Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 6 (2013), 263–278
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

On double fuzzy preuniformity

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Communicated by Salvador Romaguera Bonilla

Abstract

In this study, we introduce the notion of double fuzzy uniform space as a view point of the entourage approach in a strictly two-sided commutative quantale based on powersets of the form $L^{X\times X}$. We investigate the relations between double fuzzy preuniformity, double fuzzy topology, double fuzzy interior operator, and double fuzzy preproximity.

Keywords: Double fuzzy topology; double fuzzy interior operator; proximity; uniformity.
2010 MSC: Primary 54A05, 54A40, 54E15

1. Introduction

Since Chang [3] and Goguen [11] introduced the fuzzy theory into topology, many authors have discussed various aspects of fuzzy topology[1, 12]. In a Chang-Goguen topology (L-topology), open sets were fuzzy, but the topology comprising those open sets was a crisp subset of L^X. Fuzzification of openness was initiated by Höhle [13] in 1980 and later developed to L-subset of L^X (namely, L-fuzzy topology) independently by Kubiak [22] and Šostak [30] in 1985.

In 1983 Atanassov introduced the concept of intuitionistic fuzzy set [2]. Using this type of generalized fuzzy set, Çoker [6] and Çoker and Demirci [7] defined the notion of intuitionistic fuzzy topological space. Samanta and Mondal [27, 28], introduced the notion of intuitionistic gradation of openness as a generalization of intuitionistic fuzzy topological spaces [7] and L-fuzzy topological spaces.

Working under the name "intuitionistic" did not continue because doubts were thrown about the suitability of this term, especially when working in the case of complete lattice L. These doubts were quickly ended in 2005 by Gutierrez Garcia and Rodabaugh [8]. They proved that this term is unsuitable in mathematics and applications. They concluded that they work under the name "double".

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Received 2012-03-16

It is well-known that (quasi-) uniformity is a very important concept close to topology and a convenient tool for investigating topology. Uniformities in fuzzy sets, different approaches as follows the entourage approach of Lowen [24] and Höhle [15, 16] based on powersets of the form $L^{X\times X}$, the uniform covering of Kotze [21], the uniform operator approach of Rodabaugh [26] as a generalization of Hutton [17] based on the powersets of the form $(L^X)^{L^X}$, the unification approach of Gutierrez Garcia et al. [9, 10]. Recently, Gutierrez Garcia et al.[10] introduced L-valued Hutton uniformity where a quadruple $(L, \leq, \otimes, *)$ is defined by a GL-monoid $(L, *)$ dominated by $\otimes$, a cl-monoid $(L, \leq, \otimes)$ as an extension of a completely distributive lattice [17, 19, 21, 23] or the unit interval [25, 29] or t-norms [15].

In this paper, we introduce the notion of lattice valued double fuzzy uniform spaces as a view point of the entourage approach of Lowen [24] and Höhle [15, 16] in a strictly two-sided commutative quantale (stsc-quantale, for short) based on powersets of the form $L^{X\times X}$. We investigate the relations between double fuzzy uniformity, double fuzzy topology, double fuzzy interior operator and double fuzzy preproximity.

2. Preliminaries

Throughout this paper, let X be a nonempty set and $L = (L, \leq, \vee, \wedge, 0_L, 1_L)$ be a completely distributive lattice with the bottom element 0_L and the top element 1_L. For each $\alpha \in L$, let $\underline{\alpha}$ and $\widetilde{\alpha}$ denote the constant fuzzy subsets of X and $X \times X$ with value α, respectively. The second lattice belonging to the context of our work is denoted by M and $M_0 = M \setminus \{0_M\}$ and $M_1 = M \setminus \{1_M\}$.

Definition 2.1. [14, 16, 26] A triple $L = (L, \leq, \odot)$ is called a strictly two-sided, commutative quantale (stsc-quantale, for short) iff it satisfies the following properties:

(L1) $(L, \odot)$ is a commutative semigroup.

(L2) $a \odot 1_L = a$ and $a \odot 0_L = 0_L$, for all $a \in L$.

(L3) $\odot$ is distributive over arbitrary joins:

$$a \odot (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \odot b_i), \forall a \in L, \forall \{b_i\}_{i\in I} \subseteq L.$$

Remark [14, 16, 26](1) A complete lattice satisfying the infinite distributive law is a stsc-quantale. In particular, the unit interval $([0, 1], \leq, \wedge, 0, 1)$ is a stsc-quantale.

(2) Every left-continuous t-norm T on $([0, 1], \leq, t)$ with $\odot = t$ is a stsc-quantale.

(3) Every GL-monoid is a stsc-quantale.

(4) Let $(L, \leq, \odot)$ be a stsc-quantale. For each $x, y \in L$, we define

$$x \longmapsto y = \bigvee\{z \in L | x \odot z \leq y\}.$$

Then it satisfies Galois correspondence;i.e.

$$x \odot z \leq y \Longleftrightarrow z \leq x \longmapsto y, \forall x, y, z \in L.$$

In this paper, we always assume that $(L, \leq, \odot, \oplus, \star)$ is a stsc-quantale with an order-reversing involution $\star$ defined by

$$x \oplus y = (x^\star \odot y^\star)^\star, x^\star = x \mapsto 0_L$$

unless otherwise specified.

Lemma 2.2. *[20] For each $x, y, z, w, x_i, y_i \in L$, we have the following properties:*

(1) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z, x \mapsto y \leq x \mapsto z$ and $y \mapsto x \geq z \mapsto x$.

(2) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.

(3) $x \oplus (\bigwedge_{i\in I} y_i) = \bigwedge_{i\in I}(x \oplus y_i)$.

(4) $x \mapsto (\bigwedge_{i\in I} y_i) = \bigwedge_{i\in I}(x \mapsto y_i)$.

(5) $(\bigvee_{i\in I} x_i) \mapsto y = \bigwedge_{i\in I}(x_i \mapsto y)$.
(6) $x \mapsto (\bigvee_{i\in I} y_i) \geq \bigvee_{i\in I}(x \mapsto y_i)$.
(7) $(\bigwedge_{i\in I} x_i) \mapsto y \geq \bigvee_{i\in I}(x_i \mapsto y)$.
(8) $\bigwedge_{i\in I} x_i^\star = (\bigvee_{i\in I} x_i)^\star$ *and* $\bigvee_{i\in I} x_i^\star = (\bigwedge_{i\in I} x_i)^\star$.
(9) $(x \odot y) \mapsto z = x \mapsto (y \mapsto z) = y \mapsto (x \mapsto z)$.
(10) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w) \leq (x \odot z) \vee (y \odot w)$.
(11) $x \odot (x \mapsto y) \leq y$ *and* $x \mapsto y \leq (y \mapsto z) \mapsto (x \mapsto z)$.
(12) $y \odot z \leq x \mapsto (x \odot y \odot z)$ *and* $x \odot (x \odot y \mapsto z) \leq y \mapsto z$.
(13) $x \mapsto y = y^\star \mapsto x^\star$.
(14) $x \odot (x^\star \oplus y^\star) \leq y^\star$.
(15) $x \odot y = (x \mapsto y^\star)^\star, x \oplus y = x^\star \mapsto y$.
(16) $(x \oplus z) \odot y \leq x \oplus (y \odot z)$.
(17) $x \odot y \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$.
(18) $x \mapsto (y \oplus z) \leq (x \mapsto y)^\star \mapsto z$.
(19) $(x \mapsto y) \oplus (z \mapsto w) \leq (x \odot z) \mapsto (y \oplus w)$.
(20) $(x \mapsto y) \odot (z \mapsto w) \leq (x \oplus z) \mapsto (y \oplus w)$.
(21) $(x \mapsto y) \odot (z \mapsto w) \leq (x \odot z) \mapsto (y \odot w)$.
(22) $(x \mapsto y) \vee (z \mapsto w) \leq (x \wedge z) \mapsto (y \vee w) \leq (x \wedge z) \mapsto (y \oplus w)$.
(23) $(x \mapsto y) \vee (z \mapsto w) \leq (x \odot z) \mapsto (y \vee w)$.
(24) $(x \mapsto y) \wedge (z \mapsto w) \leq (x \vee z) \mapsto (y \vee w) \leq (x \vee z) \mapsto (y \oplus w)$.
(25) $(x \mapsto y) \wedge (z \mapsto w) \leq (x \wedge z) \mapsto (y \wedge w)$.
(26) $x \mapsto y \leq (x \odot z) \mapsto (y \odot z)$ *and* $(x \mapsto y) \odot (y \mapsto z) \leq x \mapsto z$.

All algebraic operations on L can be extended to the set L^X (resp., $L^{X\times X}$) by pointwisely, for all $x \in X$, (resp., for all $(x, y) \in X \times X$)
(1) $\lambda \leq \mu$ iff $\lambda(x) \leq \mu(x)$ (resp., $u \leq v$ iff $u(x, y) \leq v(x, y)$)
(2) $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x)$ (resp., $(u \odot v)(x, y) = u(x, y) \odot v(x, y)$)
(3) $(\lambda \mapsto \mu)(x) = \lambda(x) \mapsto \mu(x)$ (resp., $(u \mapsto v)(x, y) = u(x, y) \mapsto v(x, y)$)

Definition 2.3. [4] The pair (τ, τ^*) of maps $\tau, \tau^* : L^X \longrightarrow M$ is called a double fuzzy topology on X if it satisfies the following conditions:
(O1) $\tau(\lambda) \leq (\tau^*(\lambda) \mapsto 0_M)$, $\forall \lambda \in L^X$.
(O2) $\tau(\underline{0}) = \tau(\underline{1}) = 1_M$, $\tau^*(\underline{0}) = \tau^*(\underline{1}) = 0_M$.
(O3) $\tau(\lambda_1 \odot \lambda_2) \geq \tau(\lambda_1) \odot \tau(\lambda_2)$ and $\tau^*(\lambda_1 \odot \lambda_2) \leq \tau^*(\lambda_1) \oplus \tau^*(\lambda_2)$, for each $\lambda_1, \lambda_2 \in L^X$.
(O4) $\tau(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \tau(\lambda_i)$ and $\tau^*(\bigvee_{i\in\Gamma} \lambda_i) \leq \bigvee_{i\in\Gamma} \tau^*(\lambda_i)$, for each $\lambda_i \in L^X, i \in \Gamma$.
The triplet (X, τ, τ^*) is called a double fuzzy topological space. τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Let (X, τ_1, τ_1^*) and (Y, τ_2, τ_2^*) be two double fuzzy topological spaces. A map $\varphi : X \longrightarrow Y$ is called LF-continuous if

$$\tau_1(\varphi^{\leftarrow}(\mu)) \geq \tau_2(\mu) \text{ and } \tau_1^*(\varphi^{\leftarrow}(\mu)) \leq \tau_2^*(\mu), \forall \mu \in L^Y.$$

Definition 2.4. [4] A map $\mathcal{I} : L^X \times M_0 \times M_1 \longrightarrow L^X$ is called a double fuzzy interior operator if it satisfies the following conditions: $\forall r \in M_0, s \in M_1$ such that $r \leq (s \longmapsto 0_M)$,
(I1) $\mathcal{I}(\underline{1}, r, s) = \underline{1}$.
(I2) $\mathcal{I}(\lambda, r, s) \leq \lambda$.
(I3) If $\lambda \leq \mu$, then $\mathcal{I}(\lambda, r, s) \leq \mathcal{I}(\mu, r, s)$.
(I4) If $r \leq r'$ and $s \geq s'$, then $\mathcal{I}(\lambda, r', s') \leq \mathcal{I}(\lambda, r, s)$.
(I5) $\mathcal{I}(\lambda \odot \mu, r \odot r', s \oplus s') \geq \mathcal{I}(\lambda, r, s) \odot \mathcal{I}(\mu, r', s')$.
The pair $(X, \mathcal{I})$ is called a double fuzzy interior space.
A double fuzzy interior space $(X, \mathcal{I})$ is called topological if

$$\mathcal{I}(\mathcal{I}(\lambda, r, s), r, s) = \mathcal{I}(\lambda, r, s), \forall \lambda \in L^X, r \in M_0, s \in M_1 \text{ with } r \leq (s \longmapsto 0_M).$$

Let $(X, \mathcal{I}_1)$ and $(Y, \mathcal{I}_2)$ be two double fuzzy interior spaces. A map $\varphi : X \longrightarrow Y$ is called I-map iff

$$\varphi^{\leftarrow}(\mathcal{I}_2(\mu, r, s)) \leq \mathcal{I}_1(\varphi^{\leftarrow}(\mu), r, s), \forall \mu \in L^Y, r \in M_0 \text{ and } s \in M_1.$$

Theorem 2.5. *[4] Let (X, τ, τ^*) be a double fuzzy topological space. For each $\lambda \in L^X, r \in M_0$ and $s \in M_1$ with $r \leq (s \mapsto 0_M)$, we define an operator $\mathcal{I}_{\tau,\tau^*} : L^X \times M_0 \times M_1 \longrightarrow L^X$ as follows:*

$$\mathcal{I}_{\tau,\tau^*}(\lambda, r, s) = \bigvee \{\mu \in L^X \mid \mu \leq \lambda, \tau(\mu) \geq r \text{ and } \tau^*(\mu) \leq s\}.$$

Then $(X, \mathcal{I}_{\tau,\tau^*})$ is a topological double fuzzy interior space and if $r = \bigvee\{r' \in M_0 \mid \mathcal{I}(\lambda, r', s') = \lambda\}$ and $s = \bigwedge\{s' \in M_1 \mid \mathcal{I}(\lambda, r', s') = \lambda\}$, then $\mathcal{I}(\lambda, r, s) = \lambda$.

Theorem 2.6. *[4] Let $(X, \mathcal{I})$ be a double fuzzy interior space. Define the mappings $\tau_{\mathcal{I}}, \tau^*_{\mathcal{I}} : L^X \longrightarrow M$ by*

$$\tau_{\mathcal{I}}(\lambda) = \bigvee \{r \in M_0 \mid \mathcal{I}(\lambda, r, s) = \lambda\}$$

$$\tau^*_{\mathcal{I}}(\lambda) = \bigwedge \{s \in M_1 \mid \mathcal{I}(\lambda, r, s) = \lambda\}$$

Then the pair $(\tau_{\mathcal{I}}, \tau^*_{\mathcal{I}})$ is a double fuzzy topology on X.

Definition 2.7. [5] The pair (δ, δ^*) of maps $\delta, \delta^* : L^X \times L^X \longrightarrow M$ is called a double fuzzy preproximity on X if it satisfies the following conditions:

(P1) $\delta(\lambda, \mu) \geq \delta^*(\lambda, \mu) \longmapsto 0_M$.
(P2) $\delta(\underline{1}, \underline{0}) = \delta(\underline{0}, \underline{1}) = 0_M$ and $\delta^*(\underline{0}, \underline{1}) = \delta^*(\underline{1}, \underline{0}) = 1_M$.
(P3) If $\delta(\lambda, \mu) \neq 1_M$ and $\delta^*(\lambda, \mu) \neq 0_M$, then $\lambda \leq \mu \longmapsto \underline{0}$.
(P4) If $\lambda_1 \leq \lambda_2$, then $\delta(\lambda_1, \mu) \leq \delta(\lambda_2, \mu)$ and $\delta^*(\lambda_1, \mu) \geq \delta^*(\lambda_2, \mu)$.
(P5) $\delta(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \leq \delta(\lambda_1, \rho_1) \oplus \delta(\lambda_2, \rho_2)$ and
$\delta^*(\lambda_1 \odot \lambda_2, \rho_1 \oplus \rho_2) \geq \delta^*(\lambda_1, \rho_1) \odot \delta^*(\lambda_2, \rho_2)$.

The triplet (X, δ, δ^*) is called a double fuzzy preproximity space. Also, we call $\delta(\lambda, \mu)$ a gradation of nearness and $\delta^*(\lambda, \mu)$ a gradation of non-nearness between λ and μ. A double fuzzy preproximity (δ, δ^*) is called a double fuzzy quasi-proximity if

(P6) $\delta(\lambda, \mu) \geq \bigwedge_{\nu \in L^X} \{\delta(\lambda, \nu) \oplus \delta(\nu \mapsto \underline{0}, \mu)\}$ and
$\delta^*(\lambda, \mu) \leq \bigvee_{\nu \in L^X} \{\delta^*(\lambda, \nu) \odot \delta^*(\nu \mapsto \underline{0}, \mu)\}$.

A double fuzzy preproximity space is called principal provided that

(P7) $\delta(\bigvee_{i \in \Gamma} \lambda_i, \mu) \leq \bigvee_{i \in \Gamma} \delta(\lambda_i, \mu)$ and $\delta^*(\bigvee_{i \in \Gamma} \lambda_i, \mu) \geq \bigwedge_{i \in \Gamma} \delta^*(\lambda_i, \mu)$.

A double fuzzy quasi-proximity is called double fuzzy proximity if

(P8) $\delta(\lambda, \mu) = \delta(\mu, \lambda)$ and $\delta^*(\lambda, \mu) = \delta^*(\mu, \lambda)$.

Definition 2.8. [5] Let $(X, \delta_1, \delta_1^*)$ and $(Y, \delta_2, \delta_2^*)$ be two double fuzzy preproximity spaces. A map $\varphi : (X, \delta_1, \delta_1^*) \longrightarrow (Y, \delta_2, \delta_2^*)$ is called double fuzzy preproximally continuous if

$$\delta_1(\lambda, \mu) \leq \delta_2(\varphi^{\rightarrow}(\lambda), \varphi^{\rightarrow}(\mu))$$

and

$$\delta_1^*(\lambda, \mu) \geq \delta_2^*(\varphi^{\rightarrow}(\lambda), \varphi^{\rightarrow}(\mu)), \forall \lambda, \mu \in L^X.$$

or equivalently

$$\delta_2(\nu, \rho) \geq \delta_1(\varphi^{\leftarrow}(\nu), \varphi^{\leftarrow}(\rho))$$

and

$$\delta_2^*(\nu, \rho) \leq \delta_1^*(\varphi^{\leftarrow}(\nu), \varphi^{\leftarrow}(\rho)), \forall \nu, \rho \in L^Y.$$

Theorem 2.9. *[5] Let (X, δ, δ^*) be a double fuzzy preproximity space. Define a map $\mathcal{I}_{\delta,\delta^*} : L^X \times M_0 \times M_1 \longrightarrow L^X$ by*

$$\mathcal{I}_{\delta,\delta^*}(\lambda, r, s) = \bigvee \{\rho \in L^X \mid \delta(\rho, \lambda \mapsto \underline{0}) < r \mapsto 0_M \text{ and } \delta^*(\rho, \lambda \mapsto \underline{0}) > s \mapsto 0_M\}$$

Then it satisfies the following properties:

(1) The pair $(X, \mathcal{I}_{\delta,\delta^*})$ is a double fuzzy interior space.

(2) If (X, δ, δ^*) is a double fuzzy quasi-proximity space and L is a chain, then $(X, \mathcal{I}_{\delta,\delta^*})$ is topological.

3. Lattice valued double fuzzy uniformity

Definition 3.1. The pair $(\mathcal{U},\mathcal{U}^*)$ of maps $\mathcal{U},\mathcal{U}^* : L^{X\times X} \longrightarrow M$ is called a double fuzzy preuniformity on X if it satisfies the following conditions:

(U1) $\mathcal{U}(u) \leq \mathcal{U}^*(u) \longmapsto 0_M, \forall u \in L^{X\times X}$.

(U2) $\mathcal{U}(\widetilde{1}) = 1_M$ and $\mathcal{U}^*(\widetilde{1}) = 0_M$.

(U3) If $u \leq v$, then $\mathcal{U}(u) \leq \mathcal{U}(v)$ and $\mathcal{U}^*(u) \geq \mathcal{U}^*(v)$.

(U4) $\mathcal{U}(u \odot v) \geq \mathcal{U}(u) \odot \mathcal{U}(v)$ and $\mathcal{U}^*(u \odot v) \leq \mathcal{U}^*(u) \oplus \mathcal{U}^*(v), \forall u, v \in L^{X\times X}$.

(U5) If $\mathcal{U}(u) \neq 0_M$ and $\mathcal{U}^*(u) \neq 1_M$, then $1_\Delta \leq u$, where

$$1_\Delta(x, y) = \begin{cases} 1_L, & \text{if } x = y, \\ 0_L, & \text{if } x \neq y. \end{cases}$$

The preuniformity $(\mathcal{U},\mathcal{U}^*)$ is called quasi-uniformity if

(QU) $\mathcal{U}(u) \leq \bigvee\{\mathcal{U}(v) \mid v \circ v \leq u\}$ and $\mathcal{U}^*(u) \geq \bigwedge\{\mathcal{U}^*(v) \mid v \circ v \leq u\}, \forall u \in L^{X\times X}$

$$\text{where, } \ v \circ v(x, y) = \bigvee_{z\in X} (v(x, z) \odot v(z, y)), \forall x, y \in X.$$

A quasi-uniformity $(\mathcal{U},\mathcal{U}^*)$ is called uniformity if

(U) $\mathcal{U}(u) \leq \mathcal{U}(u^s)$ and $\mathcal{U}^*(u) \geq \mathcal{U}^*(u^s)$, where $u^s(x, y) = u(y, x)$.

The triplet $(X,\mathcal{U},\mathcal{U}^*)$ is called double fuzzy uniform space.

Remark. Let $(\mathcal{U},\mathcal{U}^*)$ be a double fuzzy quasi-uniformity on X.

(1) Since $u \wedge v \geq u \odot v$, by (U3) and (U4), $\mathcal{U}(u \wedge v) \geq \mathcal{U}(u) \odot \mathcal{U}(v)$ and $\mathcal{U}^*(u \wedge v) \leq \mathcal{U}^*(u) \oplus \mathcal{U}^*(v)$.

(2) Define $\mathcal{U}^s(u) = \mathcal{U}(u^s)$ and $(\mathcal{U}^*)^s(u) = \mathcal{U}^*(u^s)$ for all $u \in L^{X\times X}$. Then $(\mathcal{U}^s, (\mathcal{U}^*)^s)$ is a double fuzzy quasi-uniformity on X.

(3) Let $(\mathcal{U},\mathcal{U}^*)$ be a double fuzzy uniformity on X. Since $\mathcal{U}(u) \leq \mathcal{U}(u^s) \leq \mathcal{U}((u^s)^s) = \mathcal{U}(u)$ and $\mathcal{U}^*(u) \geq \mathcal{U}^*(u^s) \geq \mathcal{U}^*((u^s)^s) = \mathcal{U}^*(u)$, we have $\mathcal{U}(u) = \mathcal{U}(u^s)$ and $\mathcal{U}^*(u) = \mathcal{U}^*(u^s)$, for all $u \in L^{X\times X}$.

Example. Let $X = \{x, y, z\}$ be a set and $L = M = [0, 1]$. Define binary operations $\odot, \oplus, \mapsto$ on $[0, 1]$ (where the operation $\odot$ is called a Lukasiewicz t-norm and the operation $\oplus$ is called a Lukasiewicz t-conorm) by $x \odot y = max\{x + y - 1, 0\}, x \oplus y = min\{x + y, 1\}, x \mapsto y = min\{1 - x + y, 1\}$.

Define $w, v \in [0, 1]^{X\times X}$ as follows:

$w(x, x) = w(y, y) = w(z, z) = 1, w(x, y) = 0.5, w(y, z) = 0.6, w(x, z) = 0.6,$

$w(y, x) = 0.7, w(z, x) = 0.6, w(z, y) = 0.8, v(x, x) = v(y, y) = v(z, z) = 1,$

$v(x, y) = 0.5, v(y, z) = v(x, z) = v(y, x) = v(z, x) = 0.6, v(z, y) = 0.4.$

Define maps $\mathcal{U}_1,\mathcal{U}_1^*,\mathcal{U}_2,\mathcal{U}_2^* : [0, 1]^{X\times X} \longrightarrow [0, 1]$ as follows:

$$\mathcal{U}_1(u) = \begin{cases} 1_M, & \text{if } u = \widetilde{1} \\ 0.6, & \text{if } \widetilde{1} \neq u \geq w \\ 0.3, & \text{if } w \odot w \leq u, u \ngeq w \\ 0_M, & \text{otherwise,} \end{cases} \quad \mathcal{U}_1^*(u) = \begin{cases} 0_M, & \text{if } u = \widetilde{1} \\ 0.4, & \text{if } \widetilde{1} \neq u \geq w \\ 0.7, & \text{if } w \odot w \leq u, u \ngeq w \\ 1_M, & \text{otherwise.} \end{cases}$$

$$\mathcal{U}_2(u) = \begin{cases} 1_M, & \text{if } u = \widetilde{1} \\ 0.6, & \text{if } \widetilde{1} \neq u \geq v \\ 0.5, & \text{if } v \odot v \leq u, u \ngeq v \\ 0_M, & \text{otherwise,} \end{cases} \quad \mathcal{U}_2^*(u) = \begin{cases} 0_M, & \text{if } u = \widetilde{1} \\ 0.4, & \text{if } \widetilde{1} \neq u \geq v \\ 0.5, & \text{if } v \odot v \leq u, u \ngeq v \\ 1_M, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{U}_1,\mathcal{U}_1^*)$ is a double fuzzy quasi-uniformity on X, but $(\mathcal{U}_2,\mathcal{U}_2^*)$ is not a double fuzzy quasi-uniformity on X because

$0_M = \mathcal{U}_2(v \odot v \odot v) \ngeq \mathcal{U}_2(v \odot v) \odot \mathcal{U}_2(v) = 0.5 \odot 0.6 = 0.1$

$1_M = \mathcal{U}_2^*(v \odot v \odot v) \nleq \mathcal{U}_2^*(v \odot v) \oplus \mathcal{U}_2(v) = 0.4 \oplus 0.5 = 0.9.$

Definition 3.2. The pair $(\mathcal{B}, \mathcal{B}^*)$ of maps $\mathcal{B}, \mathcal{B}^* : L^{X\times X} \longrightarrow M$ is called a double fuzzy uniform base on X if it satisfies the following conditions:

(B1) $\mathcal{B}(u) \leq \mathcal{B}^*(u) \longmapsto 0_M, \forall u \in L^{X\times X}$.

(B2) $\mathcal{B}(\widetilde{1}) = 1_M$ and $\mathcal{B}^*(\widetilde{1}) = 0_M$.

(B3) $\mathcal{B}(u \odot v) \geq \mathcal{B}(u) \odot \mathcal{B}(v)$ and $\mathcal{B}^*(u \odot v) \leq \mathcal{B}^*(u) \oplus \mathcal{B}^*(v)$.

(B4) If $\mathcal{B}(u) \neq 0_M$ and $\mathcal{B}^*(u) \neq 1_M$, then $1_\Delta \leq u$.

(B5) $\mathcal{B}(u) \leq \bigvee\{\mathcal{B}(v) \mid v \circ v \leq u\}$ and $\mathcal{B}^*(u) \geq \bigwedge\{\mathcal{B}^*(v) \mid v \circ v \leq u\}, \forall u \in L^{X\times X}$.

(B6) $\mathcal{B}(u) \leq \bigvee\{\mathcal{B}(v) \mid v \leq u^s\}$ and $\mathcal{B}^*(u) \geq \bigwedge\{\mathcal{B}^*(v) \mid v \leq u^s\}, \forall u \in L^{X\times X}$.

Trivially, every uniformity is a uniform base.

Theorem 3.3. *Let $(\mathcal{B}, \mathcal{B}^*)$ be a double fuzzy uniform base on X. Define the maps $\mathcal{U}_\mathcal{B}, \mathcal{U}^*_{\mathcal{B}^*} : L^{X\times X} \longrightarrow M$ as*

$$\mathcal{U}_\mathcal{B}(u) = \bigvee_{v\leq u} \mathcal{B}(v) \qquad \text{and} \qquad \mathcal{U}^*_{\mathcal{B}^*}(u) = \bigwedge_{v\leq u} \mathcal{B}^*(v)$$

Then the pair $(\mathcal{U}_\mathcal{B}, \mathcal{U}^*_{\mathcal{B}^*})$ is a double fuzzy uniformity on X.

Proof. (U1)-(U3) are trivial from (B1)-(B3).

(U4) is obtained from the following inequalities.

$$\begin{aligned}
\mathcal{U}_\mathcal{B}(u_1) \odot \mathcal{U}_\mathcal{B}(u_2) &= (\bigvee_{v_1\leq u_1} \mathcal{B}(v_1)) \odot (\bigvee_{v_2\leq u_2} \mathcal{B}(v_2)) \\
&= \bigvee\{\mathcal{B}(v_1) \odot \mathcal{B}(v_2) \mid v_1 \leq u_1, v_2 \leq u_2\} \\
&\leq \bigvee\{\mathcal{B}(v_1 \odot v_2) \mid v_1 \odot v_2 \leq u_1 \odot u_2\} \\
&\leq \bigvee\{\mathcal{B}(v) \mid v \leq u_1 \odot u_2\} = \mathcal{U}_\mathcal{B}(u_1 \odot u_2).
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}^*_{\mathcal{B}^*}(u_1) \oplus \mathcal{U}^*_{\mathcal{B}^*}(u_2) &= (\bigwedge_{v_1\leq u_1} \mathcal{B}^*(v_1)) \oplus (\bigwedge_{v_2\leq u_2} \mathcal{B}^*(v_2)) \\
&= \bigwedge\{\mathcal{B}^*(v_1) \oplus \mathcal{B}^*(v_2) \mid v_1 \leq u_1, v_2 \leq u_2\} \\
&\geq \bigwedge\{\mathcal{B}^*(v_1 \odot v_2) \mid v_1 \odot v_2 \leq u_1 \odot u_2\} \\
&\geq \bigwedge\{\mathcal{B}^*(v) \mid v \leq u_1 \odot u_2\} = \mathcal{U}^*_{\mathcal{B}^*}(u_1 \odot u_2).
\end{aligned}$$

(U5) Let $\mathcal{U}_\mathcal{B}(u) \neq 0_M$ and $\mathcal{U}^*_{\mathcal{B}^*}(u) \neq 1_M$, then there exists $v \leq u$ such that $\mathcal{U}_\mathcal{B}(u) \geq \mathcal{B}(v) \neq 0_M$ and $\mathcal{U}^*_{\mathcal{B}^*}(u) \leq \mathcal{B}^*(v) \neq 1_M$ and by (B4), $1_\Delta \leq v \leq u$.

(QU) Since $\bigvee\{\mathcal{U}_\mathcal{B}(v) \mid v \circ v \leq u\} \geq \bigvee\{\mathcal{B}(v) \mid v \circ v \leq u\} \geq \mathcal{B}(v)$ and $\bigwedge\{\mathcal{U}^*_{\mathcal{B}^*}(v) \mid v \circ v \leq u\} \leq \bigwedge\{\mathcal{B}^*(v) \mid v \circ v \leq u\} \leq \mathcal{B}^*(v)$, then we have

$$\begin{aligned}
\mathcal{U}_\mathcal{B}(u) &= \bigvee\{\mathcal{B}(w) \mid w \leq u\} \\
&\leq \bigvee\{\bigvee\{\mathcal{U}_\mathcal{B}(v) \mid v \circ v \leq w\} \mid w \leq u\} \leq \bigvee\{\mathcal{U}_\mathcal{B}(v) \mid v \circ v \leq u\}.
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{U}^*_{\mathcal{B}^*}(u) &= \bigwedge\{\mathcal{B}(w) \mid w \leq u\} \\
&\geq \bigwedge\{\bigwedge\{\mathcal{U}^*_{\mathcal{B}^*}(v) \mid v \circ v \leq w\} \mid w \leq u\} \geq \bigwedge\{\mathcal{U}^*_{\mathcal{B}^*}(v) \mid v \circ v \leq u\}.
\end{aligned}$$

(U) is obtained from the following inequalities.

$$\begin{aligned}\mathcal{U}_{\mathcal{B}}(u) &= \bigvee\{\mathcal{B}(v) \mid v \leq u\}\\ &\leq \bigvee\{\bigvee\{\mathcal{B}(w) \mid w \leq v^s\} \mid v \leq u\} \leq \bigvee\{\mathcal{B}(w) \mid w \leq u^s\} = \mathcal{U}_{\mathcal{B}}(u^s).\end{aligned}$$

and

$$\begin{aligned}\mathcal{U}^*_{\mathcal{B}^*}(u) &= \bigwedge\{\mathcal{B}^*(v) \mid v \leq u\}\\ &\geq \bigwedge\{\bigwedge\{\mathcal{B}^*(w) \mid w \leq v^s\} \mid v \leq u\} \geq \bigwedge\{\mathcal{B}^*(w) \mid w \leq u^s\} = \mathcal{U}^*_{\mathcal{B}^*}(u^s).\end{aligned}$$

□

Lemma 3.4. *[18] Let $(\mathcal{U},\mathcal{U}^*)$ be a double fuzzy uniformity on X. For each $u \in L^{X\times X}$ and $\lambda \in L^X$, the image $u[\lambda]$ of λ with respect to u is the L-subset of X defined by:*

$$u[\lambda](x) = \bigvee_{y\in X} (\lambda(y) \odot u(y,x)), \forall x \in X.$$

For each $u, u_1, u_2 \in L^{X\times X}$ and $\lambda, \rho, \lambda_i \in L^X$, we have following properties.

(1) If $\mathcal{U}(u) \neq 0_M$ and $\mathcal{U}^*(u) \neq 1_M$, then $\lambda \leq u[\lambda]$.
(2) If $\mathcal{U}(u) \neq 0_M$ and $\mathcal{U}^*(u) \neq 1_M$, then $u \leq u \circ u$.
(3) $(u_1 \circ u_2)[\lambda] = u_1[u_2[\lambda]]$.
(4) $u[\bigvee_i \lambda_i] = \bigvee_i u[\lambda_i]$.
(5) $(u_1 \odot u_2)[\lambda_1 \odot \lambda_2] \leq u_1[\lambda_1] \odot u_2[\lambda_2]$.
(6) $(u_1 \odot u_2)[\lambda_1 \oplus \lambda_2] \leq u_1[\lambda_1] \oplus u_2[\lambda_2]$.
(7) $u[(u^s[\rho]) \mapsto \underline{0}] \leq \rho \mapsto \underline{0}$.
A lattice L is called $s-$compact if $\bigvee_{j\in J} c_j \geq a$ for all $c_j, a \in L$, there exists $j_0 \in J$ such that $c_{j_0} \geq a$.

Theorem 3.5. *Let $(\mathcal{U},\mathcal{U}^*)$ be a double fuzzy preuniformity on X and L be an $s-$compact lattice. Define the maps $\tau_{\mathcal{U}}, \tau^*_{\mathcal{U}^*} : L^X \longrightarrow L$ by:*

$$\tau_{\mathcal{U}}(\lambda) = \bigwedge_{x\in X}\{(\lambda(x) \mapsto 0_L) \vee \bigvee_{u[x]\leq\lambda} \mathcal{U}(u)\}$$

and

$$\tau^*_{\mathcal{U}^*}(\lambda) = \bigvee_{x\in X}\{\lambda(x) \wedge \bigwedge_{u[x]\leq\lambda} \mathcal{U}^*(u)\},$$

where $u[x](y) = u(y,x)$. Then $(\tau_{\mathcal{U}}, \tau^*_{\mathcal{U}^*})$ is a double fuzzy topology on X.

Proof. (O1) Since by (U1), it is trivial that $\tau_{\mathcal{U}}(\lambda) \leq \tau^*_{\mathcal{U}^*}(\lambda) \mapsto 0_L$.

(O2) It is trivial by (U2).

(O3) By Lemma 2.2 (10),we have

$$\begin{aligned}\tau_{\mathcal{U}}(\lambda_1) \odot \tau_{\mathcal{U}}(\lambda_2) &= (\bigwedge_{x\in X}\{(\lambda_1(x) \mapsto 0_L) \vee \bigvee_{u_1[x]\leq\lambda_1} \mathcal{U}(u_1)\}) \odot (\bigwedge_{y\in X}\{(\lambda_2(y) \mapsto 0_L) \vee \bigvee_{u_2[y]\leq\lambda_2} \mathcal{U}(u_2)\})\\ &\leq \bigwedge_{x\in X}(\{(\lambda_1(x) \mapsto 0_L) \vee \bigvee_{u_1[x]\leq\lambda_1} \mathcal{U}(u_1)\} \odot \{(\lambda_2(x) \mapsto 0_L) \vee \bigvee_{u_2[x]\leq\lambda_2} \mathcal{U}(u_2)\})\\ &\leq \bigwedge_{x\in X}\{((\lambda_1(x) \mapsto 0_L) \oplus (\lambda_2(x) \mapsto 0_L)) \vee (\bigvee_{u_1\odot u_2[x]\leq\lambda_1\odot\lambda_2} \mathcal{U}(u_1 \odot u_2))\}\\ &= \bigwedge_{x\in X}\{((\lambda_1 \odot \lambda_2) \mapsto \underline{0})(x) \vee \bigvee_{u_1\odot u_2[x]\leq\lambda_1\odot\lambda_2} \mathcal{U}(u_1 \odot u_2)\} \leq \tau_{\mathcal{U}}(\lambda_1 \odot \lambda_2)\end{aligned}$$

$$\begin{aligned}
\tau^*_{\mathcal{U}^*}(\lambda_1)\oplus\tau^*_{\mathcal{U}^*}(\lambda_2) &= (\bigvee_{x\in X}\{\lambda_1(x)\wedge\bigwedge_{u_1[x]\leq\lambda_1}\mathcal{U}^*(u_1)\})\oplus(\bigvee_{y\in X}\{\lambda_2(y)\wedge\bigwedge_{u_2[y]\leq\lambda_2}\mathcal{U}^*(u_2)\})\\
&\geq \bigvee_{x\in X}(\{\lambda_1(x)\wedge\bigwedge_{u_1[x]\leq\lambda_1}\mathcal{U}^*(u_1)\}\oplus\{\lambda_2(x)\wedge\bigwedge_{u_2[x]\leq\lambda_2}\mathcal{U}^*(u_2)\})\\
&\geq \bigvee_{x\in X}\{(\lambda_1\odot\lambda_2)(x)\wedge(\bigwedge_{u_1[x]\leq\lambda_1}\mathcal{U}^*(u_1)\oplus\bigwedge_{u_2[x]\leq\lambda_2}\mathcal{U}^*(u_2))\}\\
&\geq \bigvee_{x\in X}\{(\lambda_1\odot\lambda_2)(x)\wedge(\bigwedge_{u_1\odot u_2[x]\leq\lambda_1\odot\lambda_2}\mathcal{U}^*(u_1\odot u_2))\}\geq\tau^*_{\mathcal{U}^*}(\lambda_1\odot\lambda_2)
\end{aligned}$$

(O4) Since L is completely distributive and $s-$compact, then we have:

$$\begin{aligned}
\tau_{\mathcal{U}}(\bigvee_{i\in\Gamma}\lambda_i) &= \bigwedge_{x\in X}\{((\bigvee_{i\in\Gamma}\lambda_i(x))\mapsto 0_L)\vee\bigvee_{u[x]\leq\bigvee_{i\in\Gamma}\lambda_i}\mathcal{U}(u)\}\\
&= \bigwedge_{x\in X}\{(\bigwedge_{i\in\Gamma}(\lambda_i(x)\mapsto 0_L))\vee\bigvee_{u[x]\leq\bigvee_{i\in\Gamma}\lambda_i}\mathcal{U}(u)\}\\
&= \bigwedge_{x\in X}(\bigwedge_{i\in\Gamma}((\lambda_i(x)\mapsto 0_L)\vee\bigvee_{u[x]\leq\bigvee_{i\in\Gamma}\lambda_i}\mathcal{U}(u)))\\
&= \bigwedge_{i\in\Gamma}(\bigwedge_{x\in X}((\lambda_i(x)\mapsto 0_L)\vee\bigvee_{u[x]\leq\bigvee_{i\in\Gamma}\lambda_i}\mathcal{U}(u)))\\
&\geq \bigwedge_{i\in\Gamma}(\bigwedge_{x\in X}((\lambda_i(x)\mapsto 0_L)\vee\bigvee_{u[x]\leq\lambda_i}\mathcal{U}(u)))\\
&= \bigwedge_{i\in\Gamma}\tau_{\mathcal{U}}(\lambda_i)
\end{aligned}$$

$$\begin{aligned}
\tau^*_{\mathcal{U}^*}(\bigvee_{i\in\Gamma}\lambda_i) &= \bigvee_{x\in X}\{(\bigvee_{i\in\Gamma}\lambda_i(x))\wedge\bigwedge_{u[x]\leq\bigvee_{i\in\Gamma}\lambda_i}\mathcal{U}^*(u)\}\\
&= \bigvee_{i\in\Gamma}(\bigvee_{x\in X}(\lambda_i(x)\wedge\bigwedge_{u[x]\leq\bigvee_{i\in\Gamma}\lambda_i}\mathcal{U}^*(u)))\\
&\leq \bigvee_{i\in\Gamma}(\bigvee_{x\in X}(\lambda_i(x)\wedge\bigwedge_{u[x]\leq\lambda_i}\mathcal{U}^*(u)))\\
&= \bigvee_{i\in\Gamma}\tau^*_{\mathcal{U}^*}(\lambda_i).
\end{aligned}$$

□

Let $\lambda\in L^X$. We define $u_\lambda\in L^{X\times X}$ by

$$u_\lambda(x,y)=\begin{cases}1_L, & \text{if } x=y,\\ \lambda(x)\odot\lambda(y), & \text{if } x\neq y.\end{cases}$$

Theorem 3.6. *Let $(\mathcal{U},\mathcal{U}^*)$ be a double fuzzy uniformity on X. Define the maps $\mathcal{T}_{\mathcal{U}},\mathcal{T}^*_{\mathcal{U}^*}:L^X\longrightarrow M$ by:*

$$\mathcal{T}_{\mathcal{U}}(\lambda)=\begin{cases}1_M, & \text{if } \lambda=\underline{0}\\ \mathcal{U}(u_\lambda), & \text{if } \lambda\neq\underline{0}\end{cases}\qquad \mathcal{T}^*_{\mathcal{U}^*}(\lambda)=\begin{cases}0_M, & \text{if } \lambda=\underline{0}\\ \mathcal{U}^*(u_\lambda), & \text{if } \lambda\neq\underline{0}\end{cases}$$

Then $(\mathcal{T}_{\mathcal{U}},\mathcal{T}^*_{\mathcal{U}^*})$ is a double fuzzy topology on X.

Proof. (O1) It is trivial.

(O2) $\mathcal{T}_{\mathcal{U}}(\underline{0}) = 1_M, \mathcal{T}_{\mathcal{U}}(\underline{1}) = \mathcal{U}(u_{\underline{1}}) = \mathcal{U}(\widetilde{1}) = 1_M$ and $\mathcal{T}^*_{\mathcal{U}^*}(\underline{0}) = 0_M, \mathcal{T}^*_{\mathcal{U}^*}(\underline{1}) = \mathcal{U}^*(u_{\underline{1}}) = \mathcal{U}^*(\widetilde{1}) = 0_M$.

(O3) Let $\lambda_1, \lambda_2 \in L^X$ be given. Since $u_{(\lambda_1 \odot \lambda_2)} = u_{\lambda_1} \odot u_{\lambda_2}$, we have

$$\mathcal{T}_{\mathcal{U}}(\lambda_1 \odot \lambda_2) = \mathcal{U}(u_{(\lambda_1 \odot \lambda_2)}) = \mathcal{U}(u_{\lambda_1} \odot u_{\lambda_2}) \geq \mathcal{U}(u_{\lambda_1}) \odot \mathcal{U}(u_{\lambda_2}) = \mathcal{T}_{\mathcal{U}}(\lambda_1) \odot \mathcal{T}_{\mathcal{U}}(\lambda_2)$$
$$\mathcal{T}^*_{\mathcal{U}^*}(\lambda_1 \odot \lambda_2) = \mathcal{U}^*(u_{\lambda_1} \odot u_{\lambda_2}) \leq \mathcal{U}^*(u_{\lambda_1}) \oplus \mathcal{U}^*(u_{\lambda_2}) = \mathcal{T}^*_{\mathcal{U}^*}(\lambda_1) \oplus \mathcal{T}^*_{\mathcal{U}^*}(\lambda_2)$$

(O4) Since $(\bigvee_j \lambda_j(x)) \odot (\bigvee_j \lambda_j(y)) \geq (\bigvee_j \lambda_j(x) \odot \lambda_j(y))$, we have $u_{\lambda_j} \leq \vee_j u_{\lambda_j} \leq u_{\vee_j \lambda_j}$. So,

$$\mathcal{T}_{\mathcal{U}}(\bigvee_{i \in \Gamma} \lambda_i) = \mathcal{U}(u_{\bigvee_i \lambda_i}) \geq \mathcal{U}(u_{\lambda_i}) = \mathcal{T}_{\mathcal{U}}(\lambda_i), \quad \forall i \in \Gamma.$$

Hence, $\mathcal{T}_{\mathcal{U}}(\bigvee_{i \in \Gamma} \lambda_i) \geq \bigwedge_{i \in \Gamma} \mathcal{T}_{\mathcal{U}}(\lambda_i)$.

$$\mathcal{T}^*_{\mathcal{U}^*}(\bigvee_{i \in \Gamma} \lambda_i) = \mathcal{U}^*(u_{\bigvee_i \lambda_i}) \leq \mathcal{U}^*(u_{\lambda_i}) = \mathcal{T}^*_{\mathcal{U}^*}(\lambda_i), \quad \forall i \in \Gamma.$$

Hence, $\mathcal{T}^*_{\mathcal{U}^*}(\bigvee_{i \in \Gamma} \lambda_i) \leq \bigvee_{i \in \Gamma} \mathcal{T}^*_{\mathcal{U}^*}(\lambda_i)$. □

Theorem 3.7. *Let the pair $(\mathcal{U}, \mathcal{U}^*)$ be a double fuzzy preuniformity on X. Define the maps $\delta_{\mathcal{U}}, \delta^*_{\mathcal{U}^*} : L^X \times L^X \longrightarrow M$ as follows:*

$$\delta_{\mathcal{U}}(\lambda, \mu) = \begin{cases} (\bigvee\{\mathcal{U}(\omega) \mid \omega \in \Theta_{\lambda,\mu}\}) \mapsto 0_M, & \text{if } \Theta_{\lambda,\mu} \neq \emptyset, \\ 1_M, & \text{if } \Theta_{\lambda,\mu} = \emptyset. \end{cases}$$

and

$$\delta^*_{\mathcal{U}^*}(\lambda, \mu) = \begin{cases} (\bigwedge\{\mathcal{U}^*(\omega) \mid \omega \in \Theta_{\lambda,\mu}\}) \mapsto 0_M, & \text{if } \Theta_{\lambda,\mu} \neq \emptyset, \\ 0_M, & \text{if } \Theta_{\lambda,\mu} = \emptyset. \end{cases}$$

where $\Theta_{\lambda,\mu} = \{\omega \in L^{X \times X} \mid \omega[\lambda] \leq \mu \mapsto \underline{0}\}$. Then the pair $(\delta_{\mathcal{U}}, \delta^*_{\mathcal{U}^*})$ is a double fuzzy preproximity on X.

Proof. (P1) Since by (U1), $\mathcal{U}(w) \leq \mathcal{U}^*(w) \mapsto 0_M$, we have $\delta_{\mathcal{U}}(\lambda, \mu) \geq \delta^*_{\mathcal{U}^*}(\lambda, \mu) \mapsto 0_M$.

(P2) By the definitions, $\delta_{\mathcal{U}}(\underline{1}, \underline{0}) = \delta_{\mathcal{U}}(\underline{0}, \underline{1}) = 0_M$ and $\delta^*_{\mathcal{U}^*}(\underline{1}, \underline{0}) = \delta^*_{\mathcal{U}^*}(\underline{0}, \underline{1}) = 1_M$.

(P3) Let $\delta_{\mathcal{U}}(\lambda, \mu) \neq 1_M$ and $\delta^*_{\mathcal{U}^*}(\lambda, \mu) \neq 0_M$, then by the definition there exist $w \in \Theta_{\lambda,\mu}$ such that $\mathcal{U}(w) \neq 0_M$ and $\mathcal{U}^*(w) \neq 1_M$. By Lemma 3.4 (1), $\lambda \leq w[\lambda] \leq \mu \mapsto \underline{0}$.

(P4) Let $\lambda_1, \lambda_2 \in L^X$ be given with $\lambda_1 \leq \lambda_2$. Then, $\Theta(\lambda_2, \mu) \subseteq \Theta(\lambda_1, \mu)$. Thus, $\delta_{\mathcal{U}}(\lambda_1, \mu) \leq \delta_{\mathcal{U}}(\lambda_2, \mu)$ and $\delta^*_{\mathcal{U}^*}(\lambda_1, \mu) \geq \delta^*_{\mathcal{U}^*}(\lambda_2, \mu)$.

(P5) Since by Lemma 3.4 (5), $(w_1 \odot w_2)[\lambda_1 \odot \lambda_2] \leq w_1[\lambda_1] \odot w_2[\lambda_2]$, we have

$$\begin{aligned}
\delta_{\mathcal{U}}(\lambda_1, \mu_1) \oplus \delta_{\mathcal{U}}(\lambda_2, \mu_2) &= [(\bigvee\{\mathcal{U}(w_1) \mid w_1 \in \Theta_{\lambda_1,\mu_1}\}) \mapsto 0_M] \\
&\oplus [(\bigvee\{\mathcal{U}(w_2) \mid w_2 \in \Theta_{\lambda_2,\mu_2}\}) \mapsto 0_M] \\
&= (\bigwedge\{\mathcal{U}(w_1) \mapsto 0_M \mid w_1[\lambda_1] \leq \mu_1 \mapsto \underline{0}\}) \\
&\oplus (\bigwedge\{\mathcal{U}(w_2) \mapsto 0_M \mid w_2[\lambda_2] \leq \mu_2 \mapsto \underline{0}\}) \\
&= \bigwedge\{(\mathcal{U}(w_1) \mapsto 0_M) \oplus (\mathcal{U}(w_2) \mapsto 0_M) \mid w_i[\lambda_i] \leq \mu_i \mapsto \underline{0}, i = 1, 2\} \\
&= \bigwedge\{(\mathcal{U}(w_1) \odot \mathcal{U}(w_2)) \mapsto 0_M) \mid w_i[\lambda_i] \leq \mu_i \mapsto \underline{0}, i = 1, 2\} \\
&\geq \bigwedge\{\mathcal{U}(w_1 \odot w_2) \mapsto 0_M \mid (w_1 \odot w_2)[\lambda_1 \odot \lambda_2] \leq \odot_{i=1,2}(\mu_i \mapsto \underline{0})\} \\
&\geq \bigwedge\{\mathcal{U}(w) \mapsto 0_M \mid w[\lambda_1 \odot \lambda_2] \leq (\mu_1 \oplus \mu_2) \mapsto \underline{0}\} \\
&= (\bigvee\{\mathcal{U}(w) \mid w \in \Theta_{\lambda_1 \odot \lambda_2, \mu_1 \oplus \mu_2}\}) \mapsto 0_M \\
&= \delta_{\mathcal{U}}(\lambda_1 \odot \lambda_2, \mu_1 \oplus \mu_2).
\end{aligned}$$

and

$$\begin{aligned}
\delta^*_{\mathcal{U}^*}(\lambda_1,\mu_1)\odot\delta^*_{\mathcal{U}^*}(\lambda_2,\mu_2) &= [(\bigwedge\{\mathcal{U}^*(w_1)\mid w_1\in\Theta_{\lambda_1,\mu_1}\})\mapsto 0_M]\\
&\odot\ [(\bigwedge\{\mathcal{U}^*(w_2)\mid w_2\in\Theta_{\lambda_2,\mu_2}\})\mapsto 0_M]\\
&= (\bigvee\{\mathcal{U}^*(w_1)\mapsto 0_M\mid w_1[\lambda_1]\le\mu_1\mapsto\underline{0}\})\\
&\odot\ (\bigvee\{\mathcal{U}^*(w_2)\mapsto 0_M\mid w_2[\lambda_2]\le\mu_2\mapsto\underline{0}\})\\
&= \bigvee\{(\mathcal{U}^*(w_1)\mapsto 0_M)\odot(\mathcal{U}^*(w_2)\mapsto 0_M)\mid w_i[\lambda_i]\le\mu_i\mapsto\underline{0}, i=1,2\}\\
&= \bigvee\{(\mathcal{U}^*(w_1)\oplus\mathcal{U}^*(w_2))\mapsto 0_M)\mid w_i[\lambda_i]\le\mu_i\mapsto\underline{0}, i=1,2\}\\
&\le \bigvee\{\mathcal{U}^*(w_1\odot w_2)\mapsto 0_M\mid (w_1\odot w_2)[\lambda_1\odot\lambda_2]\le\odot_{i=1,2}(\mu_i\mapsto\underline{0})\}\\
&\le \bigvee\{\mathcal{U}^*(w)\mapsto 0_M\mid w[\lambda_1\odot\lambda_2]\le(\mu_1\oplus\mu_2)\mapsto\underline{0}\}\\
&= (\bigwedge\{\mathcal{U}^*(w)\mid w\in\Theta_{\lambda_1\odot\lambda_2,\mu_1\oplus\mu_2}\})\mapsto 0_M=\delta^*_{\mathcal{U}^*}(\lambda_1\odot\lambda_2,\mu_1\oplus\mu_2).
\end{aligned}$$

Therefore, the pair $(\delta_{\mathcal{U}},\delta^*_{\mathcal{U}^*})$ is a double fuzzy preproximity on X induced by the preuniformity $(\mathcal{U},\mathcal{U}^*)$. □

Theorem 3.8. *Let $(\mathcal{U},\mathcal{U}^*)$ be a double fuzzy preuniformity on X. Define a map $\mathcal{I}_{\mathcal{U},\mathcal{U}^*}: L^X\times M_0\times M_1\longrightarrow L^X$ by*

$$\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda,r,s)=\bigvee\{\mu\in L^X\mid \bigvee_{u[\mu]\le\lambda}\mathcal{U}(u)\ge r \text{ and } \bigwedge_{u[\mu]\le\lambda}\mathcal{U}^*(u)\le s\}$$

Then we have the following properties:

(1) The map $\mathcal{I}_{\mathcal{U},\mathcal{U}^*}$ is a double fuzzy interior operator on X.

(2) $\mathcal{I}_{\delta_{\mathcal{U}},\delta^*_{\mathcal{U}^*}}=\mathcal{I}_{\mathcal{U},\mathcal{U}^*}$.

Proof. (I1) Since $u[\underline{1}]\le\underline{1}$, then $\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\underline{1},r,s)=\underline{1}$.

(I2) Since $r\in M_0$ and $s\in M_1$, there exists $u\in L^{X\times X}$ such that $\mathcal{U}(u)\ne 0_M$ and $\mathcal{U}^*(u)\ne 1_M$. Then by Lemma 3.4 (1), $\mu\le u[\mu]\le\lambda$. Hence, $\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda,r,s)\le\lambda$.

(I3) and (I4) are trivial from the definition.

(I5) is clear from the following inequality.

$$\begin{aligned}
\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda_1,r_1,s_1)\odot\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda_2,r_2,s_2) &= (\bigvee\{\mu_1\mid \bigvee_{u_1[\mu_1]\le\lambda_1}\mathcal{U}(u_1)\ge r_1, \bigwedge_{u_1[\mu_1]\le\lambda_1}\mathcal{U}^*(u_1)\le s_1\})\\
&\odot\ (\bigvee\{\mu_2\mid \bigvee_{u_2[\mu_2]\le\lambda_2}\mathcal{U}(u_2)\ge r_2, \bigwedge_{u_2[\mu_2]\le\lambda_2}\mathcal{U}^*(u_2)\le s_2\})\\
&= \bigvee\{\mu_1\odot\mu_2\mid \bigvee_{u_i[\mu_i]\le\lambda_i}\mathcal{U}(u_i)\ge r_i, \bigwedge_{u_i[\mu_i]\le\lambda_i}\mathcal{U}^*(u_i)\le s_i, i=1,2\}\\
&\le \bigvee\{\mu_1\odot\mu_2\mid (\bigvee_{u_1[\mu_1]\le\lambda_1}\mathcal{U}(u_1))\odot(\bigvee_{u_2[\mu_2]\le\lambda_2}\mathcal{U}(u_2))\ge r_1\odot r_2\\
&\text{and } (\bigwedge_{u_1[\mu_1]\le\lambda_1}\mathcal{U}^*(u_1))\oplus(\bigwedge_{u_2[\mu_2]\le\lambda_2}\mathcal{U}^*(u_2))\le s_1\oplus s_2\}\\
&= \bigvee\{\mu_1\odot\mu_2\mid \bigvee_{u_i[\mu_i]\le\lambda_i}\mathcal{U}(u_1\odot u_2)\ge r_1\odot r_2\\
&\text{and } \bigwedge_{u_i[\mu_i]\le\lambda_i}\mathcal{U}^*(u_1\odot u_2)\le s_1\oplus s_2, i=1,2\}
\end{aligned}$$

Hence we have

$$\begin{aligned}\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda_1,r_1,s_1)\odot\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda_2,r_2,s_2) &\leq \bigvee\{\mu \mid \bigvee_{u[\mu]\leq\lambda_1\odot\lambda_2}\mathcal{U}(u)\geq r_1\odot r_2\\ &\text{and} \bigwedge_{u[\mu]\leq\lambda_1\odot\lambda_2}\mathcal{U}^*(u)\leq s_1\oplus s_2\}\\ &= \mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda_1\odot\lambda_2, r_1\odot r_2, s_1\oplus s_2).\end{aligned}$$

(2) It is trivial from the following implication:
$\delta_{\mathcal{U}}(\mu,\lambda\mapsto \underline{0}) < r\mapsto 0_M$ and $\delta^*_{\mathcal{U}^*}(\mu,\lambda\mapsto\underline{0}) > s\mapsto 0_M$ if and only if $\bigvee\{\mathcal{U}(w)\mid w[\mu]\leq\lambda\}\geq r$ and $\bigwedge\{\mathcal{U}^*(w)\mid w[\mu]\leq\lambda\}\leq s$. □

Theorem 3.9. *Let $(\mathcal{U},\mathcal{U}^*)$ be a double fuzzy preuniformity on X. Define a map $\mathcal{I}_{\mathcal{U},\mathcal{U}^*}: L^X\times M_0\times M_1\longrightarrow L^X$ by*

$$\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda,r,s)=\bigvee\{\mu\in L^X \mid u[\mu]\leq\lambda, \mathcal{U}(u)\geq r \text{ and } \mathcal{U}^*(u)\leq s\}$$

Then, the operator $\mathcal{I}_{\mathcal{U},\mathcal{U}^*}$ is a double fuzzy interior operator on X. Furthermore, if L is $s-$compact, then Theorems 3.8 and 3.9 are coincided.

Proof. (I1) Since $u[\underline{1}]\leq\underline{1}$, then $\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\underline{1},r,s)=\underline{1}$.

(I2) Since $r\in M_0$ and $s\in M_1$, then $\mathcal{U}(u)\neq 0_M$ and $\mathcal{U}^*(u)\neq 1_M$. By Lemma 3.4 (1), $\mu\leq u[\mu]\leq\lambda$ and hence $\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda,r,s)\leq\lambda$.

(I3) Let $\lambda_1\leq\lambda_2$ be given. By the definition, $\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda_1,r,s)\leq\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda_2,r,s)$.

(I4) Let $r\leq r'$, $s\geq s'$ be given. Since, $u[\mu]\leq\lambda$, $\mathcal{U}(u)\geq r'\geq r$, $\mathcal{U}^*(u)\leq s'\leq s$, then $\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda,r',s')\leq\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda,r,s)$.

(I5)

$$\begin{aligned}\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda,r,s)\odot\mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\mu,r',s') &= (\bigvee\{\nu\mid u_1[\nu]\leq\lambda, \mathcal{U}(u_1)\geq r \text{ and } \mathcal{U}^*(u_1)\leq s\})\\ &\odot (\bigvee\{\rho\mid u_2[\rho]\leq\mu, \mathcal{U}(u_2)\geq r' \text{ and } \mathcal{U}^*(u_2)\leq s'\})\\ &\leq \bigvee\{\nu\odot\rho\mid (u_1\odot u_2)[\nu\odot\rho]\leq\lambda\odot\mu, \mathcal{U}(u_1\odot u_2)\geq r\odot r'\\ &\text{and } \mathcal{U}^*(u_1\odot u_2)\leq s\oplus s'\}\\ &\leq \bigvee\{\gamma\mid u[\gamma]\leq\lambda\odot\mu, \mathcal{U}(u)\geq r\odot r' \text{ and } \mathcal{U}^*(u)\leq s\oplus s'\}\\ &= \mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda\odot\mu, r\odot r', s\oplus s').\end{aligned}$$

Hence, $\mathcal{I}_{\mathcal{U},\mathcal{U}^*}$ is a double fuzzy interior operator on X.

The second part of the proof can be seen easily. □

Theorem 3.10. *Let $(X,\mathcal{U},\mathcal{U}^*)$ be a double fuzzy preuniformity on X. Define the maps $\tau_{\mathcal{I}_{\mathcal{U},\mathcal{U}^*}}, \tau^*_{\mathcal{I}_{\mathcal{U},\mathcal{U}^*}}: L^X\longrightarrow M$ as follows:*

$$\tau_{\mathcal{I}_{\mathcal{U},\mathcal{U}^*}}(\lambda)=\bigvee\{r\in M_0\mid \mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda,r,s)=\lambda\},$$

$$\tau^*_{\mathcal{I}_{\mathcal{U},\mathcal{U}^*}}(\lambda)=\bigwedge\{s\in M_1\mid \mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\lambda,r,s)=\lambda\}.$$

Then the pair $(\tau_{\mathcal{I}_{\mathcal{U},\mathcal{U}^*}}, \tau^*_{\mathcal{I}_{\mathcal{U},\mathcal{U}^*}})$ is a double fuzzy topology on X.

Proof. It is straightforward from Theorem 2.6 and Theorems 3.8-3.9. □

Definition 3.11. Let $(X,\mathcal{U},\mathcal{U}^*)$ and $(Y,\mathcal{V},\mathcal{V}^*)$ be two double fuzzy preuniform spaces and $\varphi: X\longrightarrow Y$ be a function. Then φ is said to be double fuzzy preuniformly continuous iff

$$\mathcal{U}((\varphi\times\varphi)^{\leftarrow}(v))\geq\mathcal{V}(v) \quad\text{and}\quad \mathcal{U}^*((\varphi\times\varphi)^{\leftarrow}(v))\leq\mathcal{V}^*(v),\quad \forall v\in L^{Y\times Y}.$$

Lemma 3.12. *Let $\varphi : X \longrightarrow Y$ be a function. For each $v, v_1, v_2 \in L^{Y\times Y}$ and $\lambda \in L^X$, we have the following properties:*

(1) If φ is surjective, then $\varphi^{\leftarrow}(v[\varphi^{\rightarrow}(\lambda)]) = (\varphi\times\varphi)^{\leftarrow}(v)[\lambda]$.
(2) $(\varphi\times\varphi)^{\leftarrow}(v^s)[\lambda] = ((\varphi\times\varphi)^{\leftarrow}(v))^s[\lambda]$.
(3) $(\varphi\times\varphi)^{\leftarrow}(v_1\odot v_2) = (\varphi\times\varphi)^{\leftarrow}(v_1)\odot(\varphi\times\varphi)^{\leftarrow}(v_2)$.
(4) $(\varphi\times\varphi)^{\leftarrow}(v)\circ(\varphi\times\varphi)^{\leftarrow}(v) \le (\varphi\times\varphi)^{\leftarrow}(v\circ v)$.

Theorem 3.13. *Let $\varphi : (X,\mathcal{U},\mathcal{U}^*) \longrightarrow (Y,\mathcal{V},\mathcal{V}^*)$ be a double fuzzy uniformly continuous map and surjective. Then, $\varphi : (X,\mathcal{I}_{\mathcal{U},\mathcal{U}^*}) \longrightarrow (Y,\mathcal{I}_{\mathcal{V},\mathcal{V}^*})$ is an I-map.*

Proof. Put $\lambda = \varphi^{\leftarrow}(\gamma)$ from Lemma 3.12 (1), $v[\gamma]\le\rho$ implies

$$(\varphi\times\varphi)^{\leftarrow}(v)[\varphi^{\leftarrow}(\gamma)] = \varphi^{\leftarrow}(v[\varphi^{\rightarrow}(\varphi^{\leftarrow}(\gamma))]) \le \varphi^{\leftarrow}(v[\gamma]) \le \varphi^{\leftarrow}(\rho)$$

Since, $\mathcal{U}((\varphi\times\varphi)^{\leftarrow}(v)) \ge \mathcal{V}(v)$ and $\mathcal{U}^*((\varphi\times\varphi)^{\leftarrow}(v)) \le \mathcal{V}^*(v)$, we have

$$\begin{array}{rcl}
\varphi^{\leftarrow}(\mathcal{I}_{\mathcal{V},\mathcal{V}^*}(\rho,r,s)) & = & \varphi^{\leftarrow}(\bigvee\{\gamma\in L^Y \mid v[\gamma]\le\rho, \mathcal{V}(v)\ge r \text{ and } \mathcal{V}^*(v)\le s\}) \\
& = & \bigvee\{\varphi^{\leftarrow}(\gamma) \mid v[\gamma]\le\rho, \mathcal{V}(v)\ge r \text{ and } \mathcal{V}^*(v)\le s\}) \\
& \le & \bigvee\{\varphi^{\leftarrow}(\gamma) \mid (\varphi\times\varphi)^{\leftarrow}(v)[\varphi^{\leftarrow}(\gamma)]\le\varphi^{\leftarrow}(\rho), \mathcal{U}((\varphi\times\varphi)^{\leftarrow}(v))\ge r \\
& \text{and} & \mathcal{U}^*((\varphi\times\varphi)^{\leftarrow}(v))\le s\} \\
& \le & \bigvee\{\lambda\in L^X \mid (\varphi\times\varphi)^{\leftarrow}(v)[\lambda]\le\varphi^{\leftarrow}(\rho), \mathcal{U}((\varphi\times\varphi)^{\leftarrow}(v))\ge r \\
& \text{and} & \mathcal{U}^*((\varphi\times\varphi)^{\leftarrow}(v))\le s\} \\
& = & \mathcal{I}_{\mathcal{U},\mathcal{U}^*}(\varphi^{\leftarrow}(\rho),r,s).
\end{array}$$

□

Theorem 3.14. *Let $(X,\mathcal{U},\mathcal{U}^*)$, $(Y,\mathcal{V},\mathcal{V}^*)$ be two double fuzzy preuniform spaces and $\varphi : X \longrightarrow Y$ be an injective double fuzzy uniformly continuous function. Then $\varphi : (X,\mathcal{T}_{\mathcal{U}},\mathcal{T}^*_{\mathcal{U}^*}) \longrightarrow (Y,\mathcal{T}_{\mathcal{V}},\mathcal{T}^*_{\mathcal{V}^*})$ is LF-continuous.*

Proof. If $\lambda = \underline{0}$, it is trivial. Let $\lambda \ne \underline{0}$. Since φ is injective, we obtain the following:

$$(\varphi\times\varphi)^{\leftarrow}(u_\lambda)(x_1,x_2) = \begin{cases} 1_L, & \text{if } \varphi(x_1)=\varphi(x_2) \\ \lambda(\varphi(x_1))\odot\lambda(\varphi(x_2)), & \text{if } \varphi(x_1)\ne\varphi(x_2). \end{cases}$$

$$(\varphi\times\varphi)^{\leftarrow}(u_\lambda)(x_1,x_2) = \begin{cases} 1_L, & \text{if } x_1=x_2 \\ \varphi^{\leftarrow}(\lambda)(x_1)\odot\varphi^{\leftarrow}(\lambda)(x_2), & \text{if } x_1\ne x_2.. \end{cases}$$

$$(\varphi\times\varphi)^{\leftarrow}(u_\lambda)(x_1,x_2) = u_{\varphi^{\leftarrow}(\lambda)}(x_1,x_2).$$

So, $(\varphi\times\varphi)^{\leftarrow}(u_\lambda) = u_{\varphi^{\leftarrow}(\lambda)}$. Then, we have

$$\mathcal{T}_{\mathcal{U}}(\varphi^{\leftarrow}(\lambda)) = \mathcal{U}(u_{\varphi^{\leftarrow}(\lambda)}) = \mathcal{U}((\varphi\times\varphi)^{\leftarrow}(u_\lambda)) \ge \mathcal{V}(u_\lambda) = \mathcal{T}_{\mathcal{V}}(\lambda)$$

$$\mathcal{T}^*_{\mathcal{U}^*}(\varphi^{\leftarrow}(\lambda)) = \mathcal{U}^*(u_{\varphi^{\leftarrow}(\lambda)}) = \mathcal{U}^*((\varphi\times\varphi)^{\leftarrow}(u_\lambda)) \le \mathcal{V}^*(u_\lambda) = \mathcal{T}^*_{\mathcal{V}^*}(\lambda)$$

□

Theorem 3.15. *Let $\varphi : (X,\mathcal{U},\mathcal{U}^*) \longrightarrow (Y,\mathcal{V},\mathcal{V}^*)$ be double fuzzy preuniformly continuous function, then $\varphi : (X,\tau_{\mathcal{U}},\tau^*_{\mathcal{U}^*}) \longrightarrow (Y,\tau_{\mathcal{V}},\tau^*_{\mathcal{V}^*})$ is LF-continuous.*

Proof. First, we show that $\varphi^{\leftarrow}(v[\varphi(x)]) = (\varphi\times\varphi)^{\leftarrow}(\nu)[x]$ from:

$$\begin{aligned}\varphi^{\leftarrow}(v[\varphi(x)])(z) &= v[\varphi(x)](\varphi(z))\\ &= v(\varphi(z),\varphi(x))\\ &= (\varphi\times\varphi)^{\leftarrow}(v)(z,x) = (\varphi\times\varphi)^{\leftarrow}(v)[x](z), \quad \forall z\in X.\end{aligned}$$

Thus $v[\varphi(x)] \leq \lambda$ implies $\varphi^{\leftarrow}(v[\varphi(x)]) = (\varphi\times\varphi)^{\leftarrow}(v)[x] \leq \varphi^{\leftarrow}(\lambda)$. Hence,

$$\begin{aligned}\tau_{\mathcal{V}}(\lambda) &= \bigwedge_y\{(\lambda(y)\mapsto 0_L)\vee \bigvee_{v[y]\leq\lambda}\mathcal{V}(v)\}\\ &\leq \bigwedge_x\{(\lambda(\varphi(x))\mapsto 0_L)\vee \bigvee_{v[\varphi(x)]\leq\lambda}\mathcal{V}(v)\}\\ &\leq \bigwedge_x\{(\varphi^{\leftarrow}(\lambda)\mapsto \underline{0})(x)\vee \bigvee_{(\varphi\times\varphi)^{\leftarrow}(v)[x]\leq\varphi^{\leftarrow}(\lambda)}\mathcal{U}((\varphi\times\varphi)^{\leftarrow}(v))\}\\ &\leq \tau_{\mathcal{U}}(\varphi^{\leftarrow}(\lambda)).\end{aligned}$$

and

$$\begin{aligned}\tau^*_{\mathcal{V}^*}(\lambda) &= \bigvee_y\{\lambda(y)\wedge \bigwedge_{v[y]\leq\lambda}\mathcal{V}^*(v)\}\\ &\geq \bigvee_x\{\lambda(\varphi(x))\wedge \bigwedge_{v[\varphi(x)]\leq\lambda}\mathcal{V}^*(v)\}\\ &\geq \bigvee_x\{\varphi^{\leftarrow}(\lambda)(x)\wedge \bigwedge_{(\varphi\times\varphi)^{\leftarrow}(v)[x]\leq\varphi^{\leftarrow}(\lambda)}\mathcal{U}^*((\varphi\times\varphi)^{\leftarrow}(v))\}\\ &\geq \tau^*_{\mathcal{U}^*}(\varphi^{\leftarrow}(\lambda)).\end{aligned}$$

□

Theorem 3.16. *Let $\varphi : (X,\mathcal{U},\mathcal{U}^*) \longrightarrow (Y,\mathcal{V},\mathcal{V}^*)$ be double fuzzy preuniformly continuous function, then $\varphi : (X,\delta_{\mathcal{U}},\delta^*_{\mathcal{U}^*}) \longrightarrow (Y,\delta_{\mathcal{V}},\delta^*_{\mathcal{V}^*})$ is preproximally continuous.*

Proof. Let $\nu,\rho \in L^Y$ be given. If $\Theta_{\nu,\rho} = \emptyset$, it is trivial. Let $\Theta_{\nu,\rho} \neq \emptyset$. If $w\in\Theta_{\nu,\rho}$, then $(\varphi\times\varphi)^{\leftarrow}(w) \in \Theta(\varphi^{\leftarrow}(\nu),\varphi^{\leftarrow}(\rho))$. Hence,

$$\begin{aligned}\delta_{\mathcal{V}}(\nu,\rho) &= (\bigvee\{\mathcal{V}(w)\mid w\in\Theta_{\nu,\rho}\})\mapsto 0_M\\ &\geq (\bigvee\{\mathcal{U}((\varphi\times\varphi)^{\leftarrow}(w))\mid (\varphi\times\varphi)^{\leftarrow}(w)\in\Theta(\varphi^{\leftarrow}(\nu),\varphi^{\leftarrow}(\rho))\})\mapsto 0_M\\ &\geq (\bigvee\{\mathcal{U}(v)\mid v\in\Theta(\varphi^{\leftarrow}(\nu),\varphi^{\leftarrow}(\rho))\})\mapsto 0_M\\ &= \delta_{\mathcal{U}}(\varphi^{\leftarrow}(\nu),\varphi^{\leftarrow}(\rho)).\end{aligned}$$

and

$$\begin{aligned}\delta^*_{\mathcal{V}^*}(\nu,\rho) &= (\bigwedge\{\mathcal{V}^*(w)\mid w\in\Theta_{\nu,\rho}\})\mapsto 0_M\\ &\leq (\bigwedge\{\mathcal{U}^*((\varphi\times\varphi)^{\leftarrow}(w))\mid (\varphi\times\varphi)^{\leftarrow}(w)\in\Theta(\varphi^{\leftarrow}(\nu),\varphi^{\leftarrow}(\rho))\})\mapsto 0_M\\ &\leq (\bigwedge\{\mathcal{U}^*(v)\mid v\in\Theta(\varphi^{\leftarrow}(\nu),\varphi^{\leftarrow}(\rho))\})\mapsto 0_M\\ &= \delta^*_{\mathcal{U}^*}(\varphi^{\leftarrow}(\nu),\varphi^{\leftarrow}(\rho)).\end{aligned}$$

□

Theorem 3.17. *Let $(Y,\mathcal{V},\mathcal{V}^*)$ be a double fuzzy uniform space and $\varphi : X \longrightarrow Y$ be a function. We define, for each $u \in L^{X\times X}$,*

$$\mathcal{U}(u) = \bigvee\{\mathcal{V}(v) \mid (\varphi\times\varphi)^{\leftarrow}(v) \le u\}$$

$$\mathcal{U}^*(u) = \bigwedge\{\mathcal{V}^*(v) \mid (\varphi\times\varphi)^{\leftarrow}(v) \le u\}$$

Then we have the following properties.

(1) The pair $(\mathcal{U},\mathcal{U}^*)$ is the coarsest double fuzzy uniformity on X for which φ is double fuzzy uniformly continuous.

(2) A function $\psi : (Z,\mathcal{W},\mathcal{W}^*) \longrightarrow (X,\mathcal{U},\mathcal{U}^*)$ is double fuzzy uniformly continuous iff $\varphi\circ\psi$ is double fuzzy uniformly continuous.

Proof. (1) First we will show that $(\mathcal{U},\mathcal{U}^*)$ is a double fuzzy uniformity on X.

(U1) Since $(\mathcal{V},\mathcal{V}^*)$ is a double fuzzy uniformity on Y, then $\mathcal{U}(u) \le \mathcal{U}^*(u) \mapsto 0_M$.

(U2) Since $(\varphi\times\varphi)^{\leftarrow}(\widetilde{1}) = \widetilde{1}$, then $\mathcal{U}(\widetilde{1}) = 1_M$ and $\mathcal{U}^*(\widetilde{1}) = 0_M$.

(U3) Let $u_1 \le u_2$ be given. Then by the definition, $\mathcal{U}(u_1) \le \mathcal{U}(u_2)$ and $\mathcal{U}^*(u_1) \ge \mathcal{U}^*(u_2)$.

(U4)

$$\begin{aligned}
\mathcal{U}(u_1)\odot\mathcal{U}(u_2) &= (\bigvee\{\mathcal{V}(v_1) \mid (\varphi\times\varphi)^{\leftarrow}(v_1) \le u_1\})\odot(\bigvee\{\mathcal{V}(v_2) \mid (\varphi\times\varphi)^{\leftarrow}(v_2) \le u_2\})\\
&= \bigvee\{\mathcal{V}(v_1)\odot\mathcal{V}(v_2) \mid (\varphi\times\varphi)^{\leftarrow}(v_1) \le u_1, (\varphi\times\varphi)^{\leftarrow}(v_2) \le u_2\}\\
&\le \bigvee\{\mathcal{V}(v_1\odot v_2) \mid (\varphi\times\varphi)^{\leftarrow}(v_1\odot v_2) \le u_1\odot u_2\}\\
&\le \bigvee\{\mathcal{V}(v) \mid (\varphi\times\varphi)^{\leftarrow}(v) \le u_1\odot u_2\} = \mathcal{U}(u_1\odot u_2)
\end{aligned}$$

Similarly,

$$\mathcal{U}^*(u_1\odot u_2) \le \mathcal{U}^*(u_1)\oplus\mathcal{U}^*(u_2).$$

(U5) Let $\mathcal{U}(u) \ne 0_M$ and $\mathcal{U}^*(u) \ne 1_M$, then there exists $v \in L^{X\times X}$ with $(\varphi\times\varphi)^{\leftarrow}(v) \le u$ such that $\mathcal{U}(u) \ge \mathcal{V}(v) \ne 0_M$ and $\mathcal{U}^*(u) \le \mathcal{V}^*(v) \ne 1_M$. Since $\mathcal{V}(v) \ne 0_M$ and $\mathcal{V}^*(v) \ne 1_M$, then $1_\Delta \le v$. Hence,

$$1_\Delta \le (\varphi\times\varphi)^{\leftarrow}(1_\Delta) \le (\varphi\times\varphi)^{\leftarrow}(v) \le u.$$

(QU) Suppose that there exists $u \in L^{X\times X}$ such that

$$\mathcal{U}(u) \not\le \bigvee\{\mathcal{U}(u_1) \mid u_1\circ u_1 \le u\}$$

$$\mathcal{U}^*(u) \not\ge \bigwedge\{\mathcal{U}^*(u_1) \mid u_1\circ u_1 \le u\}$$

By the definition of $(\mathcal{U},\mathcal{U}^*)$, there exists $v \in L^{Y\times Y}$ with $(\varphi\times\varphi)^{\leftarrow}(v) \le u$ such that

$$\mathcal{V}(v) \not\le \bigvee\{\mathcal{U}(u_1) \mid u_1\circ u_1 \le u\}$$

$$\mathcal{V}^*(v) \not\ge \bigwedge\{\mathcal{U}^*(u_1) \mid u_1\circ u_1 \le u\}$$

Since $(Y,\mathcal{V},\mathcal{V}^*)$ is a double fuzzy uniform space, $\bigvee\{\mathcal{V}(w) \mid w\circ w \le v\} \ge \mathcal{V}(v)$ and $\bigwedge\{\mathcal{V}^*(w) \mid w\circ w \le v\} \le \mathcal{V}^*(v)$. Hence,

$$\bigvee\{\mathcal{U}(u_1) \mid u_1\circ u_1 \le u\} \not\ge \bigvee\{\mathcal{V}(w) \mid w\circ w \le v\}$$

and

$$\bigwedge\{\mathcal{U}^*(u_1) \mid u_1\circ u_1 \le u\} \not\le \bigwedge\{\mathcal{V}^*(w) \mid w\circ w \le v\}.$$

Then there exists $w \in L^{Y\times Y}$ with $w \circ w \leq v$ such that

$$\bigvee\{\mathcal{U}(u_1) \mid u_1 \circ u_1 \leq u\} \ngeq \mathcal{V}(w)$$

and

$$\bigwedge\{\mathcal{U}^*(u_1) \mid u_1 \circ u_1 \leq u\} \nleq \mathcal{V}^*(w).$$

On the other hand, since

$$(\varphi \times \varphi)^{\leftarrow}(w) \circ (\varphi \times \varphi)^{\leftarrow}(w) \leq (\varphi \times \varphi)^{\leftarrow}(w \circ w) \leq (\varphi \times \varphi)^{\leftarrow}(v) \leq u,$$

we have

$$\bigvee\{\mathcal{U}(u_1) \mid u_1 \circ u_1 \leq u\} \geq \mathcal{U}((\varphi \times \varphi)^{\leftarrow}(w)) \geq \mathcal{V}(w).$$

$$\bigwedge\{\mathcal{U}^*(u_1) \mid u_1 \circ u_1 \leq u\} \leq \mathcal{U}^*((\varphi \times \varphi)^{\leftarrow}(w)) \leq \mathcal{V}^*(w).$$

It is a contradiction.

(U) Suppose that there exists $u \in L^{X\times X}$ such that $\mathcal{U}(u^s) \ngeq \mathcal{U}(u)$ and $\mathcal{U}^*(u^s) \nleq \mathcal{U}^*(u)$ by the definition of $(\mathcal{U},\mathcal{U}^*)$, there exists $v \in L^{Y\times Y}$ with $(\varphi \times \varphi)^{\leftarrow}(v) \leq u$ such that $\mathcal{U}(u^s) \ngeq \mathcal{V}(v)$ and $\mathcal{U}^*(u^s) \nleq \mathcal{V}^*(v)$. Since $(\mathcal{V},\mathcal{V}^*)$ is a double fuzzy uniformity on Y, $\mathcal{V}(v^s) \geq \mathcal{V}(v)$ and $\mathcal{V}^*(v^s) \leq \mathcal{V}^*(v)$. It follows that, $\mathcal{U}(u^s) \ngeq \mathcal{V}(v^s)$ and $\mathcal{U}^*(u^s) \nleq \mathcal{V}^*(v^s)$. Since $(\varphi \times \varphi)^{\leftarrow}(v^s) = ((\varphi \times \varphi)^{\leftarrow}(v))^s \leq u^s$, we have $\mathcal{U}(u^s) \geq \mathcal{V}(v^s)$ and $\mathcal{U}^*(u^s) \leq \mathcal{V}^*(v^s)$. Thus, $(\mathcal{U},\mathcal{U}^*)$ is a double fuzzy uniformity on X.

Second, it is easily proved that, by the definition of $(\mathcal{U},\mathcal{U}^*)$

$$\mathcal{U}((\varphi \times \varphi)^{\leftarrow}(v)) \geq \mathcal{V}(v) \text{ and } \mathcal{U}^*((\varphi \times \varphi)^{\leftarrow}(v)) \leq \mathcal{V}^*(v), \quad \forall v \in L^{Y\times Y}.$$

Hence, $\varphi : (X,\mathcal{U},\mathcal{U}^*) \longrightarrow (Y,\mathcal{V},\mathcal{V}^*)$ is double fuzzy uniformly continuous.

If $\varphi : (X,\mathcal{W},\mathcal{W}^*) \longrightarrow (Y,\mathcal{V},\mathcal{V}^*)$ is double fuzzy uniformly continuous, then it is proved that $\mathcal{W} \geq \mathcal{U}$ and $\mathcal{W}^* \leq \mathcal{U}^*$ from the following:

$$\begin{aligned}\mathcal{U}(u) &= \bigvee\{\mathcal{V}(v) \mid (\varphi \times \varphi)^{\leftarrow}(v) \leq u\} \\ &\leq \bigvee\{\mathcal{W}((\varphi \times \varphi)^{\leftarrow}(v)) \mid (\varphi \times \varphi)^{\leftarrow}(v) \leq u\} \leq \mathcal{W}(u)\end{aligned}$$

Similarly, $\mathcal{U}^*(u) \geq \mathcal{W}^*(u), \forall u \in L^{X\times X}$.

(2) It is clear that the composition of double fuzzy uniformly continuous maps is double fuzzy uniformly continuous.

Conversely, suppose that $\psi : (Z,\mathcal{W},\mathcal{W}^*) \longrightarrow (X,\mathcal{U},\mathcal{U}^*)$ is not double fuzzy uniformly continuous. Then there exists $u \in L^{X\times X}$ such that

$$\mathcal{W}((\psi \times \psi)^{\leftarrow}(u)) \ngeq \mathcal{U}(u) \text{ and } \mathcal{W}^*((\psi \times \psi)^{\leftarrow}(u)) \nleq \mathcal{U}^*(u).$$

By the definition of $(\mathcal{U},\mathcal{U}^*)$, there exists $v \in L^{Y\times Y}$ with $(\varphi \times \varphi)^{\leftarrow}(v) \leq u$ such that

$$\mathcal{W}((\psi \times \psi)^{\leftarrow}(u)) \ngeq \mathcal{V}(v) \text{ and } \mathcal{W}^*((\psi \times \psi)^{\leftarrow}(u)) \nleq \mathcal{V}^*(v).$$

On the other hand, since $\varphi \circ \psi : (Z,\mathcal{W},\mathcal{W}^*) \longrightarrow (Y,\mathcal{V},\mathcal{V}^*)$ is double fuzzy uniformly continuous, we have

$$\mathcal{V}(v) \leq \mathcal{W}(((\varphi \circ \psi) \times (\varphi \circ \psi))^{\leftarrow}(v)) = \mathcal{W}((\psi \times \psi)^{\leftarrow} \circ (\varphi \times \varphi)^{\leftarrow}(v))$$

and

$$\mathcal{V}^*(v) \geq \mathcal{W}^*(((\varphi \circ \psi) \times (\varphi \circ \psi))^{\leftarrow}(v)) = \mathcal{W}^*((\psi \times \psi)^{\leftarrow} \circ (\varphi \times \varphi)^{\leftarrow}(v))$$

It follows that $\mathcal{V}(v) \leq \mathcal{W}((\psi\times\psi)^{\leftarrow}(u))$ and $\mathcal{V}^*(v) \geq \mathcal{W}^*((\psi\times\psi)^{\leftarrow}(u))$. This contradicts with the assumption.

□

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