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Wavelet packet transform on $L^p(\mathbb{R})$ spaces

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Abstract

In this paper, we study the characterization of $L^p(\mathbb{R})$ spaces by using wavelet packet coefficients. We also drive few results by using wavelet packet transform which generalize some results from the literature.

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1. Introduction

Wavelet theory developed from interdisciplinary origin in the last 30 years and the literature on wavelets and their applications (on data analysis, image compression and enhancement, filtering of signals and many others) is growing rapidly. Wavelet methods are refinement of Fourier analysis whose mathematical foundation was provided by Grossman et al. [12], Meyer [19], Mallat [17, 18], Daubechies [6, 7], Coifman et al. [5] and Wickerhauser [21].

Signals in practice are often band-limited where their frequency contents are restricted to prescribed bands. For such signals, it is natural to expand them in terms of band-limited functions such as band-limited wavelets. Band-limited refinable functions play a fundamental role in the construction of band-limited wavelets (for instance see [6, 16, 22]). Among others, contributions in the theory of band-limited refinable functions and wavelets are made in [1, 2, 9, 13, 15, 16, 20].

Band-limited orthonormal refinable functions and wavelet already well studied in the literature [2, 3, 9, 15, 16]. In order to make paper self contained, we state some basic preliminaries, notations and definitions.

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2. Preliminaries

Let $\langle .,. \rangle$ and $||.||$ be usual inner product and norm of the space $L^p(\mathbb{R})$. The space $L^p(\mathbb{R})$ the class of all measurable functions f, defined on $\mathbb{R}$ for which

$$\int_\Omega |f(x)|^p \, dx < \infty,$$

and the norm of $f \in L^p(\mathbb{R})$ is defined by

$$\|f\|_p := \begin{cases} \left\{ \int_{\mathbb{R}} |f(x)|^p dx \right\}^{1/p}, & \text{for } 1 \leq p < \infty \\ \text{ess} \sup_{-\infty<x<\infty} |f(x)|, & \text{for } p = \infty. \end{cases}$$

We define the inner product of functions $f, g \in L^2(\mathbb{R})$ as $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$, the Fourier transform of $f \in L^2(\mathbb{R})$ as $\hat{f}(w) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iwx}\, dx$ and the relationship between functions and their Fourier transform as $2\pi\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$. For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the Fourier transform $\hat{f}$ of f is in $L^2(\mathbb{R})$ and satisfies the Parseval identity $\|\hat{f}\|_2^2 = 2\pi\|f\|_2^2$.

Definition 2.1 ([8])**.** A system of elements $\{f_n\}_{n\in\Lambda}$ in a Hilbert space H is called a frame for H if there exists two positive numbers A and B such that for any $f \in H$,

$$A\, \|f\|^2 \;\leq\; \sum_{n\in\Lambda} |\langle f, f_n\rangle|^2 \;\leq\; B\, \|f\|^2.$$

The numbers A and B are called frame bounds. If $A = B$, the frame is said to be tight. The frame is called exact if it ceases to be a frame whenever any single element is deleted from the frame.

The connection between frames and numerically stable reconstruction from discretized wavelet coefficients was pointed out by Grossmann et al. [12]. A wavelet function $\psi \in L^2(\mathbb{R})$, also constitutes a wavelet frame with frame bounds A and B, if for any $f \in L^2(\mathbb{R})$,

$$A\, \|f\|_2^2 \;\leq\; \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 \;\leq\; B\, \|f\|_2^2,$$

where A and B are some positive numbers and $\psi_{j,k}(x) = 2^{j/2}\psi(2^jx - k), \quad j, k \in \mathbb{Z}$. Again, if $A = B$, the frame is said to be tight.

The continuous wavelet transformation of a L^2-function f with respect to the wavelet ψ, satisfying admissibility condition, is defined as,

$$(T^{wav}f)(a, b) \;=\; |a|^{-1/2} \int_{-\infty}^{\infty} f(t)\, \overline{\psi\left(\frac{t-b}{a}\right)} dt, \quad a, b \in \mathbb{R}, a \neq 0.$$

The term wavelet denotes a family of functions of the form $\psi_{a,b} = |a|^{-1/2}\ \psi\left(\frac{t-b}{a}\right)$, obtained from a single function ψ by the operations of dilation and translation. The wavelet coefficients at discrete points $a = \frac{k}{2^j}$, $b = \frac{1}{2^j}$, are given by $\psi_{j,k} = (T^{wav}f)(\frac{k}{2^j}, \frac{1}{2^j})$.

Wavelet Packets

Though the wavelet basis has good localization in time-frequency domain but in order to get better localization for high frequency components in the wavelet decomposition, Coifman et al. [5] introduced another kinds of bases called wavelet packets. We have the following sequence of functions due to Wickerhauser [21]. For $l = 0, 1, 2, ...$,

$$\psi_{2l}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} a_k\psi_l(2x-k) \quad \text{and} \quad \psi_{2l+1}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} b_k\psi_l(2x-k), \tag{2.1}$$

where $\{a_k\}$ is the filter such that $\sum_{n\in\mathbb{Z}} a_{n-2k}a_{n-2l} = \delta_{kl}$, $\sum_{n\in\mathbb{Z}} a_n = \sqrt{2}$ and $b_k = (-1)^k a_{1-k}$. For $l = 0$ in (2.1), we get

$$\psi_0(x) \;=\; \psi_0(2x) + \psi_0(2x-1), \quad \psi_1(x) \;=\; \psi_0(2x) - \psi_0(2x-1),$$

where ψ_0 is a scaling function and may be taken as a characteristic function. If we increase l, we get the following

$$\psi_2(x) \;=\; \psi_1(2x) + \psi_1(2x-1), \quad \psi_3(x) \;=\; \psi_1(2x) - \psi_1(2x-1)$$
$$\psi_4(x) \;=\; \psi_1(4x) + \psi_1(4x-1) + \psi_1(4x-2) + \psi_1(4x-3)$$

and so on.

Here ψ_l's have a fixed scale but different frequencies. Actually, they are Walsh functions in [0,1[. The functions $\psi_l(t-k)$, for integers k, l with $l \geq 0$, form an orthonormal basis of $L^2(\mathbb{R})$.

Theorem 2.2 ([21])**.** *For every partition P of the non negative integers into the sets of the form $I_{lj} = \{2^j l, \cdots, 2^j(l+1)-1\}$, the collection of functions $\psi_{l;jk}(x) = 2^{j/2}\psi_l(2^jx-k)$, $I_{lj} \in P, k \in \mathbb{Z}$, is an orthonormal basis of $L^2(\mathbb{R})$.*

The collection of functions gives rise to many bases including Walsh, wavelet and subband basis. A wavelet packet basis of $L^2(\mathbb{R})$ is an orthonormal basis selected from among the functions $\psi_{l,jk}$.

The trigonometric system $\{e^{imz} : m \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R}, \frac{dx}{2\pi})$. Therefore, a function $f \in L^2(\mathbb{R})$ if and only if its Fourier Coefficients

$$C_m = \int_{\mathbb{R}} f(x)e^{-imx}\frac{dx}{2\pi}, \quad m \in \mathbb{Z},$$

$$\text{to satisfy } \sum_{m\in\mathbb{Z}} |c_m|^2 < \infty,$$

and

$$||f||_{L^2(\mathbb{R},\frac{dx}{2\pi})} = \Big(\sum_{m\in\mathbb{Z}} |c_m|^2\Big)^{1/2}. \tag{2.2}$$

The Littlewood-Paley G-function (cf. [4, 11, 15]), is defined by

$$G(f)(x) = \Big(\int_0^1 (1-\gamma)\Big|(f * \frac{dP_\gamma}{d\gamma})(x)\Big|^2 d\gamma\Big)^{1/2} \tag{2.3}$$

where $P_\gamma(x) = \frac{1-\gamma^2}{1-2\gamma cosx+\gamma^2}$ is Poisson kernel for the unit disk.

The discrete version of Littlewood-Paley (cf. [11, 15]) function is defined by

$$g(f)(x) = \Big(\sum_{m\in\mathbb{Z}} |\phi_{2^{-m}} * f(x)|^2\Big)^{1/2}, \tag{2.4}$$

and these expressions can be used to characterize the spaces $L^p(\mathbb{R})$, $1 < p < \infty$.

We say that a function ϕ defined on $\mathbb{R}$ belongs to the regularity class S^0, if there exist constants $C_0, C_1, \gamma > 0$ and $\epsilon > 0$ such that

$$\begin{cases} (i)\ \int_{\mathbb{R}} \phi(x)dx = 0 \\ (ii)\ |\phi(x)| \leq \frac{C_0}{(1+|x|)^{1+\gamma}}, \quad \text{for all} \quad x \in \mathbb{R} \\ (iii)\ |\phi^{'}(x)| \leq \frac{C_1}{(1+|x|)^{1+\epsilon}}, \quad \text{for all} \quad x \in \mathbb{R}. \end{cases} \tag{2.5}$$

For any function g defined on $\mathbb{R}$ and for a real number $\lambda > 0$, we consider the maximal function [10],

$$g_\lambda^*(x) = \sup_{y\in\mathbb{R}} \frac{|g(x-y)|}{(1+|y|)^\lambda}, \qquad x \in \mathbb{R}. \tag{2.6}$$

The Hardy-Littlewoood maximal function $Mf(x)$ (cf. [14]) is given by

$$Mf(x) = \sup_{\gamma>0} \frac{1}{2\gamma} \int_{|y-x|\leq\gamma} |f(y)|dy \tag{2.7}$$

for a locally integrable function f on $\mathbb{R}$ and M is bounded on $L^p(\mathbb{R})$, $\ 1 < p \leq \infty$.

Theorem 2.3 ([11, 15]). *Suppose* $1 < p,\ q < \infty$, *there exists a constant* $A_{p,q}$ *such that*

$$\Big|\Big|\Big\{ \sum_{i=1}^{\infty} (Mf_i)^q \Big\}^{1/q} \Big|\Big|_{L^p(\mathbb{R})} \leq A_{p,q} \Big|\Big|\Big\{ \sum_{i=1}^{\infty} |f_i|^q \Big\}^{1/q} \Big|\Big|_{L^p(\mathbb{R})}$$

for any sequence $\{f_i : i = 1, 2, ...\}$ *of totally integrable functions.*

Lemma 2.4 ([10, 15]). *Let* ϕ *be band limited,* $f \in S^{'}$ *and* $0 < p \leq \infty$ *such that* $\phi_{2^{-j}} * f \in L^p(\mathbb{R})$ *for all* $j \in \mathbb{Z}$. *Then for any real* $\lambda > 0$, *there exists a constant* $C_\lambda > 0$ *such that*

$$(\phi_{j,\lambda}^{**} f)(x) = C_\lambda \Big\{ M(|\phi_{2^{-j}} * f|^{1/\lambda})(x) \Big\}^\lambda, \quad x \in \mathbb{R},$$

where $(\phi_{j,\lambda}^{**} f)(x) = sup_{y\in\mathbb{R}} \frac{|(\phi_{2^{-j}} * f)(x-y)|}{(1+2^j|y|)^\lambda}$ *and* $\phi_t(x) = \frac{1}{t}\phi(\frac{x}{t})$.

Theorem 2.5 ([10, 15]). *Let* $\phi \in S^0$ *be a band limited function. Given a real number* $\lambda \geq 1$ *and* $1 < p < \infty$, *there exists a constant* $A_{p,\lambda} < \infty$ *such that*

$$\Big|\Big|\Big\{ \sum_{j\in\mathbb{Z}} |\phi_{j,\lambda}^{**} f|^2 \Big\}^{1/2} \Big|\Big|_{L^p(\mathbb{R})} \leq A_{p,\lambda} ||f||_{L^p(\mathbb{R})}, \quad \forall f \in L^p(\mathbb{R}).$$

Remark 2.6. Assuming $\phi \in S^0$ and ϕ is band limited. We assume that ϕ satisfies

$$\sum_{l} \sum_{j\in\mathbb{Z}} |\hat{\phi}_l(2^j\xi)|^2 = M \quad \text{for a.e. } \xi \in \mathbb{R}, \quad l = 1, 2, ..., n.$$

The characterization of $L^p(\mathbb{R}), 1 < p < \infty$, in terms of the function $\phi_{l;j,\lambda}^{***} f$ i.e.

$$C||f||_{L^p(\mathbb{R})} \leq ||\Big\{ \sum_{l} \sum_{j\in\mathbb{Z}} |\phi_{l;j,\lambda}^{***} f|^2 \Big\}^{1/2} ||_{L^p(\mathbb{R})} \leq D||f||_{L^p(\mathbb{R})},$$

with C and D depending on p and λ and $\lambda \geq 1$.

Given two functions f and ψ_l, $l = 1, 2, ..., n$ for which $\langle f, \psi_l\rangle$ makes sense, we define

$$(W_{\psi_l} f)(x) = \Big\{ \sum_{l} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |\langle f, \psi_{l;j,m}\rangle|^2 2^j \chi_{[2^{-j}m, 2^{-j}(m+1)]}(x) \Big\}^{1/2}. \tag{2.8}$$

We also define the observe operator $T\psi_l$, $l = 1, 2, ..., n$, is the operator mapping f into the $l^2(\mathbb{Z}\times\mathbb{Z})$ valued function,

$$(T\psi_l)(x) = \Big\{ \langle f, \psi_{l;j,m}\rangle 2^{j/2} \chi_{[2^{-j}m, 2^{-j}(m+1)]}(x), \quad j \in \mathbb{Z}, m \in \mathbb{Z},\ l = 1, 2, ..., n \Big\}.$$

Then we have,

$$W_{\psi_l} f = \sqrt{(T\psi_l)(x) \cdot (T\psi_l)(x)},$$

where $\cdot$ denotes the dot product in $l^2(\mathbb{Z}\times\mathbb{Z})$.

3. Main Results

Theorem 3.1. *Let $\psi_l \in S^0$ $(l = 1, 2, ..., n)$ be a band limited function. For $1 < p < \infty$ and $f \in L^p(\mathbb{R})$, we have,*

$$\Big|\Big|\Big\{ \sum_{l} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |\langle f, \psi_{l;j,m}\rangle|^2 2^j \chi_{[2^{-j}m, 2^{-j}(m+1)]} \Big\}^{1/2}\Big|\Big|_{L^p(\mathbb{R})} \leq C||f||_{L^p(\mathbb{R})}, \tag{3.1}$$

with C independent of f.

Proof. We start by noticing that for $f \in L^p(\mathbb{R})$, the number $\langle f, \psi_{l;j,k}\rangle$ make sense, since $\psi_l \in L^{p'}(\mathbb{R})$. In fact

$$|\langle f, \psi_{l;j,m}\rangle| \leq 2^{j(\frac{1}{p}-\frac{1}{2})} ||\psi_l||_{L^{p'}(\mathbb{R})} ||f||_{L^p(\mathbb{R})}.$$

We have

$$\begin{aligned} |\langle f, \psi_{l;j,m}\rangle| &= 2^{j/2} |\textstyle\int_{\mathbb{R}} f(x)\overline{\psi_l(2^j x - m)} dx| \\ &= 2^{-j/2} |\textstyle\int_{\mathbb{R}} f(x)\overline{\psi_{l;2^{-j}}(x - 2^{-j}m)} dx| \\ &= 2^{-j/2} |(\tilde{\psi}_{l;2^{-j}} * f)(2^{-j}m)| \\ &\leq 2^{-j/2} \sup_{y\in I_{j,m}} |(\tilde{\psi}_{l;2^{-j}} * f)(y)| \end{aligned}$$

where $I_{j,m} = [2^{-j}m, 2^{-j}(m+1)]$ and $\tilde{\psi}_l(y) = \overline{\psi_l(-y)}$. Fixing $j \in \mathbb{Z}$, we have,

$$\begin{aligned} &\textstyle\sum_l \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |\langle f, \psi_{l;j,m}\rangle|^2 2^j \chi_{[2^{-j}m, 2^{-j}(m+1)]}(x) \\ &\leq \textstyle\sum_l \sum_j \sum_m \Big\{ \sup_{y\in I_{j,m}} |(\tilde{\psi}_{l;2^{-j}} * f)(y)| \Big\}^2 \chi_{I_{j,m}}(x) \\ &\leq \Big\{ \sup_{|z|\leq 2^{-j}} |(\tilde{\psi}_{l;2^{-j}} * f)(x - z)| \Big\}^2 \\ &= \Big\{ \sup_{|z|\leq 2^{-j}} \tfrac{|(\hat{\psi}_{l;2^{-j}} * f)(x-z)|}{(1+2^j|z|)^\lambda} \Big\}^2 (1 + 2^j|z|)^{2\lambda} \\ &\leq 2^{2\lambda} [(\hat{\psi}^{**}_{l;2^{-j}} f)(x)]^2, \end{aligned}$$

for any $\lambda > 0$. Inequality (3.1), now follows by applying Theorem 2.5 with $\lambda \geq 1$. □

Theorem 3.2. *Let $\psi_l \ (l = 1, 2, ..., n)$ be a band limited wavelet packet such that $\psi_l \in S^0$. Given $p \in (1, \infty)$, there exist two constants $0 < A_p \leq B_p < \infty$ such that*

$$A_p ||f||_{L^p(\mathbb{R})} \leq ||W_{\psi_l} f||_{L^p(\mathbb{R})} \leq B_p ||f||_{L^p(\mathbb{R})} \tag{3.2}$$

for all $f \in L^p(\mathbb{R})$.

Proof. We first observe that the coefficients $\langle f, \psi_{l;j,m} \rangle$ are well defined since $\psi_l \in L^{p'}(\mathbb{R})$, where $\frac{1}{p'} + \frac{1}{p} = 1$. The inequality in the right of (3.2) is precisely inequality (3.1).

Thus, we have already obtained a constant $B_p < \infty$ such that

$$||W_{\psi_l} f||_{L^p(\mathbb{R})} \leq B_p ||f||_{L^p(\mathbb{R})}. \tag{3.3}$$

For $p = 2$, we have equality (with $B_p = 1$) because ψ_l is an orthonormal wavelet packet;

$$\begin{aligned} &\textstyle\int_{\mathbb{R}} (T_{\psi_l} f)(x).(T_{\psi_l} f)(x) dx \\ &= ||W_{\psi_l} f||^2_{L^2(\mathbb{R})} \\ &= \textstyle\int_{\mathbb{R}} \sum_l \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |\langle f, \psi_{l;j,m} \rangle|^2 2^j \chi_{j,m}(x) dx \\ &= \textstyle\sum_l \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |\langle f, \psi_{l;j,m} \rangle|^2 \\ &= ||f||^2_{L^2(\mathbb{R})}, \end{aligned}$$

where $I_{j,m} = [2^{-j} m, 2^{-j}(m + 1)]$. From this equality the polarization identity and a density argument we obtain

$$\int_{\mathbb{R}} f(x) g(x) dx = \int_{\mathbb{R}} (T_{\psi_l} f)(x).(T_{\psi_l} g)(x) dx,$$

for $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R})$, where p' is the conjugate exponent to p. Now using the duality argument, together with Hölder's inequality and (3.3) for $L^{p'}(\mathbb{R})$, we deduce

$$\begin{aligned} ||f||_{L^p(\mathbb{R})} &= \sup_{||g||_{p'} \leq 1} |\textstyle\int_{\mathbb{R}} f(x) g(x) dx| \\ &\leq \sup_{||g||_{p'} \leq 1} ||W_{\psi_l} f||_{L^p(\mathbb{R})}.||W_{\psi_l} g||_{L^{p'}(\mathbb{R})} \\ &\leq B_{p'} ||\psi_l f||_{L^p(\mathbb{R})}. \end{aligned}$$

□

Remark 3.3. Our results generalize the relevant results of Dziubański and Hernández [9] and Hernández [16].

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