big(\int_a^t W(s)ds\big) \leq \int_a^t \chi W(s)ds$, for all $t \in [a, b]$, where $\int_a^t W(s)ds = \{\int_a^t x(s)ds : x \in W\}$.*

Lemma 2.7. *If the semigroup $T(t)$ is equicontinuous $\eta \in L([0, b]; R^+)$, then the set $\{\int_0^t T(t - s)u(s)ds : \|u(s)\| \leq \eta(s) fora.e.\ \ s \in [0, b]\}$ is equcontinuous for $t \in [0, b]$.*

3. Existence Results

Definition 3.4. *A function $x : (-\infty, b] \to X$is a mild solution of the initial value problem (1.1)-(1.3) if $x_0 = \varphi, x(.)|_J \in \mathcal{PC}$ and*

$$\begin{aligned} x(t) =& T(t)(\varphi(0) - g(0, \varphi)) + g(t, x_t) + \int_0^t T(t - s)f(s, x_s, \int_0^s e(s, \tau, x_\tau)d\tau)ds \\ &+ \sum_{0<t_i<t} T(t - t_i)I_i(x_{t_i}), \qquad t \in J. \end{aligned}$$

For the system (1.1)-(1.3), for some $\alpha \in (0, 1)$, we assume that the following hypotheses are satisfied:

(Hf) The function $f : J \times \mathcal{B} \times X \to X$ satisfies the following conditions:

- (**1**) For each $x : (-\infty, b] \to X, x_0 = \varphi \in \mathcal{B}$ and $x|J \in \mathcal{PC}$, the function $t \to f(t, x_t, \int_0^t e(t, s, x_s)ds)$ is strongly measurable and $f(t, ., .)$ is continuous for a.e. $t \in J$;
- (**2**) There exists an integrable function $\alpha : J \to [0, +\infty)$ and a monotone continuous dondecreasing function $\Omega : [0, +\infty) \to (0, +\infty)$ such that $f(t, v, w) \leq \alpha(t)\Omega(\|v\|_{\mathcal{B}} + \|w\|), t \in J, \quad (v, w) \in \mathcal{B} \times X$;

(3) There exists an integrable function $\eta : J \to [0, +\infty)$ such that
$\chi(T(s)f(t, D_1, D_2)) \leq \eta(t)(\sup_{-\infty \leq \theta \leq 0} \chi(D_1(\theta)) + \chi(D_2))$
for a.e. $s, t \in J$, and any bounded subset $D_1 \subset \mathcal{PC}([-\infty, 0]; X)$ and $D_2 \subset X$, where $D_1(\theta) = \{v(\theta) : v \in D_1\}$.

(4) There exists a positive constant L_f such that
$\|f(t, v_1, w_1) - f(t, v_2, w_2)\| \leq L_f(\|v_1 - v_2\|_\varphi + \|w_1 - w_2\|)$,
where $0 < L_f < 1, (t, v_i, w_i) \in J \times \mathcal{B} \times X, i = 1, 2$.

(He) **(1)** There exists a constant $N_1 > 0$ such that
$\left\| \int_0^t [e(t, s, x) - e(t, s, y)]ds \right\| \leq N_1 \|x - y\|_{\mathcal{B}}$, for $t, s \in J, \quad x, y \in \mathcal{B}$

(2) For each $(t, s) \in J \times J$, the function $e(t, s, .) : \mathcal{B} \to X$ is continuous and for each $x \in \mathcal{B}$, the function $e(., ., x) : J \times J \to X$ is strongly measurable. There exists an integrable function $m : J \to [0, \infty)$ and constant $\gamma > 0$, such that
$\|e(s, \tau, x)\| \leq \gamma\, m(\tau)\phi(\|x\|)$, where $\phi : [0, +\infty) \to (0, +\infty)$ is a continuous nondecreasing function. Assume that the finite bound of $\int_0^\infty \gamma\, m(s)ds$ is L_0.

(Hg) **(1)** There exists $0 < \beta < 1$, such that $g(t, v) \in X_\beta = D((-A)^\beta), (-A)^\beta g(.)$ is continuous for all $(t, v) \in J \times \mathcal{B}$ and there exist positive constants c_1 and C_2, such that
$\left\|(-A)^\beta g(t, v)\right\| \leq C_1 \|v\|_{\mathcal{B}} + C_2$

(2) There exists a positive constant L_g such that,
$\left\|(-A)^\beta g(t, v_1) - (-A)^\beta g(t, v_2)\right\| \leq L_g \|v_1 - v_2\|_{\mathcal{B}}, \qquad \forall v_1, v_2 \in \mathcal{B}.$

(HI) **(1)** There exist positive constants L_i such that
$\|I_i(u) = I_i(v)\| \leq \|u - v\|_{\mathcal{B}}, \qquad \forall u, v \in \mathcal{B}.$

(2) There exist positive constants $C_i^j, i = 1, ..., n, \quad j = 1, 2$, such tht
$\|I_i(v)\| = C_i^1 \|v\|_{\mathcal{B}} + C_i^2, \quad v \in \mathcal{B}.$

(H) **(1)** $\int\limits_0^b \hat{m}(s)ds \leq \int\limits_c^\infty \frac{ds}{\Omega(s)+\phi(s)}$
where
$\mu_1 = (K_b MH + M_b + M \left\|(-A)^\beta\right\| C_1 K_b) \|\varphi\|_{\mathcal{B}} + K_b \left\|(-A)^\beta\right\| C_2(M + 1) + K_b M \sum\limits_{0<t_i<t} C_i^2$
and
$\mu_2 = K_b \left\|(-A)^\beta\right\| C_1 + K_b M \sum\limits_{0<t_i<t} C_i^1 < 1, \qquad c = \frac{\mu_1}{1-\mu_2}.$

Let $y : (-\infty, b] \to X$ be a function defined by $y_0 = \varphi$ and $y(t) = T(t)\varphi(0)$ on J. Clearly, $\|y_t\| \leq (K_b MH + M_b) \|\varphi\|_{\mathcal{B}}$, where $K_b = \sup_{0 \leq t \leq b} K(t), M_b = \sup_{0 \leq t \leq b} M(t)$.

Theorem 3.1. *If the hypotheses (Hf), (Hg), (He), (HI) and (H) are satisfied, the the initial value problem (1.1)-(1.3) has at least one mild solution.*

Proof. Let $S(b)$ be the space $S(b) = \{x : (-\infty, b] \to X | x_0 = 0, x|_J \in \mathcal{PC}\}$ endowed with the supremum norm $\|.\|_b$. Let $\Gamma : S(b) \to S(b)$ be the map defined by

$$\Gamma x(t) = \begin{cases} 0, & t \in [-\infty, 0] \\ T(t)g(0, \varphi) + g(t, x_t + y_t) + \int_0^t T(t - s)f(s, x_s + y_s, \int_0^s e(s, \tau, x_\tau + y_\tau)d\tau)ds \\ + \sum\limits_{0<t_i<t} T(t - t_i)I_i(x_{t_i} + y_{t_i}), & t \in J. \end{cases} \tag{3.1}$$

It is easy to see that $\|x_t + y_t\|_{\mathcal{B}} \leq (K_b MH + M_b) \|\varphi\|_{\mathcal{B}} + K_b \|x\|_t$, where $\|x\|_t = \sup_{0 \leq s \leq t} \|x(s)\|$. Thus Γ is well defined and with the values in $S(b)$. In addition, from the axioms of phase space, the Lebesgue dominated convergence theorem and the condition (Hf), (Hg), (He) and (HI), we can show that Γ is continuous.

Step 1. For $0 < \lambda < 1$, set $\{x \in \mathcal{PC} : x = \lambda \Gamma x\}$ is bounded.

Let x_λ be a solution of $x = \lambda\Gamma x$ for $0 < \lambda < 1$, we have
$\|x_{\lambda t} + y_t\|_{\mathcal{B}} \le (K_b M H + M_b)\,\|\varphi\|_{\mathcal{B}} + K_b\,\|x_\lambda\|_t$.
Let $v_\lambda(t) = (K_b M H + M_b)\,\|\varphi\|_{\mathcal{B}} + K_b\,\|x_\lambda\|_t$, for each $t \in J$. Then

$$\begin{aligned}
\|x_\lambda(t)\| &= \|\lambda\Gamma x_\lambda(t)\| \le \|\Gamma x(t)\| \\
&\le M\left\|(-A)^{-\beta}\right\|(C_1\,\|\varphi\|_{\mathcal{B}} + C_2) + \left\|(-A)^{-\beta}\right\| C_2 + M\sum_{0<t_i<t} C_i^2 \\
&\quad + \Big(\left\|(-A)^{-\beta}\right\| C_1 + M\sum_{0<t_i<t} C_i^1\Big) v_\lambda(s) + M\int_0^t \alpha(s)\Omega\Big(v_\lambda(s) + \int_0^s \gamma m(\tau)\phi(v_\lambda(\tau))d\tau\Big)ds \\
\|x_\lambda\|_t &\le M\left\|(-A)^{-\beta}\right\|(C_1\,\|\varphi\|_{\mathcal{B}} + C_2) + \left\|(-A)^{-\beta}\right\| C_2 + M\sum_{0<t_i<t} C_i^2 \\
&\quad + \Big(\left\|(-A)^{-\beta}\right\| C_1 + M\sum_{0<t_i<t} C_i^1\Big) v_\lambda(s) + M\int_0^t \alpha(s)\Omega\Big(v_\lambda(s) + \int_0^s \gamma m(\tau)\phi(v_\lambda(\tau))d\tau\Big)ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
v_\lambda(t) &\le (K_b M H + M_b + M\left\|(-A)^{-\beta}\right\| C_1 K_b)\,\|\varphi\|_{\mathcal{B}} + K_b\left\|(-A)^{-\beta}\right\| C_2(M+1) + K_b M\sum_{0<t_i<t} C_i^2 \\
&+ K_b\Big(\left\|(-A)^{-\beta}\right\| C_1 + M\sum_{0<t_i<t} C_i^1\Big) v_\lambda(s) + M\int_0^t \alpha(s)\Omega\Big(v_\lambda(s) + \int_0^s \gamma m(\tau)\phi(v_\lambda(\tau))d\tau\Big)ds.
\end{aligned}$$

Consequently,

$$v_\lambda(t) \le c + \frac{MK_b}{1-\mu_2}\int_0^t \alpha(s)\Omega\Big(v_\lambda(s) + \int_0^s \gamma m(\tau)\phi(v_\lambda(\tau))d\tau\Big)ds.$$

Let us take the right-hand side of the above inequality as $\beta_\lambda(t)$. Then $\beta_\lambda(0) = c$ and

$$v_\lambda(t) \le \beta_\lambda(t), 0 \le t \le b$$

and

$$\beta_\lambda'(t) \le \frac{MK_b}{1-\mu_2}\alpha(t)\Omega\Big(v_\lambda(t) + \int_0^t \gamma m(s)\phi(v_\lambda(s))ds\Big).$$

Since Ω is nondecreasing

$$\beta_\lambda'(t) \le \frac{MK_b}{1-\mu_2}\alpha(t)\Omega\Big(\beta_\lambda(t) + \int_0^t \gamma m(s)\phi(\beta_\lambda(s))ds\Big).$$

Let $W_\lambda(t) = \beta_\lambda(t) + \int_0^t \gamma m(s)\phi(\beta_\lambda(s))ds$. Then $W_\lambda(0) = \beta_\lambda(0)$ and $W_\lambda(t) \le \beta_\lambda(t)$

$$\begin{aligned}
W_\lambda'(t) &= \beta_\lambda'(t) + \gamma m(t)\phi(\beta_\lambda(t)) \\
&\le \frac{MK_b}{1-\mu_2}\alpha(t)\Omega(W_\lambda(t)) + \gamma m(t)\phi(W_\lambda(t)) \\
&\le \hat{m}(t)\Big(\Omega(W_\lambda(t)) + \phi(W_\lambda(t))\Big)
\end{aligned}$$

This implies that

$$\int_{W_\lambda(0)}^{W_\lambda(t)} \frac{ds}{\Omega(s)+\phi(s)} \le \int_0^b \hat{m}(s)ds \le \int_c^\infty \frac{ds}{\Omega(s)+\phi(s)}$$

which implies that the functions $\beta_\lambda(t)$ are bounded on J. Thus, the function $v_\lambda(t)$ are bounded on J, and $x_\lambda(.)$ are also bounded on J.
Step 2. Next, we show that Γ is χ-contraction. To clarify this, we decompose Γ in the form $\Gamma = \Gamma_1 + \Gamma_2$, for $t \geq 0$, where

$$\Gamma_1 x(t) = T(t)g(0,\varphi) + g(t, x_t + y_t) + \sum_{0<t_i<t} T(t-t_i) I_i(x_{t_i} + y_{t_i}).$$

$$\Gamma_2 x(t) = \int_0^t T(t-s) f(s, x_s + y_s, \int_0^s e(s,\tau, x_\tau + y_\tau) d\tau) ds.$$

Firstly, we show that Γ_1 is Lipschitz continuous.

Take $x_1, x_2 \in S(b)$ arbitrary, on account of Definition 2.1 and hypotheses, we get

$$\begin{aligned}\|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\| &\leq \left\|(-A)^{-\beta}\right\| L_g \|x_{1t} - x_{2t}\|_{\mathcal{B}} + M \sum_{i=1}^n L_i \|x_{1t_i} - x_{2t_i}\|_{\mathcal{B}} \\ &\leq \Big(\left\|(-A)^{-\beta}\right\| L_g K_b + M K_b \sum_{i=1}^n L_i \Big) \|x_1 - x_2\|_b \\ &\leq K_b \Big(\left\|(-A)^{-\beta}\right\| L_g + M \sum_{i=1}^n L_i \Big) \|x_1 - x_2\|_b .\end{aligned}$$

Hence Γ_1 is Lipschitz continuous, and

$$L' = K_b \Big(\left\|(-A)^{-\beta}\right\| L_g + M \sum_{i=1}^n L_i \Big).$$

Next, take bounded subset $W \subset S(b)$ arbitrary. Since analytic semigroup is equicontinuous, $T(t-s)f(s, w_s + y_s, \int_0^s e(s,\tau, w_\tau + y_\tau)d\tau)$ piecewise equicontinuous; and from Lemma 2.6 $\chi_{\mathcal{PC}}(W) = \sup\{\chi(W(t)), t \in [0,b]\}$, and from [2], Lemma 3.4.7, $\chi(\Omega(W(\tau))) \leq K_1 \chi(W(\tau))$, where K_1 is constant, we have

$$\begin{aligned}\chi(\Gamma_2(W(t))) &\leq \chi\Big(\int_0^t T(t-s) f(s, w_s + y_s, \int_0^s e(s,\tau, w_\tau + y_\tau) d\tau)\Big) ds \\ &\leq \int_0^t \eta(s) \Big(\sup_{-\infty<\theta\leq 0} \chi[W(s+\theta) + y(s+\theta)] + \chi[\int_0^s e(s,\tau, x_\tau + y_\tau) d\tau] \Big) ds \\ &\leq \int_0^t \eta(s) \sup_{-\infty<\theta\leq 0} \Big(\chi[W(s+\theta) + y(s+\theta)] + L_o \chi[\Omega(W(s+\theta) + y(s+\theta))] \Big) ds \\ &\leq \int_0^t \eta(s) \sup_{0<\tau\leq s} \Big(\chi(W(\tau)) + L_0 \chi(\Omega(W(\tau))) \Big) ds \\ &\leq \chi_{\mathcal{PC}}(W)(1 + K_1 L_0) \int_0^t \eta(s) ds \quad \text{for each bounded set } W \in \mathcal{PC}.\end{aligned}$$

Since

$$\begin{aligned}\chi_{\mathcal{PC}}(\Gamma W) &= \chi_{\mathcal{PC}}(\Gamma_1 W + \Gamma_2 W) \\ &\leq \chi_{\mathcal{PC}}(\Gamma_1 W) + \chi_{\mathcal{PC}}(\Gamma_2 W) \\ &\leq \Big(L' + \int_0^t \eta(s) ds \Big) \chi_{\mathcal{PC}}(W),\end{aligned}$$

Γ is χ-contraction. In view of Lemma 2.3, i.e. Darbo fixed point theorem, we conclude that Γ has atleast one fixed point in W. Let x be a fixed of Γ on $S(b)$, then $z = x + y$ is a mild solution of (1.1)-(1.3). So we deduce the existence of a mild solution of (1.1)-(1.3).

Theorem 3.2. *Assume that (Hf),(He),(HI),(H) are satisfied. Furthermore, we suppose*

$$(H)(1') \qquad K_b\Big[\Big\|(-A)^{-\beta}\Big\| C_1 + M\sum_{i=1}^{n} C_i^1 + M\int_0^b \alpha(s)ds \lim_{\tau\to\infty}\sup\frac{\Omega(\tau+L_0\phi(\tau))}{\tau}\Big] < 1.$$

Then the initial value problem (1.1)-(1.3) has at least one mild solution.

Proof. Proceeding in the proof of Theorem 3.1, we refer that the map Γ given by (3.1) is continuous from $S(b)$ into $S(b)$. Furthermore, there exists $k > 0$ such that $\Gamma(B_k) \subset B_k$, where $B_k = \{x \in S(b) : \|x\|_b \le k\}$. In fact, if we assume that the assertion is false, then $k > 0$ there exists $x_k \in B_k$ anf $t_k \in J$ such that $k < \|\Gamma x_k(t_k)\|$. This yields that

$$\begin{aligned}
k < \|\Gamma x_k(t_k)\| &\le M\|g(0,\varphi)\| + \Big\|(-A)^{-\beta}\Big\|(C_1\|x_{kt_k}+y_{tk}\|_{\mathcal{B}} + C_2)\\
&+ M\int_0^{t_k}\alpha(s)\Omega\Big(\|x_{ks}+y_s\|_{\mathcal{B}} + \Big\|\int_0^s e(s,\tau,x_{k\tau}+y_\tau)d\tau\Big\|\Big)ds + M\sum_{0<t_i<t}(C_i^1\|x_{kt_i}+y_{ti}\|_{\mathcal{B}} + C_i^2)\\
&\le M\|g(0,\varphi)\| + \Big\|(-A)^{-\beta}\Big\|(C_1(K_bMH+M_b)\|\varphi\|_{\mathcal{B}} + C_1K_bk + C_2)\\
&+ M\int_0^b\alpha(s)ds\ \Omega\Big[(K_bMH+M_b)\|\varphi\|_{\mathcal{B}} + K_bk + L_0\phi((K_bMH+M_b)\|\varphi\|_{\mathcal{B}} + K_bk)\Big]\\
&+ M\sum_{i=1}^{n}(C_i^1(K_bMH+M_b)\|\varphi\|_{\mathcal{B}} + C_i^1K_bk + C_i^2),
\end{aligned}$$

which implies that

$$\begin{aligned}
1 &< K_b\Big[\Big\|(-A)^{-\beta}\Big\| C_1 + M\sum_{i=1}^{n} C_i^1\Big]\\
&+ M\int_0^b\alpha(s)ds\ \lim_{k\to\infty}\sup\frac{\Omega\Big[(K_bMH+M_b)\|\varphi\|_{\mathcal{B}} + K_bk + L_0\phi((K_bMH+M_b)\|\varphi\|_{\mathcal{B}} + K_bk)\Big]}{k}\\
&\le K_b\Big[\Big\|(-A)^{-\beta}\Big\| C_1 + M\sum_{i=1}^{n} C_i^1 + M\int_0^b\alpha(s)ds\ \lim_{\tau\to\infty}\sup\frac{\tau+L_0\phi(\tau)}{\tau}\Big] < 1
\end{aligned}$$

a contradiction.

By means of lemma 2.4, as the proof of Theorem 3.1, we conclude that (1.1)-(1.3) has at least a mild solution.

4. Example

In this section, we apply some of the results established in this paper . In the next applications, $\mathcal{B}$ will be the phase space $C_0 \times L^2(h, X)$ (see [19]).

We study, the first-order neutral integrodifferential equation with unbounded delay

$$\begin{aligned}
&\frac{d}{dt}\Big(u(t,\xi) - \int_{-\infty}^{t}\int_0^\pi b(t-s,\eta,\xi)u(s,\eta)d\eta ds\Big) = \frac{\partial^2}{\partial x^2}\Big(u(t,\xi) - \int_{-\infty}^{t}\int_0^\pi b(t-s,\eta,\xi)u(s,\eta)d\eta ds\Big)\\
&+\int_0^t a(t,\xi,s-t)F(u(s,\xi),\int_0^s q(s,\tau,u_\tau)d\tau)ds, t\in[0,b], \xi\in[0,\pi], \qquad (4.1)\\
&u(t,0) = u(t,\pi), \qquad t\in[0,b], \qquad (4.2)\\
&u(\tau,\xi) = \varphi(\tau,\xi), \qquad \tau\le 0,\ 0\le\xi\le\pi, \qquad (4.3)\\
&\Delta u(t_i)(\xi) = \int_{-\infty}^{t_i} a_i(t_i-s)u(s,\xi)ds, \qquad (4.4)
\end{aligned}$$

where, $\varphi \in C_0 \times L^2(h, X), 0 < t_1 < ... < t_n < b$ and

(a) The functions $b(s,\eta,\xi)$, $\frac{\partial b(s,\eta,\xi)}{\partial \xi}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ and
$L_g := \max\left\{\left(\int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{h(s)}\left(\frac{\partial^i b(s,\eta,\xi)}{\partial \xi^i}\right)^2 d\eta ds d\xi\right)^{1/2} : i = 0,1\right\} < \infty$;

(b) The function $a(t,\xi,\theta)$ is continuous in $J \times [0,\pi] \times (-\infty, 0]$ and
$\int_{-\infty}^0 a(t,\xi,\theta)d\theta = m(t,\xi) < \infty$

(c) The function $q(.)$ is continuous such that $0 \leq q(t,s,\xi) \leq \Omega_0(|\xi|)$, where $\Omega(.)$ is positive, continuous and nondecreasing in $[0,\infty)$.

(d) The function $F(u_1,u_2) \leq \Omega_1(|u_1| + |u_2|)$, where $\Omega_1(.)$ is positive, continuous and nondecreasing in $[0,\infty)$.

(e) The functions $a_i \in C([0,\infty); R)$ and $L_i^1 := (\int_{-\infty}^0 \frac{(a_i(s))^2}{h(s)} ds)^{1/2} < \infty$ for all $i = 1,2,...,n$.

Assuming that the conditions (a)-(e) are varified, the problem (4.1)-(4.4) can be modeled as the abstract impulsive Cauchy problem (1.1)-(1.3) by defining

$$g(t,\psi)(\xi) := \int_{-\infty}^0 \int_0^\pi b(s,v,\xi)\psi(s,v)dvds, \tag{4.5}$$

$$f(t,\phi,\psi)(\xi) := \int_{-\infty}^0 a(t,\xi,\tau)F\Big(\phi(\tau,\xi), \int_0^\tau \psi(\tau,\theta,u_\theta)d\theta\Big)d\tau, \tag{4.6}$$

$$I_i(\psi)(\xi) := \int_{-\infty}^0 a_i(s)\psi(s,\xi)ds. \tag{4.7}$$

Moreover, $g(t,.), f(t,.,.)$ and $I_i, i = 1,...,n$, are bounded linear operators, the range of $g(.)$ is contained in $X_{1/2}$, $\left\|(-A)^{1/2}g(t,.)\right\| \leq L_g$, $\|I_i\| \leq L_i^1, i = 1,...,n$ and
$\|f(t,\phi,\psi)\| \leq \alpha(t)\Omega(\|\phi\|_{\mathcal{B}} + \|\psi\|)$ for every $t \in [0,b]$, where $\alpha(t) := (\int_{-\infty}^0 \frac{\mu(t,s)^2}{h(s)} ds)^{1/2}$.

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