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On the modular G-metric spaces and fixed point theorems

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Abstract

We introduce the notion of modular G–metric spaces and obtain some fixed point theorems of contractive mappings defined on modular G–metric spaces.

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1. Introduction and Preliminaries

The theory of modular spaces was initiated by Nakano [10] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [11, 12] in 1959. The notion of a modular metric on an arbitrary set an the corresponding modular space, more general than a metric space were introduced and studied recently by Chistyakof [1]. There were any authors introduced the generalization of metric spaces such as Gahler [4], which called 2–metric spaces, and Dhage [3], which called D–metric spaces. In 2003, Mustafa and Sims [5] found that most of the claims concerning the fundemental topology properties of D–metric spaces are incorrect. They [6] introduced a generalization of metric spaces, which called G–metric spaces. In this paper, we introduce the notion of a modular G-metric spaces as the following:

Definition 1.1. Let X be a nonempty set, and let $\nu : (0, \infty) \times X \times X \times X \longrightarrow [0, \infty]$ be a function satisfying;
(V1) $\nu_\lambda(x, y, z) = 0$ for all $x, y \in X$ and $\lambda > 0$ if $x = y = z$,
(V2) $\nu_\lambda(x, x, y) > 0$ for all $x, y \in X$ and $\lambda > 0$ with $x \neq y$,
(V3) $\nu_\lambda(x, x, y) \leq \nu_\lambda(x, y, z)$ for all $x, y, z \in X$ and $\lambda > 0$ with $z \neq y$,
(V4) $\nu_\lambda(x, y, z) = \nu_\lambda(x, z, y) = \nu_\lambda(y, z, x) = \cdots$ for all $\lambda > 0$ (symmetry in all three variables),
(V5) $\nu_{\lambda+\mu}(x, y, z) \leq \nu_\lambda(x, a, a) + \nu_\mu(a, y, z)$ for all $x, y, z, a \in X$ and $\lambda, \mu > 0$,

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then the function ν_λ is called a modular G-metric on X.
The note by setting $x = y = z$ and $\lambda = \mu > 0$ in (V3), (V5) and taking into account (V1), for all $x, y, z \in X$, we fined

$$\begin{aligned} 0 = \nu_{2\lambda}(x,x,x) &\le \nu_\lambda(x,a,a) + \nu_\lambda(a,x,x) \\ &\le 2\nu_\lambda(x,y,z). \end{aligned}$$

Example 1.2. The following indexed objects ν are simple examples of modulars on a set X. Let $\lambda > 0$ and $x, y, z \in X$, we have:
(a) $\nu_\lambda(x,y,z) = \infty$ if $x \neq y \neq z$, $\nu_\lambda(x,y,z) = 0$ if $x = y = z$; and if (X,G) is a G-metric space, then we also have:
(b) $\nu_\lambda(x,y,z) = \frac{G(x,y,z)}{\varphi(\lambda)}$, where $\varphi : (0,\infty) \to (0,\infty)$ is a nondecreasing function;
(c) $\nu_\lambda(x,y,z) = \infty$ if $\lambda \le G(x,y,z)$, and $\nu_\lambda(x,y,z) = 0$ if $\lambda > G(x,y,z)$;
(d) $\nu_\lambda(x,y,z) = \infty$ if $\lambda < G(x,y,z)$, and $\nu_\lambda(x,y,z) = 0$ if $\lambda \ge G(x,y,z)$.

Remark 1.3. Note that for $x, y, z \in X$ the function $0 < \lambda \longmapsto \nu_\lambda(x,y,z) \in [0,\infty]$ is nonincreasing on $(0,\infty)$. Suppose $0 < \mu < \lambda$, then (V1) and (V5) imply

$$\nu_\lambda(x,y,z) \le \nu_{\lambda-\mu}(x,x,x) + \nu_\mu(x,y,z) = \nu_\mu(x,y,z).$$

It follows that each point $\lambda > 0$ the right limit $\nu_{\lambda+0}(x,y,z) = \lim_{\varepsilon\to+0} \nu_{\lambda+\varepsilon}(x,y,z)$ and left limit $\nu_{\lambda-0}(x,y,z) = \lim_{\varepsilon\to0} \nu_{\lambda-\varepsilon}(x,y,z)$ exist in $[0,\infty)$ and following two inequalities hold:

$$\nu_{\lambda+0}(x,y,z) \le \nu_\lambda(x,y,z) \le \nu_{\lambda-0}(x,y,z).$$

Definition 1.4. Let ν be a modular G-metric on a set X. The binary relation $\overset{\nu}{\sim}$ on X defined for $x, y, z \in X$ by

$$x \overset{\nu}{\sim} y \quad \textit{if and only if} \quad \lim_{\lambda\to\infty} \nu_\lambda(x,y,z) = 0 \quad \textit{for some } z \in X \tag{1.1}$$

is, by virtue of axioms (V1), (V4) and (V5) , an equivalence relation since, if $x \overset{\nu}{\sim} y$ and $y \overset{\nu}{\sim} a$, then there exist $z_1, z_2 \in X$ such that $\lim_{\lambda\to\infty} \nu_\lambda(x,y,z_1) = 0$ and $\lim_{\lambda\to\infty} \nu_\lambda(a,y,z_2) = 0$, so $\nu_\lambda(a,y,z_2) \le \nu_{\frac{\lambda}{2}}(x,y,y) + \nu_{\frac{\lambda}{2}}(a,y,z_2) \le \nu_{\frac{\lambda}{2}}(x,y,z_1) + \nu_{\frac{\lambda}{2}}(a,y,z_2) \to 0$ as $\lambda \to \infty$, and so, $x \overset{\nu}{\sim} y$. Denote by $X/\overset{\nu}{\sim}$ the quotient-set of X with respect to $\overset{\nu}{\sim}$ and by

$$X_\nu^\circ(x) = \{y \in X : y \overset{\nu}{\sim} x\}$$

the equivalence class of the element $x \in X$ in the quotient-set $X/\overset{\nu}{\sim}$. Note, in particular, that $x \in X_\nu^\circ(x)$ and that the transitivity property of $\overset{\nu}{\sim}$ implies $x \overset{\nu}{\sim} z$ if and only if $y, z \in X_\nu^\circ(x)$ for some $x \in X$ (e.g., $x = y$ or $x = z$).

It follows from Remark 1.3 that the function $\widetilde{G} : (X/\overset{\nu}{\sim}) \times (X/\overset{\nu}{\sim}) \times (X/\overset{\nu}{\sim}) \to [0,\infty]$ given by

$$\widetilde{G}(X_\nu^\circ(x), X_\nu^\circ(y), X_\nu^\circ(z)) = \lim_{\lambda\to\infty} \nu_\lambda(x,y,z), \quad (x,y,z \in X),$$

is well defined (the limit at the right-hand side does not depend on the representatives of the representatives of the equivalence classes) and satisfies the axioms of a G-metric, except, as Example 1.2(a) shows, that it may take infinite values.

In what follows we are interested in the equivalence classes $X_\nu^\circ(x)$. Note that the quotient-pair $(X/\overset{\nu}{\sim}, \widetilde{G})$ may degenerate in interesting and important cases: e.g., in Example 1.2(c) we have $X_\nu^\circ(x) = X$ for all $x \in X$ and $\widetilde{G} \equiv 0$.

Let us fix an element $x_0 \in X$ arbitrarily and set $X_\nu = X_\nu^\circ(x_0)$. The set X_ν is call a modular set.

Theorem 1.5. *If ν is G-metric modular on X, then the modular set X_ν is a G-metric space with G-metric given by*

$$G^\circ_\nu(x,y,z) = \inf\{\lambda > 0 : \nu_\lambda(x,y,z) \le \lambda\},$$

for all $x,y,z \in X_\nu$.

Proof. Given $x,y,z \in X_\nu$, the value $G^\circ_\nu(x,y,z) \in \mathbb{R}^+$ is well defined: in fact, since $x \overset{\nu}{\sim} y$, then, by virtue of (1.1), there exists $\lambda_0 > 0$ such that $\nu_\lambda(x,y,z) \le 1$ for all $\lambda \ge \lambda_0$, and so, setting $\lambda_1 = \max\{1, \lambda_0\}$, we get

$$\nu_{\lambda_1}(x,y,z) \le 1 \le \lambda_1,$$

which together with the definition of $G^\circ_\nu(x,y,z)$ gives

$$G^\circ_\nu(x,y,z) \le \lambda_1 < \infty.$$

Given $x \in X_\nu$, (V1) implies

$$\nu_\lambda(x,x,x) = 0 < \lambda \quad \textit{for all} \quad \lambda > 0,$$

and so, $G^\circ_\nu(x,x,x) = 0$. Condition (G2) and (G3) are clear by axioms (V2) and (V3). Due to axiom (V4), the equalities $G^\circ_\nu(x,y,z) = G^\circ_\nu(x,z,y) = G^\circ_\nu(y,z,x) = \cdots$, $x,y,z \in X_\nu$, is clear.

Let us show that $G^\circ_\nu(x,y,z) \le G^\circ_\nu(x,a,a) + G^\circ_\nu(a,y,z)$ for all $x,y,z,a \in X_\nu$. In fact, by the definition of G°_ν, for any $\lambda > G^\circ_\nu(x,a,a)$ and $\mu > G^\circ_\nu(y,z,a)$ we find $\nu_\lambda(x,a,a) \le \lambda$ and $\nu_\mu(a,y,z) \le \mu$, and so, axiom (V5) implies

$$\nu_{\lambda+\mu}(x,y,z) \le \nu_\lambda(x,a,a) + \nu_\mu(a,y,z) \le \lambda + \mu.$$

It follows from the definition of G°_ν that $G^\circ_\nu(x,y,z) \le \lambda + \mu$, and it remains to pass to the limits as $\lambda \longrightarrow G^\circ_\nu(x,a,a)$ and $\mu \longrightarrow G^\circ_\nu(a,y,z)$. □

Theorem 1.6. *Let ν be a modular G-metric on a set X. put*

$$G^1_\nu(x,y,z) = \inf_{\lambda>0} \big(\lambda + \nu_\lambda(x,y,z)\big),$$

for all $x,y,z \in X_\nu$. Then G^1_ν is a G-metric on X_ν such that $G^\circ_\nu \le G^1_\nu \le 2G^\circ_\nu$.

Proof. Since, for $x,y,z \in X_\nu$, the value $\nu_\lambda(x,y,z)$ is finite due to (1.1) for $\lambda > 0$ large enough, then the set $\{\lambda + \nu_\lambda(x,y,z) : \lambda > 0\} \subset \mathbb{R}^+$ is nonempty and bounded from below, and so, $G^1_\nu(x,y,z) \in \mathbb{R}^+$. Condition (G2) and (G3) are trivial by axioms (V2) and (V3). Axiom (V4) implies the symmetry of G^1_ν.

Let us establish the triangle inequality:

$$G^1_\nu(x,y,z) \le G^1_\nu(x,a,a) + G^1_\nu(a,y,z).$$

By the definition of G^1_ν, for any $\varepsilon > 0$ we find $\lambda = \lambda(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that

$$\lambda + \nu_\lambda(x,a,a) \le G^1_\nu(x,a,a) + \varepsilon \quad \textit{and} \quad \mu + \nu_\mu(a,y,z) \le G^1_\nu(a,y,z) + \varepsilon,$$

whence, applying axiom (V5),

$$\begin{aligned} G^1_\nu(x,y,z) &\le (\lambda+\mu) + \nu_{\lambda+\mu}(x,y,z) \le \lambda + \mu + \nu_\lambda(x,a,a) + \nu_\mu(a,y,z) \\ &\le G^1_\nu(x,a,a) + \varepsilon + G^1_\nu(a,y,z) + \varepsilon, \end{aligned}$$

and it remains to take into account the arbitrariness of $\varepsilon > 0$.

Let us prove that metrics G°_ν and G^1_ν are equivalent on X_ν. In order to obtain the left-hand side inequality, suppose that $\lambda > 0$ is arbitrary. If $\nu_\lambda(x,y,z) \le \lambda$, then the definition of G°_ν implies $G^\circ_\nu \le \lambda$. Now if $\nu_\lambda(x,y,z) > \lambda$, then $G^\circ_\nu(x,y,z) \le \nu_\lambda(x,y,z)$: in fact, setting $\mu = \nu_\lambda(x,y,z)$ we find $\mu > \lambda$, and so, it follows from Remark 1.3 that $\nu_\mu(x,y,z) \le \nu_\lambda(x,y,z) = \mu$, whence $G^\circ_\nu(x,y,z) \le \mu = \nu_\lambda(x,y,z)$. Therefore, for any $\lambda > 0$ we have

$$G^\circ_\nu(x,y,z) \le \max\{\lambda, \nu_\lambda(x,y,z)\} \le \lambda + \nu_\lambda(x,y,z),$$

and so, taking the infimum over all $\lambda > 0$, we arrive at the inequality

$$G^{\circ}_{\nu}(x, y, z) \leq G^{1}_{\nu}(x, y, z).$$

To obtain the right-hand side inequality, we note that, given $\lambda > 0$ such that $G^{\circ}_{\nu}(x, y, z) < \lambda$, by the definition of G°_{ν}, we get $\nu_{\lambda}(x, y, z) \leq \lambda$, and so, $G^{1}_{\nu}(x, y, z) \leq \lambda + \nu_{\lambda}(x, y, z) \leq 2\lambda$. passing to the limit as $\lambda \to G^{\circ}_{\nu}(x, y, z)$, we get

$$G^{1}_{\nu}(x, y, z) \leq 2G^{\circ}_{\nu}(x, y, z).$$

□

Theorem 1.7. *Given a modular G-metric ν on X, $x, y, z \in X_{\nu}$ and $\lambda > 0$, we have:*
(a) if $G^{\circ}_{\nu}(x, y, z) < \lambda$, then $\nu_{\lambda}(x, y, z) \leq G^{\circ}_{\nu}(x, y, z) < \lambda$;
(b) if $\nu_{\lambda}(x, y, z) = \lambda$, then $G^{\circ}_{\nu}(x, y, z) = \lambda$;
(c) if $\lambda = G^{\circ}_{\nu}(x, y, z) > 0$, then $\nu_{\lambda+0}(x, y, z) \leq \lambda \leq \nu_{\lambda-0}(x, y, z)$.
If the function $\mu \mapsto \nu_{\mu}(x, y, z)$ is continuous from the right on $(0, \infty)$, then along with (a)-(c) we have:
(d) $G^{\circ}_{\nu}(x, y, z) \leq \lambda$ if and only if $\nu_{\lambda}(x, y, z) \leq \lambda$.
If the function $\mu \mapsto \nu_{\mu}(x, y, z)$ is continuous from the left on $(0, \infty)$, then along with (a)-(c) we have:
(e) $G^{\circ}_{\nu}(x, y, z) < \lambda$ if and only if $\nu_{\lambda}(x, y, z) < \lambda$.
If the function $\mu \mapsto \nu_{\mu}(x, y, z)$ is continuous on $(0, \infty)$, then along with (a)-(c) we have:
(f) $G^{\circ}_{\nu}(x, y, z) = \lambda$ if and only if $\nu_{\lambda}(x, y, z) = \lambda$.

Proof. (a) For any $\mu > 0$ such that $G^{\circ}_{\nu}(x, y, z) < \mu < \lambda$, by the definition of G°_{ν} and Remark 1.3, we have $\nu_{\mu}(x, y, z) \leq \mu$ and $\nu_{\lambda}(x, y, z) \leq \nu_{\mu}(x, y, z)$, whence $\nu_{\lambda}(x, y, z) \leq \mu$, and it remains to pass to the limit as $\mu \longrightarrow G^{\circ}_{\nu}(x, y, z)$.

(b) By the definition, $G^{\circ}_{\nu}(x, y, z) \leq \lambda$, and item (a) implies $G^{\circ}_{\nu}(x, y, z) = \lambda$.

(c) For any $\mu > \lambda = G^{\circ}_{\nu}(x, y, z)$, the definition of G°_{ν} implies $\nu_{\mu}(x, y, z) \leq \mu$, and so,

$$\nu_{\lambda+0}(x, y, z) = \lim_{\mu \to \lambda+0} \nu_{\mu}(x, y, z) \leq \lim_{\mu \to \lambda+0} \mu = \lambda.$$

For any $0 < \mu < \lambda$ we find $\nu_{\mu}(x, y, z) > \mu$ (otherwise, the definition of G°_{ν}, we have $\lambda = G^{\circ}_{\nu}(x, y, z) \leq \mu$), and so,

$$\nu_{\lambda-0}(x, y, z) = \lim_{\mu \to \lambda-0} \nu_{\mu}(x, y, z) \geq \lim_{\mu \to \lambda-0} \mu = \lambda.$$

(d) The implication $\Leftarrow$ follows from the definition of G°_{ν}. Let us prove the reverse implication. If $G^{\circ}_{\nu}(x, y, z) < \lambda$, then, by virtue of item (a), $\nu_{\lambda}(x, y, z) < \lambda$, and if $G^{\circ}_{\nu}(x, y, z) = \lambda$, then

$$\nu_{\lambda}(x, y, z) = \nu_{\lambda+0}(x, y, z) \leq \lambda,$$

which is a consequence of the continuity from the right of the function $\mu \mapsto \nu_{\mu}(x, y, z)$ and item (c).

(e) By virtue of item (a), it suffices to prove the implication $\Leftarrow$. The definition of G°_{ν} gives $G^{\circ}_{\nu}(x, y, z) \leq \lambda$, but if, on the contrary, $\lambda = G^{\circ}_{\nu}(x, y, z)$, then, by item (c), we would have

$$\nu_{\lambda}(x, y, z) = \nu_{\lambda-0}(x, y, z) \geq \lambda,$$

which contradicts the assumption.

(f) $\Leftarrow$ follows from (b). For the reverse assertion, the two inequalities

$$\nu_{\lambda}(x, y, z) \leq \lambda \leq \nu_{\lambda}(x, y, z)$$

follow from (c). □

2. properties

Proposition 2.1. *Let (X,ν) be a modular G-metric space, for any $x,y,z,a \in X$ it follows that:*
(1) If $\nu_\lambda(x,y,z)=0$ for all $\lambda>0$, then $x=y=z$.
(2) $\nu_\lambda(x,y,z) \le \nu_{\frac{\lambda}{2}}(x,x,y)+\nu_{\frac{\lambda}{2}}(x,x,z)$ for all $\lambda>0$.
(3) $\nu_\lambda(x,y,y) \le 2\nu_{\frac{\lambda}{2}}(x,x,y)$ for all $\lambda>0$.
(4) $\nu_\lambda(x,y,z) \le \nu_{\frac{\lambda}{2}}(x,a,z)+\nu_{\frac{\lambda}{2}}(a,y,z)$ for all $\lambda>0$.
(5) $\nu_\lambda(x,y,z) \le \frac{2}{3}\left(\nu_{\frac{\lambda}{2}}(x,y,a)+\nu_{\frac{\lambda}{2}}(x,a,z)+\nu_{\frac{\lambda}{2}}(a,y,z)\right)$ for all $\lambda>0$.
(6) $\nu_\lambda(x,y,z) \le \left(\nu_{\frac{\lambda}{2}}(x,a,a)+\nu_{\frac{\lambda}{4}}(y,a,a)+\nu_{\frac{\lambda}{4}}(z,a,a)\right)$ for all $\lambda>0$.

If (X,ω) is an ordinary modular metric space, then (X,ω) can define modular G-metric on X by
(F_s) $\nu^s_\lambda(x,y,z)=\frac{1}{3}\{\omega_\lambda(x,y)+\omega_\lambda(y,z)+\omega_\lambda(x,z)\}$,
(F_m) $\nu^m_\lambda(x,y,z)=\max\{\omega_\lambda(x,y),\omega_\lambda(y,z),\omega_\lambda(x,z)\}$, for all $\lambda>0$.
For any nonempty set X. We have seen that from any modular metric ω on X we can construct a modular G-metric (by (F_s) or (F_m)), for any modular G-metric ν_λ on X, (F_ω) $\omega^\nu_\lambda(x,y)=\nu_\lambda(x,y,y)+\nu_\lambda(x,x,y)$, for all $\lambda>0$ is readily seen to define a modular metric on X, for all $\lambda>0$, which satisfies

$$\nu_\lambda(x,y,z) \le \nu^s_\lambda(x,y,z) \le 2\nu_\lambda(x,y,z),$$

for all $\lambda>0$. Similarly,

$$\frac{1}{2}\nu_\lambda(x,y,z) \le \nu^m_\lambda(x,y,z) \le 2\nu_\lambda(x,y,z),$$

for all $\lambda>0$. Further, starting from a modular metric ω on X, we have

$$\omega^{\nu^s}_\lambda(x,y)=\frac{4}{3}\omega_\lambda(x,y),\ \textit{and}\ \omega^{\nu^m}_\lambda(x,y)=2\omega_\lambda(x,y),$$

for all $\lambda>0$.

Definition 2.2. Let (X,ν) be a modular G-metric space then for $x_0 \in X_\nu$ and $r>0$, the ν-ball with center x_0 and radius r is

$$B_\nu(x_0,r)=\{y \in X_\nu : \quad \nu_\lambda(x_0,y,y)<r \ \textit{for all}\ \lambda>0\}.$$

Proposition 2.3. *Let (X,ν) be a modular G-metric space, then for any $x_0 \in X_\nu$ and $r>0$, we have*
(1) if $\nu_\lambda(x_0,x,y)<r$, for all $\lambda>0$ then $x,y \in B_\nu(x_0,r)$.
(2) if $y \in B_\nu(x_0,r)$ then there exists a $\delta>0$ such that $B_\nu(y,\delta) \subseteq B_\nu(x_0,r)$.

Proof. (1) follow directly from (V3), while (2) follows from (V5) with $\delta=r-\nu_\lambda(x_0,y,y)$. □

It follows from Proposition 2.3 that the familly of all ν-balls

$$\beta=\{B_\nu(x,r)|x \in X, r>0\}$$

is the base of a topology $\tau(\nu_\lambda)$ on X_ν.

Proposition 2.4. *Let (X,ν) be a modular G-metric space, then for any $x_0 \in X_\nu$ and $r>0$, we have*

$$B_\nu\left(x_0,\frac{1}{3}r\right) \subseteq B_{\omega^\nu_\lambda}(x_0,r)=\{y \in X_\omega :\ \omega^\nu_\lambda(x_0,y)<r \ \textit{for all}\ \lambda>0\} \subseteq B_\nu(x_0,r).$$

Definition 2.5. Let (X,ν) be a modular G-metric space. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ν is ν-convergent to x, if it converges to x in the topology $\tau(\nu_\lambda)$.

Proposition 2.6. *Let (X, ν) be a modular G-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ν. Then the following are equivalent:*
(1) $\{x_n\}_{n\in\mathbb{N}}$ is ν-convergent to x,
(2) $\omega_\lambda^\nu(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$, i.e., $\{x_n\}_n$ converges to x relative to the modular metric ω_λ^ν.
(3) $\nu_\lambda(x_n, x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\lambda > 0$,
(4) $\nu_\lambda(x_n, x, x) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\lambda > 0$,
(5) $\nu_\lambda(x_m, x_n, x) \longrightarrow 0$ as $m, n \longrightarrow \infty$ for all $\lambda > 0$.

Proof. The equivalence of (1) and (2) follows from proposition 2.4. That (2) implies (3) (and(4)) follows from the definition of ω_λ^ν. (3) implies (4) is a consequence of (3) of proposition 2.1, while (4) entails (5) follows from (2) of proposition 2.1. Finally, that (5) implies (2) follows from (F_ω) and axiom (V3). □

Definition 2.7. Let (X, ν) be a modular G-metric space, then a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X_\nu$ is said to be ν-cauchy if for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\nu_\lambda(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq n_\varepsilon$ and $\lambda > 0$. A modular G-metric space X is said to be ν-complete if every ν-Cauchy sequence in X is a ν-convergen sequence in X.

Proposition 2.8. *Let (X, ν) be a modular G-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ν. Then the following are equivalent:*
(1) $\{x_n\}_{n\in\mathbb{N}}$ is ν-Cauchy.
(2) For every $\varepsilon > 0$, there exist $n_\varepsilon \in \mathbb{N}$ such that $\nu_\lambda(x_n, x_m, x_m) < \varepsilon$, for any $n, m \geq n_\varepsilon$ and $\lambda > 0$.
(3) $\{x_n\}_{n\in\mathbb{N}}$ is a cauchy sequence in the modular metric space (X, ω_λ^ν).

Proof. $1 \longrightarrow 2)$ It is trivial by axiom (V3). $2 \longrightarrow 3)$ By definition ω_λ^ν is trivial.
$3 \longrightarrow 2)$ By definition $\omega_\lambda^\nu(x_n, x_m)$ is trivial.
$2 \longrightarrow 1)$ By axiom (V5) and put $a = x_m$ is trivial. □

Theorem 2.9. *Let ν be a modular G-metric on a set X. Given a sequence $\{x_n\}_{n=1}^\infty \subseteq X_\nu$ and $x \in X_\nu$, we have: $G_\nu^\circ(x_n, x_n, x) \to 0$ as $n \to \infty$ if and only if $\nu_\lambda(x_n, x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$. A similar assertion holds for Cauchy sequences.*

Proof. Given arbitrary $\varepsilon > 0$. Let $\nu_\lambda(x_n, x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$. We put $\lambda = \varepsilon$ then $\nu_\varepsilon(x_n, x_n, x) \to 0$, there is a number $n_0(\varepsilon)$ such that $\nu_\varepsilon(x_n, x_n, x) \leq \varepsilon$ for all $n \geq n_0(\varepsilon)$, whence $G_\nu^\circ(x_n, x_n, x) \leq \varepsilon$ for all $n \geq n_0(\varepsilon)$.

Necessity. Let us fix $\lambda > 0$ arbitrarily. Then, for each $\varepsilon > 0$, we have: either (a) $0 < \varepsilon < \lambda$, or (b) $\varepsilon \geq \lambda$. In case (a), by the assumption, there is a number $n_0(\varepsilon)$ such that $G_\nu^\circ(x_n, x_n, x) < \varepsilon$ for all $n \geq n_0(\varepsilon)$, and so, by theorem 1.7(a), we get $\nu_\varepsilon(x_n, x_n, x) < \varepsilon$ for all $n \geq n_0(\varepsilon)$. Since $\varepsilon < \lambda$, then, in view of Remark 1.3, we find

$$\nu_\lambda(x_n, x_n, x) \leq \nu_\epsilon(x_n, x_n, x) < \varepsilon$$

for all $n \geq n_0(\varepsilon)$.

In case (b) we set $n_1(\varepsilon) = n_0(\frac{\lambda}{2})$. From Remark 1.3 and the just established fact (when $\varepsilon = \frac{\lambda}{2} < \lambda$), we get:

$$\nu_\lambda(x_n, x_n, x) \leq \nu_{\frac{\lambda}{2}}(x_n, x_n, x) < \frac{\lambda}{2} < \frac{\varepsilon}{2} < \varepsilon \quad forall \quad n \geq n_1(\varepsilon).$$

Hence, $\nu_\lambda(x_n, x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$. □

3. Fixed point theorems

In this section we will prove the existence of fixed point of contractive mapping defined on modular G–metric spaces, where the completeness is replaced with weaker conditions.

Definition 3.1. A function $T : X_\nu \longrightarrow X_\nu$ at $x \in X_\nu$ is called ν-continuous if $\nu_\lambda(x_n, x, x) \longrightarrow 0$ then $\nu_\lambda(Tx_n, Tx, Tx) \longrightarrow 0$, for all $\lambda > 0$.

Theorem 3.2. *Let (X, ν) be a modular G-metric space and let $T : X_\nu \longrightarrow X_\nu$ be a mapping such that T satisfies that*
(I1) $\nu_\lambda(Tx, Ty, Tz) \le a\nu_\lambda(x, Tx, Tx) + b\nu_\lambda(y, Ty, Ty) + c\nu_\lambda(z, Tz, Tz)$ for all $x, y, z \in X_\nu$ and $\lambda > 0$ where $0 < a + b + c < 1$,
(I2) T is ν-continuous at a point $u \in X_\nu$,
(I3) there is $x \in X_\nu$; $\{T^n(x)\}_{n\in\mathbb{N}}$ has a subsequence $\{T^{ni}(x)\}_{n\in\mathbb{N}}$ ν-converges to u. Then u is a unique fixed point.

Proof. ν-continuity of T at u implies that $\{T^{ni+1}(x)\}_{n\in\mathbb{N}}$ ν-convergent to $T(u) = u$. Suppose $T(u) \ne u$, consider the two ν-open balls $B_1 = B(u, \varepsilon)$ and $B_2 = B(Tu, \varepsilon)$ where $\varepsilon < \frac{1}{6} min\{\nu_\lambda(u, Tu, Tu), \nu_\lambda(Tu, u, u)\}$ for all $\lambda > 0$.
Since $T^{ni}(x) \longrightarrow u$ and $T^{ni+1}(x) \longrightarrow Tu$, then there exist $N_1 \in \mathbb{N}$ such that if $i > N_1$ implies $T^{ni}(x) \in B_1$ and $T^{ni+1}(x) \in B_2$. Hence our assumption implies that we must have

$$\nu_\lambda(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)) > \varepsilon \quad (i > N_1), \tag{3.1}$$

for all $\lambda > 0$. We have from (I1),

$$\begin{aligned}\nu_\lambda(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)) \le\ & a\nu_\lambda(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)) \\ & +b\nu_\lambda(T^{ni+1}(x), T^{ni+2}(x), T^{ni+2}(x)) \\ & +c\nu_\lambda(T^{ni+2}(x), T^{ni+3}(x), T^{ni+3}(x))\end{aligned}$$

for all $\lambda > 0$. By axioms of modular G-metric (V3), we have

$$\nu_\lambda(T^{ni+1}(x), T^{ni+2}(x), T^{ni+2}(x)) \le \nu_\lambda(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)), \tag{3.2}$$

$$\nu_\lambda(T^{ni+2}(x), T^{ni+3}(x), T^{ni+3}(x)) \le \nu_\lambda(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)), \tag{3.3}$$

for all $\lambda > 0$. Whence, from (3.2) and (3.3), we get

$$\nu_\lambda(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)) \le r\nu_\lambda(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)), \tag{3.4}$$

for all $\lambda > 0$ where $r = \frac{a}{(1-(b+c))}$ and $r < 1$, since $0 < a + b + c < 1$. On the other hand by inequality (3.2) and (3.4) we get

$$\nu_\lambda(T^{ni+1}(x), T^{ni+2}(x), T^{ni+2}(x)) \le r\nu_\lambda(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)), \tag{3.5}$$

for all $\lambda > 0$. For $k > j > N_1$ and by repeated application of (3.5) we have

$$\begin{aligned}\nu_\lambda(T^{n_k}(x), T^{n_k+1}(x), T^{n_k+1}(x)) &\le r\nu_\lambda(T^{n_k-1}(x), T^{n_k}(x), T^{n_k}(x)) \\ &\le r^2\nu_\lambda(T^{n_k-2}(x), T^{n_k-1}(x), T^{n_k-1}(x)) \\ &\le \cdots \\ &\le r^{n_k-n_j}\nu_\lambda(T^{n_j}(x), T^{n_j+1}(x), T^{n_j+1}(x)),\end{aligned}$$

for all $\lambda > 0$. Thus $lim_{k\longrightarrow\infty}\nu_\lambda(T^{n_k}(x), T^{n_k+1}(x), T^{n_k+1}(x)) = 0$ for all $\lambda > 0$, which contradict (3.1), hence $Tu = u$.
Suppose there is $w \in X_\nu$; $Tw = w$, then from (I1), we have

$$\nu_\lambda(u, w, w) = \nu_\lambda(Tu, Tw, Tw) \le a\nu_\lambda(u, Tu, Tu) + (b + c)\nu_\lambda(w, Tw, Tw) = 0,$$

for all $\lambda > 0$. This prove the uniqueness of u. □

Theorem 3.3. *Let (X,ν) be a ν-complete modular G-metric space and let $T : X_\nu \longrightarrow X_\nu$ be a mapping satisfies the following condition for all $x, y, z \in X_\nu$*

$$\begin{aligned}\nu_\lambda(Tx,Ty,Tz) \;\le\;& a\nu_\lambda(x,Tx,Tx)\\ &+b\nu_\lambda(y,Ty,Ty)+c\nu_\lambda(z,Tz,Tz)+d\nu_\lambda(x,y,z),\end{aligned} \tag{3.6}$$

for any $\lambda > 0$ where $0 \le a+b+c+d < 1$, then T has a unique fixed point, say u, and T is ν-continuous at u.

Proof. Let $x_0 \in X_\nu$ be an arbitrary point and define the sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_n = T^n(x_0)$. By inequality (3.6) we have

$$\nu_\lambda(x_n,x_{n+1},x_{n+1}) \le a\nu_\lambda(x_{n-1},x_n,x_n)+(b+c)\nu_\lambda(x_n,x_{n+1},x_{n+1})+d\nu_\lambda(x_{n-1},x_n,x_n),$$

for all $\lambda > 0$. Whence

$$\nu_\lambda(x_n,x_{n+1},x_{n+1}) \le \frac{a+d}{1-(b+c)}\nu_\lambda(x_{n-1},x_n,x_n),$$

for all $\lambda > 0$. Let $r = \frac{a+d}{1-(b+c)}$ then $0 \le r < 1$ since $0 \le a+b+c+d < 1$. So

$$\nu_\lambda(x_n,x_{n+1},x_{n+1}) \le r\nu_\lambda(x_{n-1},x_n,x_n)$$

, for all $\lambda > 0$. Continuing in the same argument, we will get

$$\nu_\lambda(x_n,x_{n+1},x_{n+1}) \le r^n\nu_\lambda(x_{n-1},x_n,x_n)$$

, for all $\lambda > 0$. Moreover for all $n, m \in \mathbb{N}$; $n < m$ we have by axiom (V5)

$$\begin{aligned}\nu_\lambda(x_n,x_m,x_m) &\le \nu_{\frac{\lambda}{m-n}}(x_n,x_{n+1},x_{n+1})+\nu_{\frac{\lambda}{m-n}}(x_{n+1},x_{n+2},x_{n+2})\\ &\quad+\nu_\lambda(x_{n+2},x_{n+3},x_{n+3})+\cdots+\nu_{\frac{\lambda}{m-n}}(x_{m-1},x_m,x_m)\\ &\le (r^n+r^{n+1}+\cdots+r^{m-1})\nu_\lambda(x_0,x_1,x_1)\\ &\le \frac{r^n}{1-r}\nu_\lambda(x_0,x_1,x_1),\end{aligned}$$

for all $\lambda > 0$. Hence $\nu_\lambda(x_n,x_m,x_m) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\lambda > 0$. Thus $\{x_n\}_{n\in\mathbb{N}}$ is ν-cauchy sequence. Due to the completeness of X_ν there exists $u \in X_\nu$ such that $\{x_n\}_{n\in\mathbb{N}}$ is ν-converge to u. Suppose that $Tu \ne u$, then

$$\nu_\lambda(x_n,Tu,Tu) \le a\nu_\lambda(x_{n-1},x_n,x_n)+(b+c)\nu_\lambda(u,Tu,Tu)+d\nu_\lambda(x_{n-1},u,u),$$

for all $\lambda > 0$. Taking the limit as $n \longrightarrow \infty$ then $\nu_\lambda(u,Tu,Tu) \le (b+c)\nu_\lambda(u,Tu,Tu)$ for all $\lambda > 0$. This is contradiction implies that $Tu = u$. To prove uniqueness, suppose $u \ne w$ such that $Tw = w$, then

$$\begin{aligned}\nu_\lambda(u,w,w) &\le a\nu_\lambda(u,Tu,Tu)+(b+c)\nu_\lambda(w,Tw,Tw)+d\nu_\lambda(u,w,w)\\ &= d\nu_\lambda(u,w,w),\end{aligned}$$

for all $\lambda > 0$ which implies that $u = w$. To show that T is ν-continuous at u, let $\{y_n\}_{n\in\mathbb{N}} \subseteq X_\nu$ be a sequence such that $\lim_{n\longrightarrow\infty} y_n = u$. We can deduce that

$$\begin{aligned}\nu_\lambda(u,Ty_n,Ty_n) &\le a\nu_\lambda(u,Tu,Tu)+(b+c)\nu_\lambda(y_n,Ty_n,Ty_n)+d\nu_\lambda(u,y_n,y_n)\\ &= (b+c)\nu_\lambda(y_n,Ty_n,Ty_n)+d\nu_\lambda(u,y_n,y_n)\end{aligned}$$

and since $\nu_\lambda(y_n,Ty_n,Ty_n) \le \nu_{\frac{\lambda}{2}}(y_n,u,u)+\nu_{\frac{\lambda}{2}}(u,Ty_n,Ty_n)$, for all $\lambda > 0$. We have that

$$\begin{aligned}\nu_\lambda(u,Ty_n,Ty_n)-(b+c)\nu_\lambda(u,Ty_n,Ty_n) &\le \nu_\lambda(u,Ty_n,Ty_n)-(b+c)\nu_{\frac{\lambda}{2}}(u,Ty_n,Ty_n)\\ &\le (b+c)\nu_{\frac{\lambda}{2}}(y_n,u,u)+d\nu_\lambda(u,y_n,y_n)\end{aligned}$$

for all $\lambda > 0$, whence

$$\nu_\lambda(u, Ty_n, Ty_n) \leq \frac{(b+c)}{1-(b+c)}\nu_{\frac{\lambda}{2}}(y_n, u, u) + \frac{d}{1-(b+c)}\nu_\lambda(u, y_n, y_n),$$

for all $\lambda > 0$. Taking the limit as $n \longrightarrow \infty$ from which we see that $\nu_\lambda(u, Ty_n, Ty_n) \longrightarrow 0$ and so by definition ν-continuous $Ty_n \longrightarrow u = Tu$. If is proved that T is ν-continuous at u. □

We see that if we take $d = 0$, the following theorem becomes a direct result.

Theorem 3.4. *Let (X, ν) be a ν-complete modular G-metric space and let $T : X_\nu \longrightarrow X_\nu$ be a mapping satisfies for all $x, y, z \in X_\nu$*

$$\nu_\lambda(Tx, Ty, Tz) \leq a\nu_\lambda(x, Tx, Tx) + b\nu_\lambda(y, Ty, Ty) + c\nu_\lambda(z, Tz, Tz),$$

for any $\lambda > 0$ where $0 < a + b + c < 1$, then T has a unique fixed point, say u, and T is ν-continuous at u.

The following examples support that condition (I2) and (I3) in theorem 3.2 do not guarantee the completeness of the modular G-metric space.

Example 3.5. Let $X = [0, 1)$, $\lambda \in (0, \infty)$, $T(x) = \frac{x}{4}$ and $\nu_\lambda(x, y, z) = \frac{G(x,y,z)}{\lambda}$ such that $G(x, y, z) = max\{|x - y|, |y - z|, |x - z|\}$. Then (X, ν) is modular G-metric space but not complete, since the sequence $x_n = 1 - \frac{1}{n}$ is ν-cauchy which is not ν-convergent in (X, ν). However, condition (I2) and (I3) in theorem 3.2 are satisfied.

Theorem 3.6. *Let (X, ν) be a modular G-metric space and let $T : X_\nu \longrightarrow X_\nu$ be a G-continuous mapping satisfies the following conditions:*
(II1) $\nu_\lambda(Tx, Ty, Tz) \leq k\{\nu_\lambda(x, Tx, Tx) + \nu_\lambda(y, Ty, Ty) + \nu_\lambda(z, Tz, Tz)\}$ for all $x, y, z \in M$ and $\lambda > 0$ where M is an every where dense subset of X_ν (whit respect the topology of modular G-metric convergence) and $0 < k < \frac{1}{6}$,
(II2) there is $x \in X_\nu$; $\{T^n(x)\}_{n\in\mathbb{N}} \longrightarrow u$. Then u is a unique fixed point.

Proof. It is enough to show that condition (I1) in theorem 3.2 holds for any $x, y, z \in X_\nu$ and $\lambda > 0$.
Case 1: If $x, y, z \in X_\nu \backslash M$, let $\{x_n\}_n$, $\{y_n\}_n$, and $\{z_n\}_n$ be a sequences in M such that $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $z_n \longrightarrow z$. By axioms of modular G-metric (V5), we have

$$\nu_\lambda(Tx, Ty, Tz) \leq \nu_{\frac{\lambda}{2}}(Tx, Ty, Ty) + \nu_{\frac{\lambda}{2}}(Tz, Ty, Ty)$$

for all $\lambda > 0$, also

$$\nu_{\frac{\lambda}{2}}(Tz, Ty, Ty) \leq \nu_{\frac{\lambda}{4}}(Tz, Tz_n, Tz_n) + \nu_{\frac{\lambda}{8}}(Tz_n, Ty_n, Ty_n) + \nu_{\frac{\lambda}{8}}(Ty_n, Ty, Ty) \tag{3.7}$$

for any $\lambda > 0$ and by (II1), we get

$$\nu_{\frac{\lambda}{8}}(Tz_n, Ty_n, Ty_n) \leq k\{\nu_{\frac{\lambda}{8}}(z_n, Tz_n, Tz_n) + 2\nu_{\frac{\lambda}{8}}(y_n, Ty_n, Ty_n)\} \tag{3.8}$$

for all $\lambda > 0$, again by (V5) we have

$$\nu_{\frac{\lambda}{8}}(z_n, Tz_n, Tz_n) \leq \nu_{\frac{\lambda}{16}}(z_n, z, z) + \nu_{\frac{\lambda}{32}}(z, Tz, Tz) + \nu_{\frac{\lambda}{32}}(Tz, Tz_n, Tz_n), \tag{3.9}$$

$$\nu_{\frac{\lambda}{8}}(y_n, Ty_n, Ty_n) \leq \nu_{\frac{\lambda}{16}}(y_n, y, y) + \nu_{\frac{\lambda}{32}}(y, Ty, Ty) + \nu_{\frac{\lambda}{32}}(Ty, Ty_n, Ty_n), \tag{3.10}$$

for all $\lambda > 0$. So from (3.8), (3.9) and (3.10) we get

$$\begin{aligned}\nu_{\frac{\lambda}{2}}(Tz,Ty,Ty) &\leq \nu_{\frac{\lambda}{4}}(Tz,Tz_n,Tz_n)+\nu_{\frac{\lambda}{8}}(Ty_n,Ty,Ty)\\ &\quad +k\nu_{\frac{\lambda}{16}}(z_n,z,z)+k\nu_{\frac{\lambda}{32}}(Tz,Tz_n,Tz_n)+2k\nu_{\frac{\lambda}{16}}(y_n,y,y)\\ &\quad +2k\nu_{\frac{\lambda}{32}}(Ty,Ty_n,Ty_n)+k\nu_{\frac{\lambda}{32}}(z,Tz,Tz)+2k\nu_{\frac{\lambda}{32}}(y,Ty,Ty)\\ &\leq (1+k)\nu_{\frac{\lambda}{32}}(Tz,Tz_n,Tz_n)+\nu_{\frac{\lambda}{8}}(Ty_n,Ty,Ty)\\ &\quad +k\nu_{\frac{\lambda}{16}}(z_n,z,z)+2k\nu_{\frac{\lambda}{16}}(y_n,y,y)+2k\nu_{\frac{\lambda}{32}}(Ty,Ty_n,Ty_n)\\ &\quad +k\nu_{\frac{\lambda}{32}}(z,Tz,Tz)+2k\nu_{\frac{\lambda}{32}}(y,Ty,Ty)\end{aligned} \tag{3.11}$$

for all $\lambda > 0$, similarly we deduce that

$$\begin{aligned}\nu_{\frac{\lambda}{2}}(Tx,Ty,Ty) &\leq (1+k)\nu_{\frac{\lambda}{32}}(Tx,Tx_n,Tx_n)+\nu_{\frac{\lambda}{8}}(y_n,Ty,Ty)\\ &\quad +k\nu_{\frac{\lambda}{16}}(x_n,x,x)+2k\nu_{\frac{\lambda}{16}}(y_n,y,y)\\ &\quad +2k\nu_{\frac{\lambda}{32}}(Ty,Ty_n,Ty_n)+k\nu_{\frac{\lambda}{32}}(x,Tx,Tx)+2k\nu_{\frac{\lambda}{32}}(y,Ty,Ty)\end{aligned} \tag{3.12}$$

for all $\lambda > 0$. Hence, by inequality (3.11) and (3.12) we get

$$\begin{aligned}\nu_{\lambda}(Tx,Ty,Tz) &\leq \nu_{\frac{\lambda}{2}}(Tx,Ty,Ty)+\nu_{\frac{\lambda}{2}}(Tz,Ty,Ty)\\ &\leq \{(1+k)\nu_{\frac{\lambda}{32}}(Tx,Tx_n,Tx_n)+\nu_{\frac{\lambda}{8}}(y_n,Ty,Ty)\\ &\quad +k\nu_{\frac{\lambda}{16}}(x_n,x,x)+2k\nu_{\frac{\lambda}{16}}(y_n,y,y)+2k\nu_{\frac{\lambda}{32}}(Ty,Ty_n,Ty_n)\\ &\quad +k\nu_{\frac{\lambda}{32}}(x,Tx,Tx)+2k\nu_{\frac{\lambda}{32}}(y,Ty,Ty)\}\\ &\quad +\{(1+k)\nu_{\frac{\lambda}{32}}(Tz,Tz_n,Tz_n)+\nu_{\frac{\lambda}{8}}(Ty_n,Ty,Ty)\\ &\quad +k\nu_{\frac{\lambda}{16}}(z_n,z,z)+2k\nu_{\frac{\lambda}{16}}(y_n,y,y)+2k\nu_{\frac{\lambda}{32}}(Ty,Ty_n,Ty_n)\\ &\quad +k\nu_{\frac{\lambda}{32}}(z,Tz,Tz)+2k\nu_{\frac{\lambda}{32}}(y,Ty,Ty)\}\end{aligned}$$

for all $\lambda > 0$. Since T is ν-continuous as $n \longrightarrow \infty$ in the above inequality we obtain

$$\nu_{\lambda}(Tx,Ty,Tz) \leq k\left\{\nu_{\frac{\lambda}{32}}(x,Tx,Tx)+4\nu_{\frac{\lambda}{32}}(y,Ty,Ty)+\nu_{\frac{\lambda}{32}}(z,Tz,Tz)\right\}$$

for all $\lambda > 0$.

Case 2: If $x, y \in M$, $z \in X_\nu \backslash M$, let $\{z_n\}_n$ be a sequence in M such that $z_n \longrightarrow z$ then by (V5) we have

$$\nu_{\lambda}(Tx,Ty,Tz) \leq \nu_{\frac{\lambda}{2}}(Tx,Ty,Ty)+\nu_{\frac{\lambda}{2}}(Tz,Ty,Ty)$$

for all $\lambda > 0$. On the other hand by (II1) and (V5) we have

$$\nu_{\frac{\lambda}{2}}(Tx,Ty,Ty) \leq k\left\{\nu_{\frac{\lambda}{2}}(x,Tx,Tx)+2\nu_{\frac{\lambda}{2}}(y,Ty,Ty)\right\} \tag{3.13}$$

$$\nu_{\frac{\lambda}{2}}(Tz,Ty,Ty) \leq \nu_{\frac{\lambda}{4}}(Tz,Tz_n,Tz_n)+\nu_{\frac{\lambda}{4}}(Tz_n,Ty,Ty) \tag{3.14}$$

for all $\lambda > 0$. Again by (II1) and (V5) we have

$$\nu_{\frac{\lambda}{4}}(Tz_n,Ty,Ty) \leq k\left\{\nu_{\frac{\lambda}{4}}(z_n,Tz_n,Tz_n)+2\nu_{\frac{\lambda}{4}}(y,Ty,Ty)\right\} \tag{3.15}$$

and

$$\nu_{\frac{\lambda}{4}}(z_n,Tz_n,Tz_n) \leq \nu_{\frac{\lambda}{8}}(z_n,z,z)+\nu_{\frac{\lambda}{16}}(z,Tz,Tz)+\nu_{\frac{\lambda}{16}}(Tz,Tz_n,Tz_n) \tag{3.16}$$

for all $\lambda > 0$. By inequality (3.13), (3.14), (3.15) and (3.16) we get

$$\begin{aligned}\nu_{\lambda}(Tx,Ty,Tz) &\leq k\nu_{\frac{\lambda}{2}}(x,Tx,Tx)+2k\nu_{\frac{\lambda}{2}}(y,Ty,Ty)+k\nu_{\frac{\lambda}{8}}(z_n,z,z)+k\nu_{\frac{\lambda}{16}}(z,Tz,Tz)\\ &\quad +k\nu_{\frac{\lambda}{16}}(Tz,Tz_n,Tz_n)+\nu_{\frac{\lambda}{4}}(Tz,Tz_n,Tz_n)+2k\nu_{\frac{\lambda}{4}}(y,Ty,Ty)\end{aligned}$$

for all $\lambda > 0$. Since ν is nonincreasing function we have

$$\begin{aligned}\nu_\lambda(Tx,Ty,Tz) &\leq k\nu_{\frac{\lambda}{2}}(x,Tx,Tx)+2k\nu_{\frac{\lambda}{4}}(y,Ty,Ty)+k\nu_{\frac{\lambda}{8}}(z_n,z,z)+k\nu_{\frac{\lambda}{16}}(z,Tz,Tz)\\ &+k\nu_{\frac{\lambda}{16}}(Tz,Tz_n,Tz_n)\}+\nu_{\frac{\lambda}{4}}(Tz,Tz_n,Tz_n)+2k\nu_{\frac{\lambda}{4}}(y,Ty,Ty)\end{aligned}$$

for all $\lambda > 0$. Now letting $n \longrightarrow \infty$ in the inequality, we get

$$\begin{aligned}\nu_\lambda(Tx,Ty,Tz) &\leq k\left\{\nu_{\frac{\lambda}{2}}(x,Tx,Tx)+4\nu_{\frac{\lambda}{4}}(y,Ty,Ty)+\nu_{\frac{\lambda}{16}}(z,Tz,Tz)\right\}\\ &\leq k\left\{\nu_{\frac{\lambda}{32}}(x,Tx,Tx)+4\nu_{\frac{\lambda}{32}}(y,Ty,Ty)+\nu_{\frac{\lambda}{32}}(z,Tz,Tz)\right\}\end{aligned}$$

for all $\lambda > 0$.

Case 3: If $y \in M$ and $x,z \in X_\nu \backslash M$, let $\{x_n\}$ and $\{z_n\}$ be a sequences in M such that $x_n \longrightarrow x$ and $z_n \longrightarrow z$, but by (V5) we have

$$\nu_\lambda(Tx,Ty,Tz) \leq \nu_{\frac{\lambda}{2}}(Tx,Ty,Ty)+\nu_{\frac{\lambda}{2}}(Tz,Ty,Ty) \tag{3.17}$$

$$\nu_{\frac{\lambda}{2}}(Tx,Ty,Ty) \leq \nu_{\frac{\lambda}{4}}(Tx,Tx_n,Tx_n)+\nu_{\frac{\lambda}{4}}(Tx_n,Ty,Ty) \tag{3.18}$$

for all $\lambda > 0$. Also from (II1) and (V5) we have

$$\nu_{\frac{\lambda}{4}}(Tx_n,Ty,Ty) \leq k\{\nu_{\frac{\lambda}{4}}(x_n,Tx_n,Tx_n)+2\nu_{\frac{\lambda}{4}}(y,Ty,Ty)\} \tag{3.19}$$

$$\nu_{\frac{\lambda}{4}}(x_n,Tx_n,Tx_n) \leq \nu_{\frac{\lambda}{8}}(x_n,x,x)+\nu_{\frac{\lambda}{16}}(x,Tx,Tx)+\nu_{\frac{\lambda}{16}}(Tx,Tx_n,Tx_n) \tag{3.20}$$

for all $\lambda > 0$. So, by (3.19) and (3.20), we have

$$\begin{aligned}\nu_{\frac{\lambda}{4}}(Tx_n,Ty,Ty) &\leq k\nu_{\frac{\lambda}{8}}(x_n,x,x)+k\nu_{\frac{\lambda}{16}}(x,Tx,Tx)\\ &+k\nu_{\frac{\lambda}{16}}(Tx,Tx_n,Tx_n)+2k\nu_{\frac{\lambda}{4}}(y,Ty,Ty)\end{aligned} \tag{3.21}$$

for all $\lambda > 0$. Then from (3.17) and (3.21) we have

$$\begin{aligned}\nu_{\frac{\lambda}{2}}(Tx,Ty,Ty) &\leq k\nu_{\frac{\lambda}{8}}(x_n,x,x)+k\nu_{\frac{\lambda}{16}}(x,Tx,Tx)\\ &+(1+k)\nu_{\frac{\lambda}{16}}(Tx,Tx_n,Tx_n)+2k\nu_{\frac{\lambda}{4}}(y,Ty,Ty)\end{aligned} \tag{3.22}$$

for all $\lambda > 0$. By similaly

$$\begin{aligned}\nu_{\frac{\lambda}{2}}(Tz,Ty,Ty) &\leq k\nu_{\frac{\lambda}{8}}(z_n,z,z)+k\nu_{\frac{\lambda}{16}}(z,Tz,Tz)\\ &+(1+k)\nu_{\frac{\lambda}{16}}(Tz,Tz_n,Tz_n)+2k\nu_{\frac{\lambda}{4}}(y,Ty,Ty)\end{aligned} \tag{3.23}$$

for all $\lambda > 0$. Then from (3.22) and (3.23), we get

$$\begin{aligned}\nu_\lambda(Tx,Ty,Tz) &\leq \nu_{\frac{\lambda}{2}}(Tx,Ty,Ty)+\nu_{\frac{\lambda}{2}}(Tz,Ty,Ty)\\ &\leq (1+k)\nu_{\frac{\lambda}{16}}(Tx,Tx_n,Tx_n)+2k\nu_{\frac{\lambda}{4}}(y,Ty,Ty)\\ &+k\nu_{\frac{\lambda}{8}}(x_n,x,x)+k\nu_{\frac{\lambda}{16}}(x,Tx,Tx)\\ &+(1+k)\nu_{\frac{\lambda}{16}}(Tz,Tz_n,Tz_n)+k\nu_{\frac{\lambda}{8}}(z_n,z,z)\\ &+k\nu_{\frac{\lambda}{16}}(z,Tz,Tz)+2k\nu_{\frac{\lambda}{4}}(y,Ty,Ty)\end{aligned}$$

for all $\lambda > 0$. Now letting $n \longrightarrow \infty$ in the above inequality and using the fact that T is ν-continuous, we get

$$\begin{aligned}\nu_\lambda(Tx,Ty,Tz) &\leq k\left\{\nu_{\frac{\lambda}{16}}(x,Tx,Tx)+4\nu_{\frac{\lambda}{4}}(y,Ty,Ty)+\nu_{\frac{\lambda}{16}}(z,Tz,Tz)\right\}\\ &\leq k\left\{\nu_{\frac{\lambda}{32}}(x,Tx,Tx)+4\nu_{\frac{\lambda}{32}}(y,Ty,Ty)+\nu_{\frac{\lambda}{32}}(z,Tz,Tz)\right\}\end{aligned}$$

for all $\lambda > 0$. So, in all case we have for any $x,y,z \in X_\nu$ and $\lambda > 0$

$$\nu_\lambda(Tx,Ty,Tz) \leq a\nu_{\frac{\lambda}{32}}(x,Tx,Tx)+b\nu_{\frac{\lambda}{32}}(y,Ty,Ty)+c\nu_{\frac{\lambda}{32}}(z,Tz,Tz)$$

where $a = k$, $b = 4k$, $c = k$ and $a + b + c < 1$ since $0 < k < \frac{1}{6}$ then by theorem , T has a unique fixed point. □

References

[1] V. V. Chistyakov, *Modular metric spaces I Basic concepts*, Nonlinear Anal. **72** (2010), 1-14. 1

[2] Y.J.E. Cho, R. Saadati and Gh. Sadeghi, *Quasi-Contractive Mappings in Modular Metric Spaces*, J. Appl. Math. Volume 2012, Article ID 907951, 5 pages.

[3] B.C. Dhage, *Generalized metric space and mapping with fixed point*, Bull. Calcutta. Math. **74503** (1992), 7 pages. 1

[4] S. Gahler, *2–metrische Räume und ihre topologische struktur*, Math. Nacher.. **26** (1966), 665-667. 1

[5] Z. Mustafa and B. Sims, *Some remarks concerning D–metric spaces*, Proceeding of the International Conference on Fixed Point Theory and Applications, Valencia (spain), July (2003), 198–198. 1

[6] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces1*, J. Nonlinear and convex Anal. Volume 7, Number 2, 2006, 289–297. 1

[7] Z. Mustafa, W. Shatanawi and M. Bataineh, *Existence of fixed point results in G-metric spaces*, International. J. Math. Math. Sci., Volume 2009, Article ID 283028, 10 pages.

[8] Z. Mustafa, M. Khandaqji and W. Shatanawi, *Fixed point reasults on complete G-metric spaces*, Studia Sci. Math. Hungar., **48** (2011), 304–319.

[9] Z. Mustafa, H. Obiedat and F. Awawdeh, *Some fixed point theorem for mapping on complete G-metric spaces*, Fixed Point Theory and Application, vol. 2008, Article ID 189870, 12 pages, 2008.

[10] H. Nakano, *On the stability of functional equations*, Aequationes Math. **77** (2009), 33–88. 1

[11] J. Musielak, *Orlicz Spaces and Modular Spaces*, in: Lecture Notes in Math. Vol. **1034**, Springer–verlag, Berlin, 1983. 1

[12] W. Orlicz, *Collected Papers*, Vols. **I, II**, PWN, Warszawa, 1988. 1