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Exponential growth of solutions with L_p- *norm* of a nonlinear viscoelastic hyperbolic equation

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Abstract

In this work, we consider a viscoelastic wave equation, with strong damping, nonlinear damping and source terms, with initial and Dirichlet boundary conditions. We will show the exponential growth of solutions with $L_p - norm$ if $2 \leq m < p$.

Keywords: Nonlinear damping, strong damping, viscoelasticity, nonlinear source, exponential growth.
2010 MSC: 35L05, 35L20, 58G16.

1. Introduction

In this paper we consider the following nonlinear viscoelastic hyperbolic problem

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int\limits_0^t g(t-s)\Delta u(s,x)ds \\ +a|u_t|^{m-2}u_t = b|u|^{p-2}u, x \in \Omega, t > 0 \\ u\left(0,x\right) = u_0(x), u_t(0,x) = u_1(x), x \in \Omega \\ u\left(t,x\right) = 0, x \in \Gamma, t > 0 \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, with smooth boundary Γ, a, b, w are positive constants, and $m \geq 2$, $p \geq 2$. The function $g(t)$ satisfying conditions $G1$ and $G2$.

In the physical point of view, this type of problems arise usually in viscoelasticity. It has been considered first by Dafermos [8], in 1970, where the general decay was discussed. A related problems to 1.1 have attracted a great deal of attention in the last two decades, and many results have been appeared on the existence and long time behavior of solutions. See in this directions [2],[3],[4]-[7], [14], [23], [26], [27], [30] and references therein.

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In the absence of the strong damping Δu_t, that is for $w = 0$, and when the function g vanishes identically (i.e. $g = 0$), then problem 1.1 reduced to the following initial boundary damped wave equation with nonlinear damping and nonlinear sources terms.

$$u_{tt} - \Delta u + a\,|u_t|^{m-2}\,u_t = b\,|u|^{p-2}\,u. \tag{1.2}$$

Some special cases of equation 1.2 arise in quantum field theory which describe the motion of charged mesons in an electromagnetic field.

Equation 1.2 together with initial and boundary conditions of Dirichlet type, has been extensively studied and results concerning existence, blow up and asymptotic behavior of smooth, as well as weak solutions have been established by several authors.

For $b = 0$, that is in the absence of the source term, it is well known that the damping term $a\,|u_t|^{m-2}\,u_t$ assures global existence and decay of the solution energy for arbitrary initial data (see [13] and [17]).

For $a = 0$, the source term causes finite-time blow-up of solutions with a large initial data (negative initial energy). That is to say, the norm of our solution $u(t, x)$ in the energy space reaches $+\infty$ when the time t approaches certain value T^* called " *the blow up time*", (see [1] and [16] for more details).

The interaction between the damping term $a\,|u_t|^{m-2}\,u_t$ and the source term $b\,|u|^{p-2}\,u$ makes the problem more interesting. This situation was first considered by Levine [19]-[20] in the linear damping case ($m = 2$), where he showed that solutions with negative initial energy blow up in finite time T^*. The main ingredient used in [19] and [20] is the " concavity method" where the basic idea of this method is to construct a positive function $L(t)$ of the solution and show that for some $\gamma > 0$, the function $L^{-\gamma}(t)$ is a positive concave function of t.

Georgiev and Todorova in their famous paper [10], extended Levine's result to the nonlinear damping case ($m > 2$). More precisely, in [10] and by combining the Galerkin approximation with the contraction mapping theorem, the authors showed that problem 1.2 in a bounded domain Ω with initial and boundary conditions of Dirichlet type has a unique solution in the interval $[0, T)$ provided that T is small enough. Also, they proved that the obtained solutions continue to exist globally in time if $m \geq p$ and the initial data are small enough. Whereas for $p > m$ the unique solution of problem 1.2 blows up in finite time provided that the initial data are large enough. (i.e. the initial energy is sufficiently negative).

This later result has been pushed by Messaoudi in [28] to the situation where the initial energy $E(0) < 0$. For more general result in this direction, we refer the interested reader to the works of Vitillaro [35], Levine [21] and [25].

In the presence of the viscoelastic term ($g \neq 0$) and for $w = 0$, our problem 1.1 becomes

$$\begin{cases} u_{tt} - \Delta u + \int\limits_0^t g(t-s)\Delta u(s,x)ds \\ +a|u_t|^{m-2}u_t = b|u|^{p-2}u, x \in \Omega, t > 0 \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), x \in \Omega \\ u(t,x) = 0, x \in \Gamma, t > 0 \end{cases} \tag{1.3}$$

For $a = 0$, problem 1.3 has been investigated by Berrimi and Messaoudi [3]. They established the local existence result by using the Galerkin method together with the contraction mapping theorem. Also, they showed that for a suitable initial data, then the local solution is global in time and in addition, they showed that the dissipation given by the viscoelastic integral term is strong enough to stabilize the oscillations of the solution with the same rate of decaying (exponential or polynomial) of the kernel g. Also their result has been obtained under weaker conditions than those used by Cavalcanti et *al* [6], in which a similar problem has been addressed.

Messaoudi in [23], showed that under appropriate conditions between m, p and g a blow up and global existence result, of course his work generalizes the results by Georgiev and Todorova [10] and Messaoudi [23].

One of the main direction of the research in this field seems to find the minimal dissipation such that the solutions of problems similar to 1.3 decay uniformly to zero, as time goes to infinity. Consequently, several authors introduced different types of dissipative mechanisms to stabilize these problems. For example, a localized frictional linear damping of the form $a(x)u_t$ acting in sub-domain $\overline{w} \subset \Omega$ has been considered by Cavalcanti et *al* [6]. More precisely the authors in [5] looked into the following problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s,x)ds + a(x)u_t + |u|^\gamma u = 0. \tag{1.4}$$

for $\gamma > 0$, g a positive function and $a : \Omega \to \mathbb{R}^+$ a function, which may be null on a part of the domain Ω.

By assuming $a(x) \geq a_0 > 0$ on the sub-domain $\overline{w} \subset \Omega$, the authors showed a decay result of an exponential rate, provided that the kernel g satisfies

$$-\zeta_1 g(t) \leq g'(t) \leq -\zeta_2 g(t), t \geq 0, \tag{1.5}$$

and $\|g\|_{L^1(0,\infty)}$ is small enough.

This later result has been improved by Berrimi and Messaoudi [2], in which they showed that the viscoelastic dissipation alone is strong enough to stabilize the problem even with an exponential rate.

In many existing works on this field, the following conditions on the kernel

$$g'(t) \leq -\zeta g^p(t), t \geq 0, p \geq 1, \tag{1.6}$$

is crucial in the proof of the stability.

For a viscoelastic systems with oscillating kernels, we mention the work by Rivera et *al* [29]. In that work the authors proved that if the kernel satisfies $g(0) > 0$ and decays exponentially to zero, that is for $p = 1$ in 1.6, then the solution also decays exponentially to zero. On the other hand, if the kernel decays polynomially, i.e. $(p > 1)$ in the inequality 1.6, then the solution also decays polynomially with the same rate of decay.

In the presence of the strong damping $(w > 0)$ and in the absence of the viscoelastic term $(g = 0)$, the problem (1) takes the following form

$$\begin{cases} u_{tt} - \Delta u - \omega\Delta u_t + a|u_t|^{m-2}u_t = b|u|^{p-2}u, x \in \Omega, t > 0 \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), x \in \Omega \\ u(t,x) = 0, x \in \Gamma, t > 0 \end{cases} \tag{1.7}$$

Problem 1.7 represents the wave equation with a strong damping Δu_t. When $m = 2$, this problem has been studied by Gazzola and Squassina [9]. In their work, the authors proved some results on well posedness and asymptotic behavior of solutions. They showed the global existence and polynomial decay property of solutions provided that the initial data is in the potential well.

The proof in [9] is based on a method used in [15]. Unfortunately their decay rate is not optimal, and their result has been improved by Gerbi and Said-Houari [12], by using an appropriate modification of the energy method and some differential and integral inequalities.

Introducing a strong damping term Δu_t makes the problem from that considered in [10], for this reason less results where known for the wave equation with strong damping and many problem remain unsolved. (See [9] and [11]).

In this article, we investigated problem 1.1, in which all the damping mechanism have been considered in the same time (i.e. $w > 0$, $g \neq 0$, and $m \geq 2$), these assumptions make our problem different form those studied in the literature, specially the exponential growth of solutions. We will prove that if the initial energy $E(0)$ of our solutions is negative (this means that our initial data are large enough), then our local solutions in bounded and

$$\|u_t\|_2 + \|\nabla u\|_2 \to \infty \tag{1.8}$$

as t tends to $+\infty$. In fact it will be proved that the L^p-norm of the solution grows as an exponential function. An essential tool of the proof is an idea used in [11], which based on an auxiliary function (which is a small perturbation of the total energy), in order to obtain a differential inequality leads to the exponential growth result provided that the following conditions

$$\int_0^\infty g(s)ds < \frac{p-2}{p-1}, \tag{1.9}$$

holds.

2. Assumptions, Notations and Preliminaries

We consider a viscoelastic wave equation, with strong damping, polynomial nonlinear damping and source term. Namely we looked into the following problem

$$u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s,x)ds + a|u_t|^{m-2}u_t = b|u|^{p-2}u, x \in \Omega, t > 0 \tag{2.1}$$

subjected to the following initial and boundary conditions

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), x \in \Omega \tag{2.2}$$

$$u(t,x) = 0, x \in \Gamma, t > 0 \tag{2.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, with smooth boundary Γ, and a, b, w are positive constants, $m \geq 2$, $p \geq 2$, and g is a nonnegative nonincreasing function. This type of problems are not only important from the theoretical point of view, but also arise in many physical applications and describe a great deal of models in applied science. One of the most important field of such problems arise in the models of nonlinear viscoelasticity. Many authors studied these types of problems, and several results appeared in the literature.

The energy related to problem 1.1 is

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(t), \tag{2.4}$$

where

$$J(t) = \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{b}{p}\|u(t)\|_p^p, \tag{2.5}$$

We assume that the kernel g satisfies the following conditions:
$(G1)$ $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a bounded C^1-function such that

$$g(0) > 0,\ 1 - \int_0^\infty g(s)ds = l > 0. \tag{2.6}$$

$(G2)$ $g(t) \geq 0$, $g'(t) \leq 0$, $g(t) \leq -\xi g'(t)$, $\forall t \geq 0$.
Let us denote by

$$(g \circ u)(t) = \int_0^t g(t-s)\int_\Omega |u(s) - u(t)|^2 dxds. \tag{2.7}$$

The goal of this work is the study of the Exponential Growth of the problem 1.1. We first state a local existence theorem that can be established by combining arguments of Georgiev and Todorova [10]. In fact this depends on the parameters values of the coefficients and the exponents m and p. Let us introduce the following complete metric space

$$Y_T = \left\{ \begin{array}{ll} u: & u \in C\left([0,T]; H_0^1(\Omega)\right), \\ & u_t \in C\left([0,T]; H_0^1(\Omega)\right) \cap L^m([0,T] \times \Omega) \end{array} \right\}. \tag{2.8}$$

Theorem 2.1. *Let $(u_0, u_1) \in \left(H_0^1(\Omega)\right)^2$ be given. Suppose that $m \geq 2$, $p \geq 2$ be such that*

$$\max m, p \leq \frac{2(n-1)}{n-2}, n \geq 3. \tag{2.9}$$

Then, under the conditions $(G1)$ and $(G2)$, the problem 1.1 has a unique local solution $u(t,x) \in Y_T$, for T small enough.

The following technical lemma will play an important role in the sequel.

Lemma 2.2. *[5] For any $v \in C^1\left(0, T; H^2(\Omega)\right)$ we have*

$$\begin{aligned} \int_\Omega \int_0^t g(t-s)\Delta v(s).v'(t)dsdx &= \frac{1}{2}\frac{d}{dt}(g \circ \nabla v)(t) - \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s) \int_\Omega |\nabla v(t)|^2 dxds\right] \\ &\quad -\frac{1}{2}\left(g' \circ \nabla v\right)(t) + \frac{1}{2} g(t) \int_\Omega |\nabla v(t)|^2 dx. \end{aligned}$$

In the next theorem we give the global existence result, its proof based on the potential well depth method in which the concept of so-called stable set appears, where we show that if we restrict our initial data in the stable set, then our local solution obtained is global in time, that is to say, the norm

$$\|u_t\|_2 + \|\nabla u\|_2, \tag{2.10}$$

in the energy space $L^2(\Omega) \times H_0^1(\Omega)$ of our solution is bounded by a constant independent of the time t. We will make use of arguments in [33].

Theorem 2.3. *Suppose that $(G1), (G2)$ and 2.9 hold. if $u_0 \in W$, $u_1 \in H_0^1(\Omega)$ and*

$$\frac{bC_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{2.11}$$

where C_* is the best Poincaré's constant. Then the local solution $u(t,x)$ is global in time.

Remark 2.4. Let us remark, that if there exists $t_0 \in [0, T)$ such that $u(t_0) \in W$ and $u_t(t_0) \in H_0^1(\Omega)$ and condition 2.11 holds for t_0. Then the same result of theorem 2.3 stays true.

3. Main results

Our result reads as follows.

Theorem 3.1. *Suppose that $m \geq 2$ and $m < p \leq \infty$, if $n = 1, 2$, $m < p \leq \dfrac{2(n-1)}{n-2}$, if $n \geq 3$. Assume further that $E(0) < 0$ and $\int_0^\infty g(s)ds < \dfrac{p-2}{p-1}$ holds. Then the unique local solution of problem 1.1 grows exponentially.*

Proof. We set

$$H(t) = -E(t). \tag{3.1}$$

By multiplying the first equations in 1.1 by $-u_t$, integrating over Ω, we obtain

$$\begin{aligned} &-\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\left(g \circ \nabla u\right)(t) - \frac{b}{p}\|u\|_p^p\right\} \\ &= a\|u_t\|_m^m - \frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2 + w\|\nabla u_t\|_2^2. \end{aligned} \tag{3.2}$$

By the definition of $H(t)$, 3.2 rewritten as

$$H'(t) = a\|u_t\|_m^m - \frac{1}{2}\left(g' \circ \nabla u\right)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2 + w\|\nabla u_t\|_2^2 \geq 0, \forall t \geq 0. \tag{3.3}$$

Consequently, $E(0) < 0$, we have

$$H(0) = -\frac{1}{2}\|u_1\|_2^2 - \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{b}{p}\|u_0\|_p^p > 0. \tag{3.4}$$

It's clear that by 3.1, we have

$$H(0) \leq H(t), \forall t \geq 0. \tag{3.5}$$

Using $(G2)$, to get

$$\begin{aligned} H(t) - \frac{b}{p}\|u\|_p^p &= -\left[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\left(g \circ \nabla u\right)(t)\right] \\ &\leq 0, \forall t \geq 0. \end{aligned} \tag{3.6}$$

One implies

$$0 < H(0) \leq H(t) \leq \frac{b}{p}\|u\|_p^p. \tag{3.7}$$

Let us define the functional

$$L(t) = H(t) + \varepsilon \int_\Omega u_t u dx + \varepsilon \frac{w}{2}\|\nabla u\|_2^2. \tag{3.8}$$

for ε small to be chosen later.

By taking the time derivative of 3.8, we obtain

$$\begin{aligned} L'(t) &= H'(t) + \varepsilon \int_\Omega u u_{tt}(t,x) dx + \varepsilon\|u_t\|_2^2 + \varepsilon w \int_\Omega \nabla u_t \nabla u dx \\ &= \left[w\|\nabla u_t\|_2^2 + a\|u_t\|_m^m - \frac{1}{2}\left(g' \circ \nabla u\right)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2\right] \\ &\quad + \varepsilon\|u_t\|_2^2 + \varepsilon w \int_\Omega \nabla u_t \nabla u dx + \varepsilon \int_\Omega u_{tt} u dx. \end{aligned} \tag{3.9}$$

Using the first equations in 1.1, to obtain

$$\begin{aligned} \int_\Omega u u_{tt} dx &= b\|u\|_p^p - \|\nabla u\|_2^2 - \omega \int_\Omega \nabla u_t \nabla u dx - a \int_\Omega |u_t|^{m-2} u_t u dx \\ &\quad + \int_\Omega \nabla u \int_0^t g(t-s)\nabla u(s,x) ds dx. \end{aligned} \tag{3.10}$$

Inserting 3.10 into 3.9 to get

$$\begin{aligned} L'(t) &= w\|\nabla u_t\|_2^2 + a\|u_t\|_m^m - \frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2 \\ &\quad + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 + \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u . \nabla u(s) dx ds \\ &\quad + \varepsilon b\|u\|_p^p - \varepsilon a \int_\Omega |u_t|^{m-2} u_t u dx. \end{aligned} \tag{3.11}$$

By using $(G2)$, the last equality takes the form

$$\begin{aligned} L'(t) \geq\ & w\|\nabla u_t\|_2^2 + a\|u_t\|_m^m + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 + \varepsilon b\|u\|_p^p \\ & +\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u.\nabla u(s)dxds - \varepsilon a\int_\Omega |u_t|^{m-2}u_t u d\sigma. \end{aligned} \tag{3.12}$$

To estimate the last term in the right-hand side of 3.12, we use the following Young's inequality

$$XY \leq \frac{\delta^r}{r}X^r + \frac{\delta^{-q}}{q}Y^q, X, Y \geq 0, \tag{3.13}$$

for all $\delta > 0$ be chosen later, $\frac{1}{r} + \frac{1}{q} = 1$, with $r = m$ and $q = \frac{m}{m-1}$.

So we have

$$\begin{aligned} \int_\Omega |u_t|^{m-2}u_t u dx \leq\ & \int_\Omega |u_t|^{m-1}|u|dx \\ \leq\ & \frac{\delta^m}{m}\|u\|_m^m + \left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)}\|u_t\|_m^m, \forall t \geq 0. \end{aligned} \tag{3.14}$$

Therefore, the estimate 3.12 takes the form

$$\begin{aligned} L'(t) \geq\ & w\|\nabla u_t\|_2^2 + a\|u_t\|_m^m + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 + \varepsilon b\|u\|_p^p \\ & +\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u.\nabla u(s)dxds \\ & -\varepsilon a\frac{\delta^m}{m}\|u\|_m^m - \varepsilon a(\frac{m-1}{m})\delta^{(\frac{-m}{m-1})}\|u_t\|_m^m \\ \geq\ & w\|\nabla u_t\|_2^2 + a\|u_t\|_m^m + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 + \varepsilon b\|u\|_p^p \\ & +\varepsilon\|\nabla u\|_2^2\int_0^t g(s)ds + \varepsilon\int_0^t g(t-s)\int_\Omega \nabla u(t)\left[\nabla u(s) - \nabla u(t)\right]dxds \\ & -\varepsilon a\frac{\delta^m}{m}\|u\|_m^m - \varepsilon a(\frac{m-1}{m})\delta^{(\frac{-m}{m-1})}\|u_t\|_m^m. \end{aligned} \tag{3.15}$$

Using Cauchy-Schwarz and Young's inequalities to obtain

$$\begin{aligned} L'(t) \geq\ & w\|\nabla u_t\|_2^2 + a\left(1 - \varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)}\right)\|u_t\|_m^m + \varepsilon\|u_t\|_2^2 \\ & -\varepsilon\|\nabla u\|_2^2 + \varepsilon b\|u\|_p^p + \varepsilon\|\nabla u\|_2^2\int_0^t g(s)ds \\ & -\varepsilon\int_0^t g(t-s)\|\nabla u\|_2\|\nabla u(s) - \nabla u(t)\|_2 ds - \varepsilon a\frac{\delta^m}{m}\|u\|_m^m \\ \geq\ & w\|\nabla u_t\|_2^2 + a\left(1 - \varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)}\right)\|u_t\|_m^m + \varepsilon\|u_t\|_2^2 + \varepsilon b\|u\|_p^p \\ & +\varepsilon\left(\frac{1}{2}\int_0^t g(s)ds - 1\right)\|\nabla u\|_2^2 - \varepsilon\frac{1}{2}(g \circ \nabla u(t)) - \varepsilon a\frac{\delta^m}{m}\|u\|_m^m. \end{aligned} \tag{3.16}$$

Using assumptions to substitute for $b\|u\|_p^p$. Hence, 3.16 becomes

$$\begin{aligned} L'(t) &\geq w\|\nabla u_t\|_2^2 + a\left(1-\varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)}\right)\|u_t\|_m^m + \varepsilon\|u_t\|_2^2 \\ &+\varepsilon\left(pH(t) + \frac{p}{2}\|u_t\|_2^2 + \frac{p}{2}\left(g\circ\nabla u\right)(t) + \frac{p}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right) \\ &+\varepsilon\left(\frac{1}{2}\int_0^t g(s)ds - 1\right)\|\nabla u\|_2^2 - \varepsilon\frac{1}{2}(g\circ\nabla u(t)) - \varepsilon a\frac{\delta^m}{m}\|u\|_m^m. \qquad (3.17) \\ &\geq w\|\nabla u_t\|_2^2 + a\left(1-\varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)}\right)\|u_t\|_m^m + \varepsilon\left(1+\frac{p}{2}\right)\|u_t\|_2^2 \\ &+\varepsilon a_1\|\nabla u\|_2^2 + \varepsilon a_2(g\circ\nabla u(t) - \varepsilon a\frac{\delta^m}{m}\|u\|_m^m + \varepsilon pH(t). \end{aligned}$$

where $a_1 = (\frac{1-p}{2})\int_0^\infty g(s)ds + (\frac{p-2}{2}) > 0$, $a_2 = \frac{p-1}{2} > 0$.

In order to undervalue $L'(t)$ with terms of $E(t)$ and since $p > m$, we have from the embedding $L^p(\Omega) \hookrightarrow L^m(\Omega)$,

$$\|u\|_m^m \leq C\|u\|_p^m \leq C\left(\|u\|_p^p\right)^{\frac{m}{p}}, \forall t \geq 0. \qquad (3.18)$$

for some positive constant C depending on Ω only. Since $0 < \frac{m}{p} < 1$, as in [11] we use the algebraic inequality

$$z^k \leq (z+1) \leq \left(1+\frac{1}{w}\right)(z+w), \forall z \geq 0, 0 < k \leq 1, w > 0, \qquad (3.19)$$

to find

$$\left(\|u\|_p^p\right)^{\frac{m}{p}} \leq K\left(\|u\|_p^p + H(0)\right), \forall t \geq 0, \qquad (3.20)$$

where $K = 1 + \frac{1}{H(0)} > 0$, then by 3.7 we have

$$\|u\|_m^m \leq C\left(1+\frac{b}{p}\right)\|u\|_p^p, \forall t \geq 0. \qquad (3.21)$$

Inserting 3.21 into 3.17, to get

$$\begin{aligned} L'(t) &\geq w\|\nabla u_t\|_2^2 + a\left(1-\varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)}\right)\|u_t\|_m^m + \varepsilon\left(1+\frac{p}{2}\right)\|u_t\|_2^2 \\ &+\varepsilon a_1\|\nabla u\|_2^2 + \varepsilon a_2(g\circ\nabla u)(t) - \varepsilon C_1\|u\|_p^p + \varepsilon pH(t). \qquad (3.22) \end{aligned}$$

where $C_1 = aC\frac{\delta^m}{m}\left(1+\frac{b}{p}\right) > 0$.

By using 3.1 and by the same statements as in [12], we have

$$\begin{aligned} 2H(t) &= -\|u_t\|_2^2 - \|\nabla u\|_2^2 + \int_0^t g(s)ds\|\nabla u\|_2^2 - \left(g\circ\nabla u\right)(t) + \frac{2b}{p}\|u\|_p^p \\ &\geq -\|u_t\|_2^2 - \|\nabla u\|_2^2 - \left(g\circ\nabla u\right)(t) + \frac{2b}{p}\|u\|_p^p, \forall t \geq 0. \end{aligned}$$

Adding and substituting the value $2a_3H(t)$ from 3.22, and choosing δ small enough such that $a_3 < \min a_1, a_2$, we obtain

$$\begin{aligned} L'(t) \geq \ & w\|\nabla u_t\|_2^2 + a\left(1-\varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)}\right)\|u_t\|_m^m \\ & +\varepsilon\left(1+\frac{p}{2}-a_3\right)\|u_t\|_2^2 + \varepsilon\left(a_1-a_3\right)\|\nabla u\|_2^2 \\ & +\varepsilon\left(a_2-a_3\right)(g\circ\nabla u)(t) + \varepsilon\left(\frac{2b}{p}a_3 - C_1\right)\|u\|_p^p \\ & +\varepsilon\left(p-2a_3\right)H(t). \end{aligned} \tag{3.23}$$

Now, once δ is fixed, we can choose ε small enough such that

$$1-\varepsilon\left(\frac{m-1}{m}\right)\delta^{\left(\frac{-m}{m-1}\right)} > 0, \text{ and } L(0) > 0. \tag{3.24}$$

Therefore, 3.23 takes the form

$$L'(t) \geq \varepsilon\theta\left\{H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u(t)) + \|u\|_p^p\right\}, \tag{3.25}$$

for some $\theta > 0$.

Now, using $(G2)$, Young's and Poincaré's inequalities in 3.8 to get

$$L(t) \leq \theta_1\left\{H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2\right\}, \tag{3.26}$$

for some $\theta_1 > 0$. Since, $H(t) > 0$, we have from 3.1

$$-\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{2}\left(g\circ\nabla u\right)(t) + \frac{b}{p}\|u\|_p^p > 0, \forall t \geq 0. \tag{3.27}$$

Then,

$$\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 < \frac{b}{p}\|u\|_p^p < \frac{b}{p}\|u\|_p^p + \left(g\circ\nabla u\right)(t). \tag{3.28}$$

In the other hand, using $(G1)$, to get

$$\frac{1}{2}\left(1-l\right)\|\nabla u\|_2^2 \leq \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 < \frac{b}{p}\|u\|_p^p + \left(g\circ\nabla u\right)(t). \tag{3.29}$$

Consequently,

$$\|\nabla u\|_2^2 < \frac{2b}{p}\|u\|_p^p + 2\left(g\circ\nabla u\right)(t) + 2l\|\nabla u\|_2^2, b, l > 0. \tag{3.30}$$

Inserting 3.30 into 3.26, to see that there exists a positive constant λ such that

$$L(t) \leq \lambda\left\{H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u)(t) + \frac{b}{p}\|u\|_p^p\right\}, \forall t \geq 0. \tag{3.31}$$

From inequalities 3.25 and 3.31 we obtain the differential inequality

$$\frac{L'(t)}{L(t)} \geq \mu, \text{ for some}, \mu > 0, \forall t \geq 0. \tag{3.32}$$

Integration of 3.32, between 0 and t gives us

$$L(t) \geq L(0)\exp(\mu t), \forall t \geq 0, \tag{3.33}$$

From 3.8 and for ε small enough, we have

$$L(t) \leq H(t) \leq \frac{b}{p}\|u\|_p^p. \tag{3.34}$$

By 3.33 and 3.34, we have

$$\|u\|_p^p \geq C\exp(\mu t), C > 0, \forall t \geq 0. \tag{3.35}$$

Therefore, we conclude that the solution in the L^p-norm growths exponentially. □

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