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Fixed points for Geraghty-contractions in partial metric spaces

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Abstract

We establish some fixed point theorems for mappings satisfying Geraghty-type contractive conditions in the setting of partial metric spaces and ordered partial metric spaces. Presented theorems extend and generalize many existing results in the literature. Examples are given showing that these results are proper extensions of the existing ones.

Keywords: Coincidence point, partial metric space, ordered partial metric space, Geraghty-type contractive condition, fixed point.
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1. Introduction and Preliminaries

Banach's contraction principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics.

Definition 1.1. Let S denotes the class of the functions $\beta : [0, +\infty) \to [0, 1)$ which satisfy the condition $\beta(t_n) \to 1 \Rightarrow t_n \to 0$.

The following generalization of Banach's contraction principle, proved in 1973, is due to Geraghty [18].

Theorem 1.2. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Assume that there exists $\beta \in S$ such that, for all $x, y \in X$,*

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y).$$

Then f has a unique fixed point $z \in X$ and, for any choice of the initial point $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = f(x_{n-1})$ for each $n \geq 1$ converges to the point z.

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Very recently, Amini-Harandi and Emami [4] proved the following existence theorem.

Theorem 1.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be an increasing mapping such that there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\beta \in S$ such that*

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y)$$

for all $x, y \in X$ with $x \succeq y$. Assume that either f is continuous or X is such that if an increasing sequence $\{x_n\}$ converges to x, then $x_n \preceq x$ for each $n \geq 1$. Besides, if for all $x, y \in X$, there exists $z \in X$ which is comparable to x and y, then f has a unique fixed point in X.

In the mathematical field of domain theory, attempts were made in order to equip semantics domain with a notion of distance. In particular, Matthews [27] introduced the notion of a partial metric space as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle can be generalized to the partial metric context for applications in program verification. Moreover, the existence of several connections between partial metrics and topological aspects of domain theory have been lately pointed by other authors as O'Neill [28], Bukatin and Scott [9], Bukatin and Shorina [10], Romaguera and Schellekens [39] and others (see also [19, 25, 26, 38, 41, 42] and the references therein).

After the result of Matthews [27], the interest for fixed point theory developments in partial metric spaces has been constantly growing. Indeed, many authors presented significant contributions in the directions of establishing partial metric versions of well-known fixed point theorems in classical metric spaces (see for example [8, 12, 14, 43]). Obviously, we cannot cite all these papers but we give only a partial list [1]-[7], [11, 13, 15, 16], [20]-[24], [32, 33, 36, 37, 44].

2. Partial metric spaces

The following definitions and details can be seen in [27, 28, 32, 33].

Definition 2.1. A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that, for all $x, y, z \in X$

(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p2) $p(x, x) \leq p(x, y)$,

(p3) $p(x, y) = p(y, x)$,

(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then from (p1) and (p2) follows $x = y$. But if $x = y$, $p(x, y)$ may not be 0. Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$, with respect to τ_p if $\lim_{n\to+\infty} p(x_n, x) = p(x, x)$. This will be denoted by $x_n \to x$, as $n \to +\infty$ or $\lim_{n\to+\infty} x_n = x$.
If p is a partial metric on X, then the function $p^s : X \times X \to [0, +\infty)$ given by:

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{2.1}$$

is a metric on X. Furthermore, $\lim_{n\to+\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n,m\to+\infty} p(x_n, x_m). \tag{2.2}$$

Example 2.2. A basic example of a partial metric space is the pair $([0,+\infty),p)$, where $p(x,y)=\max\{x,y\}$ for all $x,y\in[0,+\infty)$. The corresponding metric is:

$$p^s(x,y)=2\max\{x,y\}-x-y=|x-y|.$$

Example 2.3. If (X,d) is a metric space and $c\geq 0$ is arbitrary, then

$$p(x,y)=d(x,y)+c$$

defines a partial metric on X and the corresponding metric is $p^s(x,y)=2d(x,y)$.

Other examples of partial metric spaces which are interesting from a computational point of view may be found in [27].

Remark 2.4. Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot,\cdot)$ need not be continuous in the sense that $x_n\to x$ and $y_n\to y$ implies $p(x_n,y_n)\to p(x,y)$. For example, if $X=[0,+\infty)$ and $p(x,y)=\max\{x,y\}$ for $x,y\in X$, then for $\{x_n\}=\{1\}$, $p(x_n,x)=x=p(x,x)$ for each $x\geq 1$ and so, for example, $x_n\to 2$ and $x_n\to 3$ when $n\to+\infty$.

Definition 2.5. Let (X,p) be a partial metric space. Then one has the following:

(i) A sequence $\{x_n\}$ in (X,p) is called a Cauchy sequence if $\lim_{n,m\to\ +\infty}p(x_n,x_m)$ exists (and is finite).

(ii) The space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x\in X$ such that

$$p(x,x)=\lim_{n\to\ +\infty}p(x_n,x)=\lim_{n,m\to\ +\infty}p(x_n,x_m).$$

Lemma 2.6 ([27, 32])**.** *Let (X,p) be a partial metric space. Then one has the following:*

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(b) The space (X,p) is complete if and only if the metric space (X,p^s) is complete.

Definition 2.7. Let X be a nonempty set. Then, $(X,p,\preceq)$ is called an ordered partial metric space if:

(i) (X,p) is a partial metric space,

(ii) $(X,\preceq)$ is a partially ordered set.

3. Main results

We begin with the following auxiliary lemmas which are useful to prove some fixed point theorems in a partial metric space.

Lemma 3.1 ([33], Lemma 2)**.** *Let (X,p) be a partial metric space and $\{x_n\}\subset X$. If $x_n\to x\in X$ and $p(x,x)=0$, then $\lim\limits_{n\to+\infty}p(x_n,z)=p(x,z)$ for all $z\in X$.*

Lemma 3.2 ([17], Lemma 2.8)**.** *Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X such that:*

$$\lim_{n\to+\infty}p(x_{n+1},x_n)=0.$$

If $\{x_{2n}\}$ is not a Cauchy sequence in (X,p), then there exist $\epsilon>0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers, with $m_k<n_k$, such that the following four sequences:

$$\{p(x_{2m_k},x_{2n_k})\},\quad \{p(x_{2m_k},x_{2n_k+1})\},\quad \{p(x_{2m_k-1},x_{2n_k})\},\quad \{p(x_{2m_k-1},x_{2n_k+1})\}$$

tend to ϵ as $k\to+\infty$.

Definition 3.3 ([40], Definition 2.2)**.** Let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. The mapping f is α-admissible if for all $x, y \in X$ such that $\alpha(x, y) \geq 1$, we have $\alpha(fx, fy) \geq 1$.

Definition 3.4. Let (X, p) be a partial metric space and let $\alpha : X \times X \to [0, +\infty)$. X is called α-regular if for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.

The following theorem is one of our main results.

Theorem 3.5. *Let (X, p) be a complete partial metric space and let $\alpha : X \times X \to [0, +\infty)$ be a function. Let $f : X \to X$ be a self mapping. Suppose that there exists $\beta \in S$ such that*

$$\alpha(x, fx)\alpha(y, fy)p(fx, fy) \leq \beta(M(x, y))M(x, y) \tag{3.1}$$

for all $x, y \in X$, where

$$M(x, y) = \max\left\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}\left[p(x, fy) + p(fx, y)\right]\right\}.$$

Assume also that the following conditions hold:

(i) f is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, we have $\alpha(x, fx) \geq 1$;

(iv) $\alpha(x, fx) \geq 1$ for all $x \in Fix(f)$.

Then f has a unique fixed point z in X.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by

$$x_n = fx_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Since, by hypothesis, f is α-admissible, we obtain

$$\alpha(fx_0, fx_1) = \alpha(x_1, x_2) \geq 1, \quad \alpha(fx_1, fx_2) = \alpha(x_2, x_3) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x_n = x_{n+1} = fx_n$ and so x_n is a fixed point of f. Now, we assume $p(x_{n+1}, x_n) > 0$ for each $n \in \mathbb{N} \cup \{0\}$. First, we will prove that the sequence $\{p(x_{n+1}, x_n)\}$ is decreasing and tends to 0 as $n \to +\infty$. By (3.1), for each $n \in \mathbb{N}$, we have:

$$\begin{aligned} p(x_{n+2}, x_{n+1}) &= p(fx_{n+1}, fx_n) \\ &\leq \alpha(x_{n+1}, fx_{n+1})\alpha(x_n, fx_n)p(fx_{n+1}, fx_n) \\ &\leq \beta(M(x_{n+1}, x_n))M(x_{n+1}, x_n) \\ &< M(x_{n+1}, x_n), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &M(x_{n+1}, x_n) \\ &= \max\left\{p(x_{n+1}, x_n), p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1}), \frac{1}{2}\left[p(x_{n+1}, x_{n+1}) + p(x_{n+2}, x_n)\right]\right\}. \end{aligned}$$

Since in a partial metric space we have

$$p(x_{n+1}, x_{n+1}) + p(x_{n+2}, x_n) \leq p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n),$$

then we get

$$M(x_{n+1}, x_n) \leq \max\{p(x_{n+1}, x_n), p(x_{n+2}, x_{n+1})\}.$$

If $M(x_{n+1}, x_n) = p(x_{n+2}, x_{n+1})$, by (3.2), we have a contradiction. Then $M(x_{n+1}, x_n) = p(x_{n+1}, x_n)$. Again using (3.2) it follows $0 < p(x_{n+2}, x_{n+1}) < p(x_{n+1}, x_n)$. Hence, the sequence $\{p(x_{n+1}, x_n)\}$ is decreasing and bounded from below, thus it converges to some $r \geq 0$. Suppose that $r > 0$. By (3.2), we have

$$\frac{p(x_{n+2}, x_{n+1})}{p(x_{n+1}, x_n)} \leq \beta(p(x_{n+1}, x_n)) \leq 1,$$

for all $n \in \mathbb{N} \cup \{0\}$ which yields that $\lim_{n\to+\infty} \beta(p(x_{n+1}, x_n)) = 1$. On the other hand, since $\beta \in S$, we have $\lim_{n\to+\infty} p(x_{n+1}, x_n) = 0$ and so $r = 0$.

In order to prove that $\{x_n\}$ is a Cauchy sequence in (X, p), suppose the contrary, that is, $\{x_n\}$ is not a Cauchy sequence. Using Lemma 3.2, we know that there exist $\epsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers, with $m_k < n_k$, such that the following four sequences

$$\{p(x_{2m_k}, x_{2n_k})\}, \{p(x_{2m_k}, x_{2n_k+1})\}, \{p(x_{2m_k-1}, x_{2n_k})\}, \{p(x_{2m_k-1}, x_{2n_k+1})\}$$

tend to ϵ as $k \to +\infty$.

Putting, in the contractive condition (3.1), $x = x_{2m_k-1}$ and $y = x_{2n_k}$, it follows that:

$$\begin{aligned} p(x_{2m_k}, x_{2n_k+1}) &\leq \alpha(x_{2m_k-1}, fx_{2m_k-1})\alpha(x_{2m_k}, fx_{2m_k})p(fx_{2m_k-1}, fx_{2m_k}) \\ &\leq \beta(M(x_{2m_k-1}, x_{2n_k}))M(x_{2m_k-1}, x_{2n_k}) \\ &< M(x_{2m_k-1}, x_{2n_k}), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} M(x_{2m_k-1}, x_{2n_k}) = \max\{&p(x_{2m_k-1}, x_{2n_k}), p(x_{2m_k-1}, x_{2m_k}), p(x_{2n_k}, x_{2n_k+1}), \\ &\frac{1}{2}[p(x_{2m_k-1}, x_{2n_k+1}) + p(x_{2m_k}, x_{2n_k})]\}. \end{aligned}$$

Letting $k \to +\infty$, we get $M(x_{2m_k-1}, x_{2n_k}) \to \epsilon$. From (3.3) we have:

$$\frac{p(x_{2m_k}, x_{2n_k+1})}{M(x_{2m_k-1}, x_{2n_k})} \leq \beta(M(x_{2m_k-1}, x_{2n_k})) \leq 1, \text{ for all } k \in \mathbb{N}.$$

From the previous inequality, as $k \to +\infty$, we obtain

$$\lim_{k\to+\infty} \beta(M(x_{2m_k-1}, x_{2n_k})) = 1.$$

Since $\beta \in S$, we have $\lim_{k\to+\infty} M(x_{2m_k-1}, x_{2n_k}) = 0$, which is a contradiction. This implies that $\epsilon = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in (X, p). Since (X, p) is complete, it follows that the sequence $\{x_n\}$ converges to some $z \in X$. We say

$$p(z, z) = \lim_{n\to+\infty} p(x_n, z) = \lim_{n,m\to+\infty} p(x_n, x_m) = 0. \tag{3.4}$$

Now, we show that z is a fixed point of f. If $p(z, fz) > 0$, using condition (iii) and (3.1) with $x = x_n$ and $y = z$, we get

$$\begin{aligned} p(x_{n+1}, fz) &\leq \alpha(x_n, fx_n)\alpha(z, fz)p(fx_{n_k}, fz) \\ &\leq \beta(M(x_n, z))M(x_n, z). \end{aligned}$$

Now, for n large enough we have $M(x_n, z) = p(z, fz)$ and so, from the previous inequality and Lemma 3.1, we obtain

$$1 = \lim_{n\to+\infty} \frac{p(x_{n+1}, fz)}{M(x_n, z)} = \lim_{n\to+\infty} \beta(M(x_n, z)) \leq 1.$$

This implies $\lim_{n\to+\infty} M(x_n, z) = 0$, a contradiction. Thus, $p(z, fz) = 0$ and hence $fz = z$, that is, z is a fixed point of f.

Assume that u and v, with $u \neq v$, are two fixed points of f. Then

$$0 < p(u, v) \leq \alpha(u, fu)\alpha(v, fv)p(fu, fv) \leq \beta(M(u, v))M(u, v) < M(u, v),$$

where

$$M(u, v) = \max\left\{p(u, v), p(u, fu), p(v, fv)), \frac{1}{2}[p(u, fv)) + p(fu, v)]\right\} = p(u, v).$$

It follows $0 < p(u, v) < M(u, v) = p(u, v)$, a contradiction. Therefore, we get $u = v$ and this completes the proof. □

The following theorem is another main result of this paper.

Theorem 3.6. *Let (X, p) be a complete partial metric space and let $\alpha : X \times X \to [0, +\infty)$ be a function. Let $f : X \to X$ be a self mapping. Suppose that there exists $\beta \in S$ such that*

$$\alpha(x, y)p(fx, fy) \leq \beta(M(x, y))M(x, y) \tag{3.5}$$

for all $x, y \in X$, where

$$M(x, y) = \max\left\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}[p(x, fy) + p(fx, y)]\right\}.$$

Assume also that the following conditions hold:

(i) f is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) X is α-regular and for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, we have $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$;

(iv) $\alpha(x, y) \geq 1$ for all $x, y \in Fix(f)$.

Then f has a unique fixed point $z \in X$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by

$$x_n = fx_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Proceeding as in the proof of Theorem 3.5, we deduce that $\{x_n\}$ is a Cauchy sequence in (X, p) such that $p(x_{n+1}, x_n) \to 0$, as $n \to +\infty$. Since (X, p) is complete, it follows that the sequence $\{x_n\}$ converges to some $z \in X$ such that

$$p(z, z) = \lim_{n\to+\infty} p(x_n, z) = \lim_{n,m\to+\infty} p(x_n, x_m) = 0. \tag{3.6}$$

Now, we show that z is a fixed point of f. Since X is α-regular, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, z) \geq 1$ for all $k \in \mathbb{N}$. If $p(z, fz) > 0$, using (3.5) with $x = x_{n_k}$ and $y = z$, we get that

$$\begin{aligned} p(x_{n_k+1}, fz) &\leq \alpha(x_{n_k}, z)p(fx_{n_k}, fz) \\ &\leq \beta(M(x_{n_k}, z))M(x_{n_k}, z). \end{aligned}$$

Now, for k large enough, we have $M(x_{n_k}, z) = p(z, fz)$ and so, from the previous inequality and Lemma 3.1, we obtain

$$1 = \lim_{k\to+\infty} \frac{p(x_{n_k+1}, fz)}{M(x_{n_k}, z)} = \lim_{k\to+\infty} \beta(M(x_{n_k}, z)) \leq 1.$$

This implies $\lim_{k\to+\infty} M(x_{n_k}, z)) = 0$, which is a contradiction. Thus, $p(z, fz) = 0$ and hence $fz = z$, that is, z is a fixed point of f.

Assume that u and v, with $u \neq v$, are two fixed point of f. Then

$$0 < p(u, v) \leq \alpha(u, v)p(fu, fv) \leq \beta(M(u, v))M(u, v) < M(u, v),$$

where

$$M(u, v) = \max\left\{p(u, v), p(u, fu), p(v, fv)), \frac{1}{2}\left[p(u, fv)) + p(fu, v)\right]\right\} = p(u, v).$$

It follows $0 < p(u, v) < M(u, v) = p(u, v)$, which is a contradiction. Therefore, we get $u = v$ and this completes the proof. □

Example 3.7. Let $X = [0, 1]$, $d(x, y) = |x - y|$ for all $x, y \in X$, $p(x, y) = \max\{x, y\}$ for all $x, y \in X$, $\beta(t) = \frac{e^{-t}}{(t+1)}$ for each $t > 0$ and $\beta(0) = 1/2$. Let

$$\alpha(x, y) = \begin{cases} \frac{1}{4} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0). \end{cases}$$

The mapping $f : X \to X$ defined by $f(x) = \dfrac{x}{3}$ is α-admissible, but it does not satisfy the conditions of Geraghty's theorem in the metric space (X, d). Indeed, taking $x = 1$ and $y = 0$, we have

$$d(f1, f0) = d(\frac{1}{3}, 0) = |\frac{1}{3} - 0| = \frac{1}{3}$$

and

$$\beta(d(1, 0))\, d(1, 0) = \beta(|1 - 0|)\, |1 - 0| = \beta(1) = \frac{1}{2e}.$$

Since $\dfrac{1}{3} > \dfrac{1}{2e}$, Geraghty's theorem cannot be used to prove the existence of a fixed point of f.
Also we note that the mapping f does not satisfy the condition of Theorem 3.1 of [17] with respect to the partial metric defined above, because of

$$p(f1, f0) = \frac{1}{3} > \frac{1}{2\,e} = \beta(p(1, 0))\, p(1, 0).$$

On the other hand, taking $x, y \in X$ with, for example, $x \geq y$ and $x > 0$, then:

$$M(x, y) = \max\left\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}\left[p(x, fy) + p(fx, y)\right]\right\} = x$$

$$\alpha(x, y)p(fx, fy) = \frac{1}{12}x$$

and

$$\beta(M(x, y))\, M(x, y) = \beta(x)\, x = \frac{e^{-x}}{x + 1}\, x.$$

Now, from $1/12 < 1/2\,e \leq e^{-x}/(x + 1)$ for all $x \in [0, 1]$, we get that (3.5) holds.
Since the conditions (i)-(iv) of Theorem 3.6 are satisfied, then f has a unique fixed point (z=0).

4. Fixed points in ordered partial metric spaces

The existence of fixed points in partially ordered sets has been considered in [34]. Later on, some generalizations of [34] are given in [17, 29, 30, 31, 33, 37, 40]. Several applications of these results to matrix equations are presented in [34]. Moreover, some applications to periodic boundary value problems and to some particular problems are given, respectively, in [29, 30].

The following theorem ensures the existence of a fixed point for self-mappings in the setting of ordered partial metric spaces.

Theorem 4.1. *Let $(X, p, \preceq)$ be a complete ordered partial metric space and $\alpha : X \times X \to [0, +\infty)$ be a function. Let $f : X \to X$ be a non-decreasing mapping. Suppose that there exists $\beta \in S$ such that*

$$p(fx, fy) \leq \beta(M(x, y))M(x, y) \tag{4.1}$$

for all $x, y \in X$ with $x \preceq y$, where

$$M(x, y) = \max\left\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}\left[p(x, fy) + p(fx, y)\right]\right\}.$$

Assume also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;

(ii) X is such that, if a non-decreasing sequence $\{x_n\}$ converges to x, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$;

(iii) x, y are comparable whenever $x, y \in Fix(f)$.

Then f has a unique fixed point $z \in X$.

Proof. Define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

The reader can show easily that f is an α-admissible mapping and so (i) of Theorem 3.6 holds. The condition (i) above ensures that (ii) of Theorem 3.6 holds. Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$. By the definition of α, we have $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. By (ii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$ and so $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$ and hence X is α-regular. Further, $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$. Hence (iii) of Theorem 3.6 holds. The same considerations show that (iii) of this theorem implies (iv) of Theorem 3.6. Thus the hypotheses (i)-(iv) of Theorem 3.6 are satisfied. Also the contractive condition (3.5) is satisfied, because of $\alpha(x, y) = 1$ for all $x, y \in X$ such that $x \preceq y$ and $\alpha(x, y) = 0$ if $x \npreceq y$. Hence by Theorem 3.6, f have a unique fixed point. □

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References

[1] T. Abdeljawad, E. Karapinar and K. Tas, *A generalized contraction principle with control functions on partial metric spaces*, Comput. Math. Appl. **63** (2012), 716–719. 1

[2] I. Altun and A. Erduran, *Fixed point theorems for monotone mappings on partial metric spaces*, Fixed Point Theory Appl. **2011** (2011), Article ID 508730, 10 pages, doi:10.1155/2011/508730.

[3] I. Altun, F. Sola and H. Simsek, *Generalized contractions on partial metric spaces*, Topology Appl. **157** (2010), 2778–2785.

[4] A. Amini-Harandi and H. Emami, *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Anal. **72** (2010), 2238–2242. 1

[5] H. Aydi, *Fixed point theorems for generalized weakly contractive condition in ordered partial metric spaces*, J. Nonlinear Anal. Optimization: Theory and Applications **2** (2011), 33–48.

[6] H. Aydi, *Common fixed point results for mappings satisfying (ψ, ϕ)-weak contractions in ordered partial metric spaces*, Int. J. Math. Stat. **12** (2012), 53–64.

[7] H. Aydi, M. Abbas and C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology Appl. **159** (2012), 3234–3242. 1

[8] V. Berinde and F. Vetro, *Common fixed points of mappings satisfying implicit contractive conditions*, Fixed Point Theory Appl., **2012**:105 (2012). 1

[9] M.A. Bukatin and J.S. Scott, *Towards computing distances between programs via Scott domains*, in: Logical Foundations of Computer Science, Lecture Notes in Computer Science (eds. S. Adian and A. Nerode), vol. **1234**, Springer (Berlin, 1997), 33–43. 1

[10] M.A. Bukatin and S.Y. Shorina, *Partial metrics and co-continuous valuations*, in: Foundations of Software Science and Computation Structures, Lecture Notes in Computer Science (ed. M. Nivat), vol. **1378**, Springer (Berlin, 1998), 33–43. 1

[11] K.P. Chi, E. Karapinar and T.D. Thanh, *A generalized contraction principle in partial metric spaces*, Math. Comput. Modelling **55** (2012), 1673– 1681. 1

[12] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267–273. 1

[13] Lj. Ćirić, B. Samet, H. Aydi and C. Vetro, *Common fixed points of generalized contractions on partial metric spaces and an application*, Appl. Math. Comput. **218** (2011), 2398–2406. 1

[14] Lj. Ćirić, *On contraction type mappings*, Math. Balkanica **1** (1971), 52–57. 1

[15] C. Di Bari and P. Vetro, *Fixed points for weak φ-contractions on partial metric spaces*, Int. J. of Engineering, Contemporary Mathematics and Sciences **1** (2011), 5–13. 1

[16] C. Di Bari, Z. Kadelburg, H. Nashine and S. Radenović, *Common fixed points of g-quasicontractions and related mappings in 0-complete partial metric spaces*, Fixed Point Theory Appl., **2012**:113 (2012). 1

[17] D. Dukić, Z. Kadelburg and S. Radenović, *Fixed points of Geraghty-type mappings in various generalized metric spaces*, Abstr. Appl. Anal. **2011**, Article ID 561245, 13 pages 3.2, 3.7, 4

[18] M. Geraghty, *On contractive mappings*, Proc. Am. Math. Soc. **40** (1973), 604–608. 1

[19] R. Heckmann, *Approximation of metric spaces by partial metric spaces*, Appl. Categ. Structures **7** (1999), 71–83. 1

[20] D. Ilić, V. Pavlović and V. Rakočević, *Some new extensions of Banach's contraction principle to partial metric space*, Appl. Math. Lett. **24** (2011), 1326–1330. 1

[21] E. Karapinar, *Weak ϕ-contraction on partial metric spaces*, J. Comput. Anal. Appl. **14** (2012), 206–210.

[22] E. Karapinar, *A note on common fixed point theorems in partial metric spaces*, Miskolc Math. Notes **12** (2011), 185–191.

[23] E. Karapinar and I.M. Erhan, *Fixed point theorems for operators on partial metric spaces*, Appl. Math. Lett. **24** (2011), 1900–1904.

[24] E. Karapinar and U. Yuksel, *Some common fixed point theorems in partial metric spaces*, J. Appl. Math. **2011** (2011), Article ID 263621, 17 pages. 1

[25] R.D. Kopperman, S.G. Matthews and H. Pajoohesh, *What do partial metrics represent?*, Notes distributed at the 19th Summer Conference on Topology and its Applications, University of CapeTown (2004). 1

[26] H.P.A. Kunzi, H. Pajoohesh and M.P. Schellekens, *Partial quasi-metrics*, Theoret. Comput. Sci. **365** (2006), 237–246. 1

[27] S.G. Matthews, *Partial metric topology*, in: Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. **728** (1994), 183–197. 1, 2, 2, 2.6

[28] S.J. O'Neill, *Partial metrics, valuations and domain theory*, in: Proc. 11th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. **806** (1996), 304–315. 1, 2

[29] J.J. Nieto and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), 223–239. 4

[30] J.J. Nieto and R. Rodríguez-López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin. (Engl. Ser.) **23** (2007), 2205–2212. 4

[31] J.J. Nieto, R.L. Pouso and R. Rodríguez-López, *Fixed point theorems in ordered abstract spaces*, Proc. Amer. Math. Soc. **132** (2007), 2505–2517. 4

[32] S. Oltra and O. Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. Istit. Mat. Univ. Trieste **36** (2004), 17-26. 1, 2, 2.6

[33] D. Paesano and P. Vetro, *Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces*, Topology Appl. **159** (2012), 911–920. 1, 2, 3.1, 4

[34] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2003), 1435–1443. 4

[35] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. **14** (1971), 121-124.

[36] S. Romaguera, *Fixed point theorems for generalized contractions on partial metric spaces*, Topology Appl. **159** (2012), 194–199. 1

[37] S. Romaguera, *Matkowski's type theorems for generalized contractions on (ordered) partial metric spaces*, Appl. Gen. Topol. **12** (2011), 213–220. 1, 4

[38] S. Romaguera and M. Schellekens, *Duality and quasi-normability for complexity space*, Appl. Gen. Topol. **3** (2002), 91–112. 1

[39] S. Romaguera and M. Schellekens, *Partial metric monoids and semivaluation spaces*, Topology Appl. **153** (2005), 948–962. 1

[40] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165. 3.3, 4

[41] M.P. Schellekens, *A characterization of partial metrizability: domains are quantifiable*, Theoret. Comput. Sci. **305** (2003), 409–432. 1

[42] M.P. Schellekens, *The correspondence between partial metrics and semivaluations*, Theoret. Comput. Sci. **315** (2004), 135–149. 1

[43] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Amer. Math. Soc. **136** (2008), 1861-1869. 1

[44] F. Vetro and S. Radenović, *Nonlinear ψ-quasi-contractions of Ćirić-type in partial metric spaces*, Appl. Math. Comput. **219** (2012), 1594–1600. 1