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Convergence and stability analysis of modified backward time centered space approach for non-dimensionalizing parabolic equation

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Abstract

The present paper is motivated by the desire to obtain the numerical solution of the heat equation. A finite-difference schemes is introduced to obtain the solution. The convergence and stability analysis of the proposed approach is discussed and compared.

Keywords: Convergence analysis, Finite-difference schemes, Heat equation, Stability.
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1. Introduction

The heat equations play an important role in applied mathematics and engineering. They arise for example, in the study of conduction processes, thermoelasticity, chemical diffusion, and control theory [1, 2, 3, 4, 5, 6, 7, 8]. Recently, a lot of attention has been devoted to the study of heat equations. In this paper, we develop a numerical framework to obtain the numerical solution of the following equation

$$u_t - \alpha u_{xx} = 0. \tag{1.1}$$

Furthermore, stability and convergence analysis will be discussed.

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2. Background material and preliminaries

As we know, the heat equation is derived from Fourier's law and conservation of energy. By Fourier's law, the flow rate of heat energy through a surface is proportional to the negative temperature gradient across the surface,

$$q = -k\Delta u, \tag{2.1}$$

where k is the thermal conductivity and u is the temperature. In one dimension, we have

$$q = -ku_x, \tag{2.2}$$

A change in internal energy per unit volume in the material, ΔQ is proportional to the change in temperature, Δu. In the other words

$$\Delta Q = c_p \rho \Delta u, \tag{2.3}$$

where c_p is the specific heat capacity and ρ is the mass density of the material. Choosing zero energy at absolute zero temperature, this can be rewritten as

$$Q = c_p \rho u, \tag{2.4}$$

The increase in internal energy in a small spatial region of the material, over the time period is given by

$$\begin{aligned} Q &= c_p\rho \int_{x-\Delta x}^{x+\Delta x} u(\xi, t+\Delta t) - u(\xi, t-\Delta t) d\xi && (2.5)\\ &= c_p\rho \int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \frac{\partial u}{\partial x} d\xi d\tau. && (2.6)\end{aligned}$$

Consequently, by Fourier's law in $[x-\Delta x, x+\Delta x]$ we obtain

$$k\int_{t-\Delta t}^{t+\Delta t} \frac{\partial u}{\partial x}(x+\Delta x, \tau) - \frac{\partial u}{\partial x}(x-\Delta x, \tau) d\tau = k\int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \frac{\partial^2 u}{\partial x^2} d\xi d\tau. \tag{2.7}$$

Again by the fundamental theorem of calculus and by conservation of energy, we obtain

$$\int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \Big(c_p\rho \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2}\Big) d\xi d\tau = 0. \tag{2.8}$$

By the fundamental lemma of the calculus of variations, the integrand must vanish identically

$$c_p\rho \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0. \tag{2.9}$$

Which can be written as

$$u_t = \frac{k}{c_p\rho} u_{xx}, \tag{2.10}$$

if

$$\alpha = \frac{k}{c_p\rho}, \tag{2.11}$$

consequently, one will set

$$u_t = \alpha u_{xx}. \tag{2.12}$$

3. Analysis of the proposed approach

In this section we introduce our numerical method for solving the heat equation. Consider heat equation as

$$u_t = \alpha u_{xx}, \quad 0 < t < T, \; 0 < x < L, \tag{3.1}$$

with the initial and boundary conditions in the following form

$$u(x,0) = \xi(x), \quad u(0,t) = 0, \quad u(L,t) = \eta(t). \tag{3.2}$$

Let $u(x,t)$ represent the exact solution of a partial differential equation with independent variables x and t. Define $t_n = n\Delta t,\ n = 0,1,2,\cdots,N,\ \ x_i = i\Delta x,\ i = 0,1,2,\cdots,M$, where $\Delta x = \frac{L}{M}$ and $\Delta t = \frac{T}{N}$ are space and time steps, respectively. We suppose that u_i^n as the numerical solution in the difference method and

$$\triangle_t\, u(x_i,t_k) = u(x_i,t_{k+1}) - u(x_i,t_k). \tag{3.3}$$

Using the forward time formula at x_i and the centered space formula at t_{n+1}, difference approximation of the Eq. (3.1) is

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{(\Delta x)^2}, \tag{3.4}$$

or

$$-su_{i-1}^{n+1} + (1+2s)u_i^{n+1} - su_{i+1}^{n+1} = u_i^n, \tag{3.5}$$

where $s = \frac{\alpha(\Delta t)}{(\Delta x)^2} > 0$.

In partial differential equation (PDE), the local truncation error is defined as the difference of the finite difference scheme and partial differential equation. The finite difference scheme is consistent if the limit of the local truncation error is zero as Δx and/or Δt approach zero.

Using the Taylor expanding around (x_i, t_{k+1}) for Eq. (3.5) and summarizing the formula, the following relation can be obtained

$$\begin{aligned} &- s\Big[2u_i^n + 2\frac{\partial u}{\partial t}\Delta t + \frac{1}{2!}\Big(\frac{\partial^2 u}{\partial x^2}\Delta x^2 + \frac{\partial^2 u}{\partial t^2}\Delta t^2\Big) + \frac{2}{3!}\Big(\frac{\partial^3 u}{\partial t^3}\Delta t^3 + 3\frac{\partial^3 u}{\partial x^2 \partial t}\Delta x^2 \Delta t\Big)\Big] \\ &+ \frac{2}{4!}\Big(\frac{\partial^4 u}{\partial x^4}\Delta x^4 + \frac{\partial^4 u}{\partial t^4}\Delta t^4 + 6\frac{\partial^4 u}{\partial x^2 \partial t^2}\Delta x^2 \Delta t^2\Big) \\ &+ (1+2s)\Big(u_i^n + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2i}\frac{\partial^2 u}{\partial t^2}\Delta t^2 + \cdots\Big) - u_i^n = 0, \end{aligned} \tag{3.6}$$

because of

$$\frac{\partial u}{\partial t}\big((1+2s)\Delta t - 2s\Delta t\big) = \frac{\partial u}{\partial t}(\Delta t)\frac{\partial^2 u}{\partial x^2}(-s(\Delta x)^2) = -s\Delta x^2 \frac{\partial^2 u}{\partial x^2}, \tag{3.7}$$

and

$$\begin{aligned} &\frac{\partial^4 u}{\partial x^4}\Big(-\alpha^2 \Delta t^2 - s\alpha \Delta t \Delta x^2 - \frac{s}{12}\Delta x^4 + \frac{(1+2s)}{2}\alpha^2 \Delta t^2\Big) \\ &\qquad = \frac{\partial^4 u}{\partial x^4}\alpha \Delta t \Delta x^2\Big(-s - \frac{s}{12}\frac{\Delta x^2}{\alpha \Delta t} + \frac{1}{2}\alpha\frac{\Delta t}{\Delta x^2}\Big) \\ &\qquad = \alpha \Delta t \Delta x^2 \frac{\partial^4 u}{\partial x^4}\Big(-\frac{1}{2}s - \frac{1}{12}\Big) \\ &\qquad = -\frac{\alpha}{12}\Delta t \Delta x^2 \frac{\partial^4 u}{\partial x^4}(6s+1). \end{aligned}$$

At last, we obtain this formula

$$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} - \frac{\alpha}{12}\Delta x^2 \frac{\partial^4 u}{\partial x^4}(1+6s) + O\{(\Delta x)^4\} = 0. \tag{3.8}$$

that shows the order of accuracy is $O((\Delta x)^2)$. Now if $(\Delta t, \Delta x) \to 0$, the above statement tends to Eq. (3.1) and therefore this method of finite difference is consistent with the partial differential equation.

Hereunder, we present two theorems that are used in next sections.

Theorem 1. Suppose that $B = Trid(a, b, c)$ be an arbitrary matrix. Consequently, the eigenvalues of B are

$$\lambda = b + 2\sqrt{ac}\cos\frac{k\pi}{M}, \quad k = 0, 1, \cdots, M-1. \tag{3.9}$$

Proof. Suppose that λ is an eigenvalue of B and $x = [x_1, x_2, \cdots, x_k]$ is the corresponding eigenvector of λ. We have

$$(A - \lambda I)x = 0, \tag{3.10}$$

or if define the initial condition as $x_0 = x_n = 0$, then we can write the following difference equation

$$ax_{j-1} + (b - \lambda)x_j + cx_{j+1} = 0, \quad j = 1, \cdots, n \tag{3.11}$$

Solutions of the equation are as $x_k = r^k$ that by substituting in Eq. (3.11), we have

$$cr^2 + (b - \lambda)r + a = 0, \tag{3.12}$$

Let r_1 and r_2 are roots of (3.12). Then the general answer of (3.11) is as $x_k = c_1 r_1^k + c_2 r_2^k$. According to initial conditions,

$$\begin{aligned} x_0 &= 0 \Longrightarrow \alpha_1 = -\alpha_2, \\ x_{n+1} &= 0 \Longrightarrow \left(\frac{r_1}{r_2}\right)^{n+1} = 1 = e^{-i2k\pi},\ k = 1, \cdots N. \end{aligned} \tag{3.13}$$

Therefore

$$r_1 = \sqrt{\frac{a}{c}} e^{i\frac{k\pi}{N+1}}, \quad r_2 = \sqrt{\frac{a}{c}} e^{-i\frac{k\pi}{N+1}}. \tag{3.14}$$

In the other words

$$\lambda = b + c\sqrt{\frac{a}{c}}\left(e^{i\frac{k\pi}{N+1}} + e^{-i\frac{k\pi}{N+1}}\right) = b + 2\sqrt{ac}\,\cos\left(\frac{k\pi}{N+1}\right),\ k = 1, \cdots, N, \tag{3.15}$$

and finally the last relation can be obtained as

$$\lambda = b + 2\sqrt{ac}\,\cos\left(\frac{k\pi}{M}\right),\ k = 0, 1, \cdots, M-1. \tag{3.16}$$

Theorem 2. If B be a square matrix, then these relations are equivalent [5]

(1) $\lim_{k\to\infty} B^k = 0$,

(2)$(\forall v)$ $\lim_{k\to\infty} B^k v = 0$,

(3)$\rho(B) < 1$,

(4)$\exists \|.\|$ $s.t$ $||B|| < 1$, where $\rho = (\max \lambda_i)$ and (λ_i)s are the eigenvalues of B.

4. Stability

Theorem 3. A finite one-step difference scheme $P_{h,\tau} u_i^k = 0$ for a first order PDE is stable if there exist numbers $\tau_0 > 0$ and $h_0 > 0$ such that for any $T > 0$ there exists a constant C_T such that

$$\|u^k\| \le \|u^0\| C_T, \tag{4.1}$$

for $0 < k\tau < T, 0 < h < h_0, 0 < \tau < \tau_0$. Therefore, the scheme is stable if the truncation error doesn't growth. That is

If $E^n = [e_1^n, e_2^n, \cdots, e_{M-1}^n]^T$, then $\|E^{n+1}\|_\infty \le \|E^n\|_\infty$, $n = 0, 1, 2, \cdots$.

Proof. The difference approximation form of the Eq. (3.5) is

$$-se_{i-1}^{n+1} + (1+2s)e_i^{n+1} - se_{i+1}^{n+1} = e_i^n, \quad i = 1, 2, \cdots, M-1. \tag{4.2}$$

Substituting values of i, as follows

$$\begin{cases} i = 1 : -se_0^{n+1} + (1+2s)e_1^{n+1} - se_2^{n+1} = e_1^n, \\ \\ i = 2 : -se_1^{n+1} + (1+2s)e_2^{n+1} - se_3^{n+1} = e_2^n, \\ \\ \qquad\qquad \vdots \\ \\ i = m-1 : -se_{m-2}^{n+1} + (1+2s)e_{m-1}^{n+1} - se_m^{n+1} = e_{m-1}^n, \end{cases} \tag{4.3}$$

Consequently, we can obtain

$$\underbrace{\begin{bmatrix} (1+2s) & -s & 0 & \dots & 0 \\ -s & (1+2s) & -s & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -s & (1+2s) \end{bmatrix}}_{\text{A}} \underbrace{\begin{bmatrix} e_1^{n+1} \\ \vdots \\ \vdots \\ e_{m-1}^{n+1} \end{bmatrix}}_{E^{n+1}} = \underbrace{\begin{bmatrix} e_1^n \\ \vdots \\ \vdots \\ e_{m-1}^n \end{bmatrix}}_{E^n}.$$

where $A = Trid(-s, 1+2s, -s)$.

According to Theorem (3) we have

$$\lambda(A) = (1+2s) + 2s\cos(\frac{k\pi}{M}) = 1 + 4s\cos^2(\frac{k\pi}{M}) > 1, \tag{4.4}$$

and $\lambda(A^{-1}) < 1$. On the other hand

$$E^{n+1} = A^{-1}E^n. \tag{4.5}$$

Corresponding the Theorem (3) and Eq. (4.5) the following relation can be obtained

$$\|E^{n+1}\| \le \|E^n\|. \tag{4.6}$$

Thus, the truncation error doesn't growth and the scheme is unconditionally stable.

5. Convergence

Theorem 4. A finite difference scheme approximating of a PDE is convergent if u_i^k, tends to $u(x,t)$, at a fixed point or along a t-level as Δx and Δt both tend to zero.

Proof. We assume that $\widetilde{u}_i^j$ is the approximate solution of (3.1,3.2) and

$$e_i^j = \widetilde{u}_i^j - u_i^j, \tag{5.1}$$

is the error of method for $i = 0, 1, \cdots, M, j = 0, 1, \cdots, N$.

Substituting (5.1) in difference Eq. (3.5), we have

$$e_i^n = -se_{i-1}^{n+1} + (1+2s)e_i^{n+1} - se_{i+1}^{n+1} + s\left(\hat{u}_{i+1}^{n+1} - 2\hat{u}_i^{n+1} + \hat{u}_{i-1}^{n+1}\right) - \left(\hat{u}_i^{n+1} - \hat{u}_i^n\right).$$

Then by using the Taylor expanding around (x_i, t_n) and Mean Value Theorem, the previous relation can be rewritten as follows

$$\begin{aligned} e_i^n &= -se_{i-1}^{n+1} + (1+2s)e_i^{n+1} - s_{i+1}^{n+1} \\ &+ s\Big(\hat{u}_i^{n+1}\Big|_{i+\theta_2}^{n} + \Delta x\frac{\partial u}{\partial x}\Big|_i^{n+1} + \frac{1}{2!}\Delta x^2\frac{\partial^2 u}{\partial x^2}\Big|_i^{n+1} + \cdots - \hat{u}_i^{n+1}\Big) \\ &- \big(\hat{u}_i^n + \Delta t\frac{\partial u}{\partial x}\Big|_i^n + \cdots - \hat{u}_i^n\big). \end{aligned}$$

or

$$e_i^n = -se_{i-1}^{n+1} + (1+2s)e_i^{n+1} - se_{i+1}^{n+1} - (\Delta t)\Big[\Big(\frac{\partial u}{\partial t}\Big|_i^{n+\theta_1}\Big) + s\Big(\frac{\partial^2 u}{\partial x^2}\Big|_{i+\theta_2}^{n+1} + \frac{\partial^2 u}{\partial x^2}\Big|_{i-\theta_3}^{n+1}\Big)(\Delta x)^2\Big]. \tag{5.2}$$

Let $|e_{max}^n| = \max_{1\le i\le M-1}|e_i^n|$ and $\bar{M} = \max R_i^n, 1 \le i \le M-1, 1 \le n \le N.$ where

$$R_i^n = \Big(\frac{\partial u}{\partial t}\Big|_i^{n+\theta_1}\Big) + s\Big(\frac{\partial^2 u}{\partial x^2}\Big|_{i+\theta_2}^{n+1} + \frac{\partial^2 u}{\partial x^2}\Big|_{i-\theta_3}^{n+1}\Big)(\Delta x)^2. \tag{5.3}$$

It is sufficient to show $|e_{max}^N| \to 0$ while $(\Delta x, \Delta t) \to 0$. According to Eq. (5.2)

$$|e_i^n| \le \big| - se_{i-1}^{n+1} + (1+2s)e_i^{n+1} - se_{i+1}^{n+1}\big| - (\Delta t)\bar{M} \le \big|e_{max}^{n+1}\big| - \Delta t\bar{M},\ n = 0, 1, \cdots, N-1. \tag{5.4}$$

Substituting values of n, as follows

$$\begin{cases} n = 0 : |e_{max}^1| \le \big|e_{max}^0\big| + \Delta t\bar{M}, \\ n = 1 : |e_{max}^2| \le \big|e_{max}^1\big| + (\Delta t)\bar{M}, \\ \qquad\qquad \vdots \\ n = N-1 : |e_{max}^n| \le n(\Delta t)\bar{M}. \end{cases} \tag{5.5}$$

Also, if $(\Delta t, \Delta x) \to 0$, then $\theta_1, \theta_2, \theta_3 \to 0$. Thus

$$\begin{aligned} &\lim_{(\Delta t,\Delta x)\to 0}\bar{M} = \lim_{(\Delta t,\Delta x)\to 0}\max R_i^n = \\ &\lim_{(\Delta t,\Delta x)\to 0}\max\Big(\frac{\partial u}{\partial t}\Big|_i^{n+\theta_1}\Big) + s\Big(\frac{\partial^2 u}{\partial x^2}\Big|_{i+\theta_2}^{n+1} + \frac{\partial^2 u}{\partial x^2}\Big|_{i-\theta_3}^{n+1}\Big)(\Delta x)^2 = 0. \end{aligned}$$

Because of $|e_{max}| \le n(\Delta t)\bar{M}$, thus $|e_{max}| \to 0$. This shows that the method is convergence to the Eq. (3.1).

6. Numerical examples

Consider the following problems

Case 1.

$$u_t = \frac{5}{42}u_{xx}, \quad 0 \le x \le 20, \quad 0 \le t \le 604.8, \tag{6.1}$$

with the boundary and initial conditions

$$u(0,t) = 0, \quad u(20,t) = 10, \quad u(x,0) = 2. \tag{6.2}$$

For $\Delta t = 67.2$ and $\Delta x = 4$, the solution obtained by our proposed approach is recorded in Table 1.

Case 2.

$$u_t = \frac{1}{4} u_{xx}, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1, \tag{6.3}$$

with the boundary and initial conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = \sin(\pi x). \tag{6.4}$$

For $\Delta t = 0.1$ and $\Delta x = 0.1$, the solution obtained by our proposed approach is recorded in Table 1.

Table 1

$n(case1)$	x	$Error$	$n(case2)$	x	$Error$
10	4	0.36455	1	0.2	0.011801
	8	0.60316		0.4	0.024666
	12	0.62005		0.6	0.024666
	16	0.39197		0.8	0.011801
50	4	0.00035	10	0.2	0.00012
	8	0.00056		0.4	0.00035
	12	0.00056		0.6	0.00035
	16	0.00035		0.8	0.00012
80	4	0	20	0.2	0
	8	0		0.4	0
	12	0		0.6	0
	16	0		0.8	0

7. Conclusion

This paper has outlined an approach for the study of a heat differential equations. We studied the numerical solution of this prototype phenomena. The explicit finite-difference schemes, were applied to the mentioned model and the proposed numerical scheme solved this model quite satisfactory. The results reveal that our proposed strategy is effective and excellent.

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