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Existence and uniqueness of fixed points in modified intuitionistic fuzzy metric spaces

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Abstract

In this paper, utilizing the concept of common limit range property, we prove integral type common fixed point theorems for two pairs of weakly compatible mappings satisfying ϕ-contractive conditions in modified intuitionistic fuzzy metric spaces. We give some examples to support the useability of our results. We extend our results to four finite families of self mappings by using the notion of pairwise commuting.

Keywords: Modified intuitionistic fuzzy metric space, weakly compatible mappings, common limit range property, fixed point.
2010 MSC: 47H10, 54H25.

1. Introduction

In 2004, Park [21] introduced a notion of intuitionistic fuzzy metric space which is based both on the idea of intuitionistic fuzzy set due to Atanassov [2], and the concept of a fuzzy metric space studied by George and Veeramani [12]. In an interesting note on intuitionistic fuzzy metric spaces, Gregori et al. [13] showed that the topology induced by fuzzy metric coincides with topology induced by intuitionistic fuzzy metric. Further, Saadati et al. [23] proposed the idea modified intuitionistic fuzzy metric spaces in which the notions of continuous t-norm and continuous t-conorm are used (also see, [8]).

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Saadati et al. [23] proved common fixed point theorems for compatible and weakly compatible mappings in modified intuitionistic fuzzy metric spaces. Consequently, Jain et al. [17] utilized the notion of compatibility of type (P) for the existence of fixed points. Sedghi et al. [25] proved some integral type fixed point theorems for weakly compatible mappings with property (E.A) which is studied by Aamri and Moutawakil [1]. Recently, Tanveer et al. [29] and Imdad et al. [15] proved some fixed point theorems for two pairs of weakly compatible mappings in modified intuitionistic fuzzy metric spaces satisfying common property (E.A) due to Liu et al. [20]. Sintunavarat and Kumam [27] introduced the notion of "common limit in the range property" which never requires the closedness of the underlying subspace. In this sequence, Imdad et al. [16] extend the notion of common limit in the range property to pairs of self mappings and proved some interesting results in Menger spaces. The study of fixed points in modified intuitionistic fuzzy metric spaces has been an area of vigorous research activity. To mention a few, we refer [5, 6, 15, 26, 29] and references cited therein.

In 2002, Branciari [3] proved an integral type fixed point theorem which generalized Banach's contraction principle. In a series of papers the authors [4, 7, 11, 19, 22, 24, 30, 31] proved several fixed point results involving more general integral type contractive conditions. In [28], Suzuki showed that a Meir-Keeler contraction of integral type is still a Meir-Keeler contraction.

In this paper, we prove common fixed point theorems for weakly compatible mappings satisfying integral type ϕ-contractive condition with common limit range property in modified intuitionistic fuzzy metric spaces. Some illustrative examples are also furnished to support our results. As an application to our main result, we derive a fixed point theorem for four finite families of self mappings which can be utilized to derive common fixed point theorems involving any finite number of mappings. Our results improve the results of Sedghi et al. [25] and the references mentioned therein.

2. Preliminaries

Lemma 2.1. *[9] Consider the set L^* and operation $\leq_{L^*}$ defined by*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, for every $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 2.2. [2] An intuitionistic fuzzy set $A_{\zeta,\eta}$ in a universe $\mathcal{U}$ is an object $A_{\zeta,\eta} = \{(\zeta_A(u), \eta_A(u) \mid u \in \mathcal{U})\}$, where, for all $u \in \mathcal{U}$, $\zeta_A(u) \in [0,1]$ and $\eta_A(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of $u \in A_{\zeta,\eta}$ and furthermore they satisfy $\zeta_A(u) + \eta_A(u) \leq 1$.

For every $z_i = (x_i, y_i) \in L^*$, if $c_i \in [0,1]$ such that $\sum_{j=1}^n c_j = 1$ then it is easy to see that

$$c_1(x_1, y_1) + \ldots + c_n(x_n, y_n) = \sum_{j=1}^n c_j(x_j, y_j) = \left(\sum_{j=1}^n c_j x_j, \sum_{j=1}^n c_j y_j\right) \in L^*.$$

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a triangular norm $* = \mathrm{T}$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $\mathrm{T} : [0,1]^2 \to [0,1]$ satisfying $\mathrm{T}(1,x) = 1 * x = x$, for all $x \in [0,1]$. A triangular co-norm $\mathrm{S} = \diamond$ is defined as an increasing, commutative, associative mapping $\mathrm{S} : [0,1]^2 \to [0,1]$ satisfying $\mathrm{S}(0,x) = 0 \diamond x = x$, for all $x \in [0,1]$. Using the lattice $(L^*, \leq_{L^*})$ these definitions can straightforwardly be extended.

Definition 2.3. [10] A triangular norm (briefly, t-norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \to L^*$ satisfying the following conditions for all $x, y, x', y' \in L^*$:

1. $\mathcal{T}(x, 1_{L^*}) = x$,
2. $\mathcal{T}(x, y) = \mathcal{T}(y, x)$,
3. $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$,

4. $x \leq_{L^*} x^{'}$ and $y \leq_{L^*} y^{'} \Rightarrow \mathcal{T}(x,y) \leq_{L^*} \mathcal{T}(x^{'},y^{'})$.

Definition 2.4. [9, 10] A continuous t-norm $\mathcal{T}$ on L^* is called continuous t-representable if and only if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0,1]$ such that, for all $x=(x_1,x_2), y=(y_1,y_2)\in L^*$,

$$\mathcal{T}(x,y)=(x_1 * y_1, x_2 \diamond y_2).$$

Now, we define a sequence $\{\mathcal{T}_n\}$ recursively by $\{\mathcal{T}^1=\mathcal{T}\}$ and

$$\mathcal{T}^n\left(x^{(1)},\ldots,x^{(n+1)}\right)=\mathcal{T}\left(\mathcal{T}^{n-1}\left(x^{(1)},\ldots,x^{(n)}\right),x^{(n+1)}\right),$$

for $n\geq 2$ and $x^{(i)}\in L^*$.

Definition 2.5. [9, 10] A negator on L^* is any decreasing mapping $\mathcal{N}: L^*\to L^*$ satisfying $\mathcal{N}(0_{L^*})=1_{L^*}$ and $\mathcal{N}(1_{L^*})=0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x))=x$, for all $x\in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N:[0,1]\to[0,1]$ satisfying $N(0)=1$ and $N(1)=0$. N_s denotes the standard negator on $[0,1]$ defined as (for all $x\in[0,1]$) $N_s(x)=1-x$.

Definition 2.6. [23] Let M,N are fuzzy sets from $X^2\times(0,\infty)$ to $[0,1]$ such that $M(x,y,t)+N(x,y,t)\leq 1$ for all $x,y\in X$ and $t>0$. The 3-tuple $(X,\mathcal{M}_{M,N},\mathcal{T})$ is said to be a modified intuitionistic fuzzy metric space (shortly, modified IFMS) if X is an arbitrary non-empty set, $\mathcal{T}$ is a continuous t-representable and $\mathcal{M}_{M,N}$ is an intutionistic fuzzy set from $X^2\times(0,\infty)\to L^*$ satisfying the following conditions for every $x,y\in X$ and $t,s>0$:

1. $\mathcal{M}_{M,N}(x,y,t)>_{L^*} 0_{L^*}$,
2. $\mathcal{M}_{M,N}(x,y,t)=1_{L^*}$ if and only if $x=y$,
3. $\mathcal{M}_{M,N}(x,y,t)=\mathcal{M}_{M,N}(y,x,t)$,
4. $\mathcal{M}_{M,N}(x,y,t+s)\geq_{L^*}\mathcal{T}(\mathcal{M}_{M,N}(x,z,t),\mathcal{M}_{M,N}(z,y,s))$,
5. $\mathcal{M}_{M,N}(x,y,\cdot):(0,\infty)\to L^*$ is continuous.

In this case $\mathcal{M}_{M,N}$ is called a modified intuitionistic fuzzy metric. Here,

$$\mathcal{M}_{M,N}(x,y,t)=(M(x,y,t),N(x,y,t)).$$

Remark 2.7. In an intuitionistic fuzzy metric space $(X,\mathcal{M}_{M,N},\mathcal{T})$, $M(x,y,\cdot)$ is non-decreasing and $N(x,y,\cdot)$ is non-increasing for all $x,y\in X$: Hence $(X,\mathcal{M}_{M,N},\mathcal{T})$ is non-decreasing function for all $x,y\in X$.

Example 2.8. [23] Let (X,d) be a metric space. Denote $\mathcal{T}(a,b)=(a_1b_1,\min\{a_2+b_2,1\})$ for all $a=(a_1,a_2)$ and $b=(b_1,b_2)\in L^*$ and let M and N be fuzzy sets on $X^2\times(0,\infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x,y,t)=(M(x,y,t),N(x,y,t))=\left(\frac{ht^n}{ht^n+md(x,y)},\frac{md(x,y)}{ht^n+md(x,y)}\right),$$

for all $h,m,n,t\in\mathbb{R}^+$. Then $(X,\mathcal{M}_{M,N},\mathcal{T})$ is a modified IFMS.

Example 2.9. [23] Let $X=\mathbb{N}$. Denote $\mathcal{T}(a,b)=(\max\{0,a_1+b_1-1\},a_2+b_2-a_2b_2)$ for all $a=(a_1,a_2)$ and $b=(b_1,b_2)\in L^*$ and let M and N be fuzzy sets on $X^2\times(0,\infty)$. Then $\mathcal{M}_{M,N}(x,y,t)$ defined as for all $x,y\in X$ and $t>0$:

$$\mathcal{M}_{M,N}(x,y,t)=(M(x,y,t),N(x,y,t))=\begin{cases}\left(\frac{x}{y},\frac{y-x}{y}\right), & \text{if } x\leq y;\\ \left(\frac{y}{x},\frac{x-y}{x}\right), & \text{if } x\leq y.\end{cases}$$

Then $(X,\mathcal{M}_{M,N},\mathcal{T})$ is a modified IFMS.

Definition 2.10. [23] Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a modified IFMS. For $t > 0$, define the open ball $\mathcal{B}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$, as

$$\mathcal{B}(x, r, t) = \{y \in X : \mathcal{M}_{M,N}(x, y, t) >_{L^*} (N_s(r), r)\}.$$

A subset $\mathcal{A} \subset X$ is called open if for each $x \in \mathcal{A}$, there exist $t > 0$ and $0 < r < 1$ such that $\mathcal{B}(x, r, t) \subseteq \mathcal{A}$. Let $\tau_{\mathcal{M}_{M,N}}$ denote the family of all open subsets of X. $\tau_{\mathcal{M}_{M,N}}$ is called the topology induced by intuitionistic fuzzy metric $\mathcal{M}_{M,N}$. Notice that this topology is Hausdorff (see [21, Remark 3.3, Theorem 3.5]).

Definition 2.11. [23] A sequence $\{x_n\}$ in a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{M}_{M,N}(x_n, y_m, t) >_{L^*} (N_s(\epsilon), \epsilon),$$

and for each $n, m \geq n_0$, where N_s is a standard negator. The sequence $\{x_n\}$ is said to be convergent to $x \in X$ in the modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ and is generally denoted by $x_n \to^{\mathcal{M}_{M,N}} x$ if $\mathcal{M}_{M,N}(x_n, y, t) \to 1_{L^*}$ whenever $n \to \infty$ for every $t > 0$. A modified IFMS is said to be complete if and only if every Cauchy sequence is convergent.

Lemma 2.12. *[23] Let $\mathcal{M}_{M,N}$ be an intuitionistic fuzzy metric. Then, for any $t > 0$, $\mathcal{M}_{M,N}(x, y, t)$ is non-decreasing with respect to t in $(L^*, \leq_{L^*})$, for all $x, y \in X$.*

Definition 2.13. [23] Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a modified IFMS. Then $\mathcal{M}_{M,N}$ is said to be continuous on $X \times X \times (0, \infty)$, if

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(x_n, y_n, t_n) = \mathcal{M}_{M,N}(x, y, t),$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times (0, \infty)$ converges to a point $\{(x, y, t)\} \in X \times X \times (0, \infty)$, that is

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(x_n, x, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(y_n, y, t) = 1_{L^*},$$

and

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(x, y, t_n) = \mathcal{M}_{M,N}(x, y, t).$$

Lemma 2.14. *[23] Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a modified IFMS. Then, $\mathcal{M}_{M,N}$ is continuous function on $X \times X \times (0, \infty)$.*

Definition 2.15. Let A and S be two mappings from a modified IFM-space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ into itself. Then the mappings are said to be

1. compatible [23] if
$$\lim_{n\to\infty} \mathcal{M}_{M,N}(ASx_n, SAx_n, t) = 1_{L^*},$$
for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X.$$

2. non-compatible [29] if there exists at least one sequence $\{x_n\}$ in X such that
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X,$$
but
$$\lim_{n\to\infty} \mathcal{M}_{M,N}(ASx_n, SAx_n, t) \neq 1_{L^*},$$
or non-existent for at least one $t > 0$.

Definition 2.16. [18] Let A and S be two self mappings of a non-empty set X. Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Every pair of compatible self mappings A and S of a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is weakly compatible. But the converse is not true in general ([23]).

Definition 2.17. [25] Let A and S be two mappings from a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ into itself. Then the mappings are said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that for all $t > 0$

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(Ax_n, z, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(Sx_n, z, t) = 1_{L^*},$$

for some $z \in X$.

Definition 2.18. [29] Two pairs (A, S) and (B, T) of self mappings of a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ are said to satisfy the common property (E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $t > 0$

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(Ax_n, z, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(Sx_n, z, t) =$$
$$\lim_{n\to\infty} \mathcal{M}_{M,N}(By_n, z, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(Ty_n, z, t) = 1_{L^*},$$

for some $z \in X$.

Inspired by Sintunavarat and Kumam [27], we extend the notion of (CLR_S) property with respect to mapping S in modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ as follows:

Definition 2.19. A pair (A, S) of self mappings of a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to satisfy the (CLR_S) property with respect to mapping S if there exists a sequence $\{x_n\}$ in X such that for all $t > 0$

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(Ax_n, z, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(Sx_n, z, t) = 1_{L^*},$$

where $z \in S(X)$.

Now, we present some examples of self mappings A and S satisfying the (CLR_S) property.

Example 2.20. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$, where $X = [0, \infty)$ and $\mathcal{M}_{M,N}(x, y, t) = \left(\frac{t}{t+|x-y|}, \frac{|x-y|}{t+|x-y|}\right)$ for all $x, y \in X$ and $t > 0$. Define the self mappings A and S on X by $A(x) = x + 6$ and $S(x) = 7x$ for all $x \in X$. Let a sequence $\{x_n\} = \{1 + \frac{1}{n}\}_{n\in\mathbb{N}}$ in X, we have

$$\mathcal{M}_{M,N}(Ax_n, 7, t) = \mathcal{M}_{M,N}(Sx_n, 7, t) = 1_{L^*},$$

where $7 \in S(X)$ and $t > 0$. Hence the mappings A and S satisfy the (CLR_S) property.

Example 2.21. The conclusion of Example 2.20 remains true if the self mappings A and S are defined on X by $A(x) = \frac{x}{9}$ and $S(x) = \frac{3x}{4}$ for all $x \in X$. Let a sequence $\{x_n\} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ in X. Since

$$\mathcal{M}_{M,N}(Ax_n, 0, t) = \mathcal{M}_{M,N}(Sx_n, 0, t) = 1_{L^*},$$

where $0 \in S(X)$ and $t > 0$. Therefore the mappings A and S satisfy the (CLR_S) property.

From the Examples 2.20-2.21, it is evident that a pair (A, S) satisfying the property (E.A) along with closedness of the subspace $S(X)$ always enjoys the (CLR_S) property.

On the lines of Imdad et al. [16], we define the (CLR_{ST}) property (with respect to mappings S and T) as follows:

Definition 2.22. Two pairs (A, S) and (B, T) of self mappings of a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ are said to satisfy the (CLR_{ST}) property with respect to mappings S and T if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that for all $t > 0$

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(Ax_n, z, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(Sx_n, z, t) =$$
$$\lim_{n\to\infty} \mathcal{M}_{M,N}(By_n, z, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(Ty_n, z, t) = 1_{L^*},$$

where $z \in S(X) \cap T(X)$.

Definition 2.23. [29] Two families of self mappings $\{A_i\}_{i=1}^m$ and $\{S_k\}_{k=1}^n$ are said to be pairwise commuting if

1. $A_iA_j = A_jA_i$ for all $i, j \in \{1, 2, \ldots, m\}$,
2. $S_kS_l = S_lS_k$ for all $k, l \in \{1, 2, \ldots, n\}$,
3. $A_iS_k = S_kA_i$ for all $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

3. Main results

Let Φ be the set of all continuous functions $\phi : L^* \to L^*$, such that $\phi(t) >_{L^*} t$ for all $t \in L^*\backslash\{0_{L^*}, 1_{L^*}\}$.

Example 3.1. [25] Let $\phi : L^* \to L^*$ defined by $\phi(t_1, t_2) = (\sqrt{t_1}, 0)$ for every $t = (t_1, t_2) \in L^*\backslash\{0_{L^*}, 1_{L^*}\}$.

Sedghi et al. [25, Theorem 2.1] proved the following theorem:

Theorem 3.2. *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a complete intuitionistic fuzzy metric space and A, B, S and T be self mappings of X satisfying the following conditions:*

$$A(X) \subset T(X), B(X) \subset S(X), \tag{3.1}$$

$$\int_0^{\mathcal{M}_{M,N}(Ax,By,t)} \varphi(s)ds \geq_{L^*} \phi\left(\int_0^{\mathcal{L}_{M,N}(x,y,t)} \varphi(s)ds\right), \tag{3.2}$$

for all $x, y \in X$, $\phi \in \Phi$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable and non-negative satisfying for each $0 < \epsilon < 1$,

$$0 < \int_0^{\epsilon} \varphi(s)ds < 1, \int_0^1 \varphi(s)ds = 1, \tag{3.3}$$

and where

$$\mathcal{L}_{M,N}(x, y, t) = \min\left\{\begin{array}{c} \mathcal{M}_{M,N}(Sx, Ty, t), \mathcal{M}_{M,N}(Ax, Sx, t), \\ \mathcal{M}_{M,N}(By, Ty, t), \mathcal{M}_{M,N}(Sx, By, t), \\ \mathcal{M}_{M,N}(Ax, Ty, t) \end{array}\right\}.$$

Suppose that the pair (A, S) or (B, T) satisfies the property (E.A), one of $A(X)$ or $B(X)$ or $S(X)$ or $T(X)$ is a closed subset of X and the pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.

We begin with the following lemma.

Lemma 3.3. *Let A, B, S and T be self mappings of a modified IFM-space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ satisfying conditions (3.2)-(3.3) of Theorem 3.2. Suppose that*

1. *the pair (A, S) satisfies the (CLR_S) property $\Big($or the pair (B, T) satisfies the (CLR_T) property$\Big)$,*
2. $A(X) \subset T(X)$ $\Big($*or* $B(X) \subset S(X)\Big)$,

3. *$T(X)$ $\Big($or $S(X)\Big)$ is a closed subset of X,*
4. *$\{By_n\}$ converges for every sequence $\{y_n\}$ in X whenever $\{Ty_n\}$ converges $\Big($or $\{Ax_n\}$ converges for every sequence $\{x_n\}$ in X whenever $\{Sx_n\}$ converges$\Big)$,*

Then the pairs (A, S) and (B, T) share the (CLR_{ST}) property.

Proof. Suppose the pair (A, S) satisfies the (CLR_S) property, then there exists a sequence $\{x_n\}$ in X such that for all $t > 0$

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \text{ where } z \in S(X).$$

Since $A(X) \subset T(X)$ (wherein $T(X)$ is a closed subset of X), for each $\{x_n\}$ in X there corresponds a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z, \text{ where } z \in S(X) \cap T(X).$$

Thus in all, we have $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. By (4), the sequence $\{By_n\}$ converges and we need to show that $By_n \to z$ as $n \to \infty$. On using inequality (3.2), with $x = x_n$, $y = y_n$, we get

$$\int_0^{\mathcal{M}_{M,N}(Ax_n,By_n,t)} \varphi(s)ds \geq_{L^*} \phi\left(\int_0^{\mathcal{L}_{M,N}(x_n,y_n,t)} \varphi(s)ds\right), \tag{3.4}$$

where

$$\mathcal{L}_{M,N}(x_n, y_n, t) = \min \left\{ \begin{array}{c} \mathcal{M}_{M,N}(Sx_n, Ty_n, t), \mathcal{M}_{M,N}(Ax_n, Sx_n, t), \\ \mathcal{M}_{M,N}(By_n, Ty_n, t), \mathcal{M}_{M,N}(Sx_n, By_n, t), \\ \mathcal{M}_{M,N}(Ax_n, Ty_n, t) \end{array} \right\}.$$

Let $\lim_{n\to\infty} \mathcal{M}_{M,N}(By_n, l, t) = 1_{L^*}$, where $l(\neq z)$ for all $t > 0$. Then, taking limit as $n \to \infty$ in (3.4), we have

$$\int_0^{\mathcal{M}_{M,N}(z,l,t)} \varphi(s)ds \geq_{L^*} \phi\left(\lim_{n\to\infty} \int_0^{\mathcal{L}_{M,N}(x_n,y_n,t)} \varphi(s)ds\right), \tag{3.5}$$

where

$$\begin{aligned} \lim_{n\to\infty} \mathcal{L}_{M,N}(x_n, y_n, t) &= \min \left\{ \begin{array}{c} \mathcal{M}_{M,N}(z, z, t), \mathcal{M}_{M,N}(z, z, t), \\ \mathcal{M}_{M,N}(l, z, t), \mathcal{M}_{M,N}(z, l, t), \\ \mathcal{M}_{M,N}(z, z, t) \end{array} \right\} \\ &= \min\{1_{L^*}, 1_{L^*}, \mathcal{M}_{M,N}(l, z, t), \mathcal{M}_{M,N}(z, l, t), 1_{L^*}\} \\ &= \mathcal{M}_{M,N}(z, l, t). \end{aligned}$$

Hence from (3.5), we obtain

$$\begin{aligned} \int_0^{\mathcal{M}_{M,N}(z,l,t)} \varphi(s)ds &\geq_{L^*} \phi\left(\int_0^{\mathcal{M}_{M,N}(z,l,t)} \varphi(s)ds\right) \\ &>_{L^*} \int_0^{\mathcal{M}_{M,N}(z,l,t)} \varphi(s)ds, \end{aligned}$$

which is a contradiction, we have $z = l$. Therefore, the pairs (A, S) and (B, T) share the (CLR_{ST}) property. □

Theorem 3.4. *Let A, B, S and T be self mappings of a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ satisfying conditions (3.2)-(3.3) of Theorem 3.2. If the pairs (A, S) and (B, T) share the (CLR_{ST}) property, then (A, S) and (B, T) have a coincidence point each. Moreover, A, B, S and T have a unique common fixed point provided the pairs (A, S) and (B, T) are weakly compatible.*

Proof. Since the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $t > 0$

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$. Hence there exist points $u, v \in X$ such that $Su = z$ and $Tv = z$. We claim that $Au = Su$. For if $Au \neq Su$, then there exists a positive real number t such that $\mathcal{M}_{M,N}(Au, Su, t) < 1_{L^*}$. By (3.2), we get

$$\int_0^{\mathcal{M}_{M,N}(Au,By_n,t)} \varphi(s)ds \geq_{L^*} \phi\left(\int_0^{\mathcal{L}_{M,N}(u,y_n,t)} \varphi(s)ds\right), \tag{3.6}$$

where

$$\mathcal{L}_{M,N}(u, y_n, t) = \min\left\{\begin{array}{c} \mathcal{M}_{M,N}(Su, Ty_n, t), \mathcal{M}_{M,N}(Au, Su, t), \\ \mathcal{M}_{M,N}(By_n, Ty_n, t), \mathcal{M}_{M,N}(Su, By_n, t), \\ \mathcal{M}_{M,N}(Au, Ty_n, t) \end{array}\right\}.$$

Letting $n \to \infty$ in (3.6), we obtain

$$\int_0^{\mathcal{M}_{M,N}(Au,z,t)} \varphi(s)ds \geq_{L^*} \phi\left(\lim_{n\to\infty}\int_0^{\mathcal{L}_{M,N}(u,y_n,t)} \varphi(s)ds\right), \tag{3.7}$$

where

$$\begin{aligned} \lim_{n\to\infty} \mathcal{L}_{M,N}(u, y_n, t) &= \min\left\{\begin{array}{c} \mathcal{M}_{M,N}(z, z, t), \mathcal{M}_{M,N}(Au, z, t), \mathcal{M}_{M,N}(z, z, t), \\ \mathcal{M}_{M,N}(z, z, t), \mathcal{M}_{M,N}(Au, z, t) \end{array}\right\} \\ &= \min\{1_{L^*}, \mathcal{M}_{M,N}(Au, z, t), 1_{L^*}, 1_{L^*}, \mathcal{M}_{M,N}(Au, z, t)\} \\ &= \mathcal{M}_{M,N}(Au, z, t). \end{aligned}$$

From (3.7), we have

$$\begin{aligned} \int_0^{\mathcal{M}_{M,N}(Au,z,t)} \varphi(s)ds &\geq_{L^*} \phi\left(\int_0^{\mathcal{M}_{M,N}(Au,z,t)} \varphi(s)ds\right) \\ &>_{L^*} \int_0^{\mathcal{M}_{M,N}(Au,z,t)} \varphi(s)ds, \end{aligned}$$

which contradicts, hence $Au = Su = z$ which shows that u is a coincidence point of the pair (A, S). Now we assert that $Bv = Tv$. Suppose that $Bv \neq Tv$, then there exists a positive real number t such that $\mathcal{M}_{M,N}(Bv, Tv, t) < 1_{L^*}$. By (3.2), it follows that

$$\int_0^{\mathcal{M}_{M,N}(Au,Bv,t)} \varphi(s)ds \geq_{L^*} \phi\left(\int_0^{\mathcal{L}_{M,N}(u,v,t)} \varphi(s)ds\right), \tag{3.8}$$

where

$$\begin{aligned} \mathcal{L}_{M,N}(u, v, t) &= \min\left\{\begin{array}{c} \mathcal{M}_{M,N}(Su, Tv, t), \mathcal{M}_{M,N}(Au, Su, t), \\ \mathcal{M}_{M,N}(Bv, Tv, t), \mathcal{M}_{M,N}(Su, Bv, t), \\ \mathcal{M}_{M,N}(Au, Tv, t) \end{array}\right\} \\ &= \min\left\{\begin{array}{c} \mathcal{M}_{M,N}(z, z, t), \mathcal{M}_{M,N}(z, z, t), \\ \mathcal{M}_{M,N}(Bv, z, t), \mathcal{M}_{M,N}(z, Bv, t), \\ \mathcal{M}_{M,N}(z, z, t) \end{array}\right\} \\ &= \min\{1_{L^*}, 1_{L^*}, \mathcal{M}_{M,N}(Bv, z, t), \mathcal{M}_{M,N}(z, Bv, t), 1_{L^*}\} \\ &= \mathcal{M}_{M,N}(z, Bv, t). \end{aligned}$$

Hence (3.8) implies

$$\begin{aligned}\int_0^{\mathcal{M}_{M,N}(z,Bv,t)} \varphi(s)ds &\geq_{L^*} \phi\left(\int_0^{\mathcal{M}_{M,N}(z,Bv,t)} \varphi(s)ds\right)\\ &>_{L^*} \int_0^{\mathcal{M}_{M,N}(z,Bv,t)} \varphi(s)ds,\end{aligned}$$

which is a contradiction, that is, $z = Bv$ and hence $Bv = Tv = z$ which shows that v is a coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible and $Au = Su$, hence $Az = ASu = SAu = Sz$. We assert that $z = Az$. Let, on the contrary, $z \neq Az$, then there exists a positive real number t such that $\mathcal{M}_{M,N}(Az, z, t) < 1_{L^*}$. By (3.2), we get

$$\int_0^{\mathcal{M}_{M,N}(Az,Bv,t)} \varphi(s)ds \geq_{L^*} \phi\left(\int_0^{\mathcal{L}_{M,N}(z,v,t)} \varphi(s)ds\right), \tag{3.9}$$

where

$$\begin{aligned}\mathcal{L}_{M,N}(z,v,t) &= \min\left\{\begin{array}{c}\mathcal{M}_{M,N}(Sz,Tv,t), \mathcal{M}_{M,N}(Az,Sz,t),\\ \mathcal{M}_{M,N}(Bv,Tv,t), \mathcal{M}_{M,N}(Sz,Bv,t),\\ \mathcal{M}_{M,N}(Az,Tv,t)\end{array}\right\}\\ &= \min\left\{\begin{array}{c}\mathcal{M}_{M,N}(Az,z,t), \mathcal{M}_{M,N}(Az,Az,t),\\ \mathcal{M}_{M,N}(z,z,t), \mathcal{M}_{M,N}(Az,z,t),\\ \mathcal{M}_{M,N}(Az,z,t)\end{array}\right\}\\ &= \min\{\mathcal{M}_{M,N}(Az,z,t), 1_{L^*}, 1_{L^*}, \mathcal{M}_{M,N}(Az,z,t), \mathcal{M}_{M,N}(Az,z,t)\}\\ &= \mathcal{M}_{M,N}(Az,z,t).\end{aligned}$$

From (3.9), we have

$$\begin{aligned}\int_0^{\mathcal{M}_{M,N}(Az,z,t)} \varphi(s)ds &\geq_{L^*} \phi\left(\int_0^{\mathcal{M}_{M,N}(Az,z,t)} \varphi(s)ds\right)\\ &>_{L^*} \int_0^{\mathcal{M}_{M,N}(Az,z,t)} \varphi(s)ds,\end{aligned}$$

which contradicts. Therefore $Az = Sz = z$ which shows that z is a common fixed point of A and S.

Also the pair (B, T) is weakly compatible, therefore $Bz = BTv = TBv = Tz$. We claim that $z = Az$. If $z \neq Az$, then there exists a positive real number t such that $\mathcal{M}_{M,N}(z, Bz, t) < 1_{L^*}$. Using (3.2), we have

$$\int_0^{\mathcal{M}_{M,N}(Au,Bz,t)} \varphi(s)ds \geq_{L^*} \phi\left(\int_0^{\mathcal{L}_{M,N}(u,z,t)} \varphi(s)ds\right), \tag{3.10}$$

where

$$\begin{aligned}\mathcal{L}_{M,N}(u,z,t) &= \min\left\{\begin{array}{c}\mathcal{M}_{M,N}(Su,Tz,t), \mathcal{M}_{M,N}(Au,Su,t),\\ \mathcal{M}_{M,N}(Bz,Tz,t), \mathcal{M}_{M,N}(Su,Bz,t),\\ \mathcal{M}_{M,N}(Au,Tz,t)\end{array}\right\}\\ &= \min\left\{\begin{array}{c}\mathcal{M}_{M,N}(z,Bz,t), \mathcal{M}_{M,N}(z,z,t),\\ \mathcal{M}_{M,N}(Bz,Bz,t), \mathcal{M}_{M,N}(z,Bz,t),\\ \mathcal{M}_{M,N}(z,Bz,t)\end{array}\right\}\\ &= \min\{\mathcal{M}_{M,N}(z,Bz,t), 1_{L^*}, 1_{L^*}, \mathcal{M}_{M,N}(z,Bz,t), \mathcal{M}_{M,N}(z,Bz,t)\}\\ &= \mathcal{M}_{M,N}(z,Bz,t).\end{aligned}$$

From (3.10), we obtain

$$\begin{aligned}\int_0^{\mathcal{M}_{M,N}(z,Bz,t)} \varphi(s)ds &\geq_{L^*} \phi\left(\int_0^{\mathcal{M}_{M,N}(z,Bz,t)} \varphi(s)ds\right)\\ &>_{L^*} \int_0^{\mathcal{M}_{M,N}(z,Bz,t)} \varphi(s)ds,\end{aligned}$$

which is a contradiction, hence $Bz = z = Tz$ which shows that z is a common fixed point of B and T. Therefore z is a common fixed point of A, B, S and T. Uniqueness of the common fixed point is an easy consequence of the inequality (3.2). □

Example 3.5. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a modified IFMS, where $X = [3, 27)$, $\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ with

$$\mathcal{M}_{M,N}(x, y, t) = \left(\frac{t}{t+ \mid x - y \mid}, \frac{\mid x - y \mid}{t+ \mid x - y \mid}\right),$$

for all $x, y \in X$ and $t > 0$. Let $\phi : L^* \to L^*$ is defined as in Example 3.1. Define the self mappings A, B, S and T by

$$A(x) = \begin{cases} 3, & \text{if } x \in \{3\} \cup (11, 27);\\ 21, & \text{if } x \in (3, 11]. \end{cases} \quad B(x) = \begin{cases} 3, & \text{if } x \in \{3\} \cup (11, 27);\\ 8, & \text{if } x \in (3, 11]. \end{cases}$$

$$S(x) = \begin{cases} 3, & \text{if } x = 3;\\ 12, & \text{if } x \in (3, 11];\\ \frac{x+1}{4}, & \text{if } x \in (11, 27). \end{cases} \quad T(x) = \begin{cases} 3, & \text{if } x = 3;\\ 20, & \text{if } x \in (3, 11];\\ x-8, & \text{if } x \in (11, 27). \end{cases}$$

Consider the sequences $\{x_n\}_{n\in\mathbb{N}} = \{11 + \frac{1}{n}\}$, $\{y_n\}_{n\in\mathbb{N}} = \{3\}$ or $\{x_n\}_{n\in\mathbb{N}} = \{3\}$, $\{y_n\}_{n\in\mathbb{N}} = \{11 + \frac{1}{n}\}$, hence the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property, that is,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 3 \in S(X) \cap T(X).$$

It is noted that $A(X) = \{3, 21\} \nsubseteq [3, 19) \cup \{20\} = T(X)$ and $B(X) = \{3, 8\} \nsubseteq [3, 7) \cup \{12\} = S(X)$. Thus, all the conditions of Theorem 3.4 are satisfied with $\varphi(s) = 1$ and 3 is a unique common fixed point of the pairs (A, S) and (B, T) which also remains a point of coincidence as well. Also all the involved mappings are even discontinuous at their unique common fixed point 3.

Theorem 3.6. *Let A, B, S and T be self mappings of a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ satisfying conditions (3.2)-(3.3) of Theorem 3.2 and all the hypotheses of Lemma 3.3. Then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof. By Lemma 3.3, the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property, therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$. The rest of the proof can be completed on the lines of the proof of Theorem 3.4. □

Example 3.7. If we replace the self mappings S and T in Example 3.5 by the following besides retaining the rest:

$$S(x) = \begin{cases} 3, & \text{if } x = 3; \\ 3 + x, & \text{if } x \in (3, 11]; \\ \frac{x+1}{4}, & \text{if } x \in (11, 27). \end{cases} \quad T(x) = \begin{cases} 3, & \text{if } x = 3; \\ 11 + x, & \text{if } x \in (3, 11]; \\ x - 8, & \text{if } x \in (11, 27). \end{cases}$$

Then $A(X) = \{3, 21\} \subset [3, 22] = T(X)$ and $B(X) = \{3, 8\} \subset [3, 14] = S(X)$. Also all the conditions of Theorem 3.6 can be easily verified with $\varphi(s) = 1$. Here, 3 is a unique common fixed point of the pairs (A, S) and (B, T). Here, it is noted that Theorem 3.4 cannot be used in the context of this example as $S(X)$ and $T(X)$ are closed subsets of X.

By setting $A = B$ and $S = T$ in Theorem 3.4, we obtain a common fixed point theorem for a pair of weakly compatible mappings satisfying common limit in the range property with respect to mapping S.

Corollary 3.8. *Let A and S be self mappings of a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$. Suppose that the pair (A, S) enjoys the (CLR_S) property satisfying*

$$\int_0^{\mathcal{M}_{M,N}(Ax,Ay,t)} \varphi(s)ds \geq_{L^*} \phi\left(\int_0^{\mathcal{L}_{M,N}(x,y,t)} \varphi(s)ds\right), \tag{3.11}$$

where

$$\mathcal{L}_{M,N}(x, y, t) = \min \left\{ \begin{array}{c} \mathcal{M}_{M,N}(Sx, Sy, t), \mathcal{M}_{M,N}(Ax, Sx, t), \\ \mathcal{M}_{M,N}(Ay, Sy, t), \mathcal{M}_{M,N}(Sx, Ay, t), \\ \mathcal{M}_{M,N}(Ax, Sy, t) \end{array} \right\},$$

for all $x, y \in X$, $\phi \in \Phi$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable and non-negative such that (3.3) holds.

Then A and S have a unique common fixed point provided the pair (A, S) is weakly compatible.

Our next theorem is proved for six self mappings in modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ employing the notion of pairwise commuting which is studied by Tanveer et al. [29].

Theorem 3.9. *Let A, B, H, R, S and T be self mappings of a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$. Suppose that the pairs (A, SR) and (B, TH) enjoy the $(CLR_{(SR)(TH)})$ property satisfying*

$$\int_0^{\mathcal{M}_{M,N}(Ax,By,t)} \varphi(s)ds \geq_{L^*} \phi\left(\int_0^{\mathcal{L}_{M,N}(x,y,t)} \varphi(s)ds\right), \tag{3.12}$$

where

$$\mathcal{L}_{M,N}(x, y, t) = \min \left\{ \begin{array}{c} \mathcal{M}_{M,N}(SRx, THy, t), \mathcal{M}_{M,N}(Ax, SRx, t), \\ \mathcal{M}_{M,N}(By, THy, t), \mathcal{M}_{M,N}(SRx, By, t), \\ \mathcal{M}_{M,N}(Ax, THy, t) \end{array} \right\},$$

for all $x, y \in X$, $\phi \in \Phi$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable and non-negative such that (3.3) holds.

Then (A, SR) and (B, TH) have a coincidence point each. Moreover, A, B, H, R, S and T have a unique common fixed point provided both the pairs (A, SR) and (B, TH) commute pairwise, that is, $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$, $BH = HB$ and $TH = HT$.

Proof. Since the pairs (A, SR) and (B, TH) are commuting pairwise, obviously both the pairs are weakly compatible. By Theorem 3.4, A, B, SR and TH have a unique common fixed point z in X. Now we show that z is the unique common fixed point of the self mappings A, B, H, R, S and T. We claim that $z = Rz$.

For if $z \neq Rz$, then there exists a positive real number t such that $\mathcal{M}_{M,N}(z, Rz, t) < 1_{L^*}$. Using (3.12), we have

$$\int_0^{\mathcal{M}_{M,N}(A(Rz),Bz,t)} \varphi(s)ds \geq_{L^*} \phi\left(\int_0^{\mathcal{L}_{M,N}(Rz,z,t)} \varphi(s)ds\right), \tag{3.13}$$

where

$$\begin{aligned}
\mathcal{L}_{M,N}(Rz,z,t) &= \min\left\{\begin{array}{c} \mathcal{M}_{M,N}(SR(Rz),THz,t), \mathcal{M}_{M,N}(A(Rz),SR(Rz),t), \\ \mathcal{M}_{M,N}(Bz,THz,t), \mathcal{M}_{M,N}(SR(Rz),Bz,t), \\ \mathcal{M}_{M,N}(A(Rz),THz,t) \end{array}\right\} \\
&= \min\left\{\begin{array}{c} \mathcal{M}_{M,N}(Rz,z,t), \mathcal{M}_{M,N}(Rz,Rz,t), \\ \mathcal{M}_{M,N}(z,z,t), \mathcal{M}_{M,N}(Rz,z,t), \\ \mathcal{M}_{M,N}(Rz,z,t) \end{array}\right\} \\
&= \min\left\{\begin{array}{c} \mathcal{M}_{M,N}(Rz,z,t), 1_{L^*}, 1_{L^*}, \\ \mathcal{M}_{M,N}(Rz,z,t), \mathcal{M}_{M,N}(Rz,z,t) \end{array}\right\} \\
&= \mathcal{M}_{M,N}(Rz,z,t).
\end{aligned}$$

From (3.13), we obtain

$$\begin{aligned}
\int_0^{\mathcal{M}_{M,N}(Rz,z,t)} \varphi(s)ds &\geq_{L^*} \phi\left(\int_0^{\mathcal{M}_{M,N}(Rz,z,t)} \varphi(s)ds\right) \\
&>_{L^*} \int_0^{\mathcal{M}_{M,N}(Rz,z,t)} \varphi(s)ds,
\end{aligned}$$

which is a contradiction, we have $Rz = z$, that implies $S(Rz) = Sz = z$. Similarly, one can prove that $z = Hz$, that is, $T(Hz) = Tz = z$. Hence $z = Az = Bz = Sz = Rz = Tz = Hz$, and z is the unique common fixed point of A, B, H, R, S and T. □

Theorem 3.10. *Let $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_g\}_{g=1}^q$ be four finite families of self mappings of a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$ such that $A = A_1A_2 \dots A_m$, $B = B_1B_2 \dots B_n$, $S = S_1S_2 \dots S_p$ and $T = T_1T_2 \dots T_q$ satisfying conditions (3.2)-(3.3) of Theorem 3.2. If the pairs (A, S) and (B, T) share the (CLR_{ST}) property then (A, S) and (B, T) have a point of coincidence each.*

Moreover, $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n, \{S_k\}_{k=1}^p$ and $\{T_g\}_{g=1}^q$ have a unique common fixed point if the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_r\}, \{T_g\})$ commute pairwise, where $i \in \{1, 2, \dots, m\}, k \in \{1, 2, \dots, p\}, r \in \{1, 2, \dots, n\}$ and $g \in \{1, 2, \dots, q\}$.

Proof. The proof is similar to that of Theorem 4.1 contained in [14], hence the details are avoided. □

By setting $A_1 = A_2 = \dots = A_m = A$, $B_1 = B_2 = \dots = B_n = B$, $S_1 = S_2 = \dots = S_p = S$ and $T_1 = T_2 = \dots = T_q = T$ in Theorem 3.10, we deduce the following:

Corollary 3.11. *Let A, B, S and T be four self mappings of a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$. Suppose that*

1. *the pairs (A^m, S^p) and (B^n, T^q) satisfy the $(CLR_{S^pT^q})$ property*
2. $$\int_0^{\mathcal{M}_{M,N}(A^mx,B^ny,t)} \varphi(s)ds \geq_{L^*} \phi\left(\int_0^{\mathcal{L}_{M,N}(x,y,t)} \varphi(s)ds\right), \tag{3.14}$$

   where

   $$\mathcal{L}_{M,N}(x,y,t) = \min\left\{\begin{array}{c} \mathcal{M}_{M,N}(S^px,T^qy,t), \mathcal{M}_{M,N}(A^mx,S^px,t), \\ \mathcal{M}_{M,N}(B^ny,T^qy,t), \mathcal{M}_{M,N}(S^px,B^ny,t), \\ \mathcal{M}_{M,N}(A^mx,T^qy,t) \end{array}\right\},$$

for all $x, y \in X$, $\phi \in \Phi$, m, n, p, q are fixed positive integers and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable and non-negative such that (3.3) *holds.*

Then A, B, S and T have a unique common fixed point provided $AS = SA$ and $BT = TB$.

Remark 3.12. The conclusions of Lemma 3.3 and Theorems 3.4, 3.6 remain true if we replace condition (3.2) by the following:

$$\int_0^{\mathcal{M}_{M,N}(Ax,By,t)} \varphi(s)ds \geq_{L^*} \phi\left(\mathcal{L}_{M,N}(x,y,t)\right),$$

where

$$\mathcal{L}_{M,N}(x,y,t) = \min\left\{ \begin{array}{c} \int_0^{\mathcal{M}_{M,N}(Sx,Ty,t)} \varphi(s)ds, \int_0^{\mathcal{M}_{M,N}(Ax,Sx,t)} \varphi(s)ds, \\ \int_0^{\mathcal{M}_{M,N}(By,Ty,t)} \varphi(s)ds, \int_0^{\mathcal{M}_{M,N}(Sx,By,t)} \varphi(s)ds, \\ \int_0^{\mathcal{M}_{M,N}(Ax,Ty,t)} \varphi(s)ds \end{array} \right\},$$

for all $x, y \in X$, $\phi \in \Phi$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable and non-negative such that (3.3) holds.

Remark 3.13. The result similar to Corollary 3.11 can also be outlined in respect of condition (3.15).

Remark 3.14. Theorem 3.4 improves the results of Sedghi et al. [25, Theorem 2.1] without any requirement on completeness (or closedness) of the underlying space (or subspaces) and containment of ranges amongst the involved mappings. Theorem 3.9 extends the results of Sedghi et al. [25, Theorem 2.1] to six self mappings whereas Theorem 3.10 and Corollary 3.11 extend our results to four finite families of self mappings.

Notice that the earlier proved results, that is, Lemma 3.3, Theorems 3.4, 3.6, 3.9, 3.10 and Corollaries 3.8, 3.11, 3.15 are also valid for $\varphi(s) = 1$. For a sample, we present our next result which is a particular case of Theorem 3.4.

Corollary 3.15. *Let A, B, S and T be self mappings of a modified IFMS $(X, \mathcal{M}_{M,N}, \mathcal{T})$. Suppose that the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property satisfying*

$$\mathcal{M}_{M,N}(Ax, By, t) \geq_{L^*} \phi\left(\mathcal{L}_{M,N}(x,y,t)\right), \tag{3.15}$$

where

$$\mathcal{L}_{M,N}(x,y,t) = \min\left\{ \begin{array}{c} \mathcal{M}_{M,N}(Sx,Ty,t), \mathcal{M}_{M,N}(Ax,Sx,t), \\ \mathcal{M}_{M,N}(By,Ty,t), \mathcal{M}_{M,N}(Sx,By,t), \\ \mathcal{M}_{M,N}(Ax,Ty,t) \end{array} \right\},$$

for all $x, y \in X$, $\phi \in \Phi$. Then (A, S) and (B, T) have a coincidence point each. Moreover, A, B, S and T have a unique common fixed point provided the pairs (A, S) and (B, T) are weakly compatible.

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