Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 7 (2014), 42–50
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Common fixed point theorems under rational contractions in complex valued metric spaces

H. K. Nashine[a,*], M. Imdad[b], M. Hasan[c]

[a]*Department of Mathematics, Disha Institute of Management and Technology, Raipur-492101(Chhattisgarh), India.*
[b]*Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India.*
[c]*Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India.*

Communicated by Z. Kadelburg

Abstract

In this paper, we prove some common fixed point theorems for a pair of mappings satisfying certain rational contractions in the frame work of complex valued metric besides discussing consequences of our main results. To illustrate our results and to distinguish them from the existing ones, we equip the paper with suitable examples.

Keywords: Common fixed point, contractive type mapping, complex valued metric space.
2010 MSC: 54H25, 47H10.

1. Introduction

The Banach contraction principle [2] is a very popular and effective tool in solving existence problems in many branches of mathematical analysis. Due to simplicity and usefulness of this classic and celebrated theorem, it has become a very popular source of existence and uniqueness theorems in different branches of mathematical analysis. This theorem provides an impressive illustration of the unifying power of functional analytic methods and their usefulness in various disciplines. This famous theorem runs as follows.

Theorem 1.1. [2]. *Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying: $d(Tx, Ty) \leq kd(x, y)$, $\forall\, x, y \in X$, where k is a constant in $(0, 1)$. Then, T has a unique fixed point $x^* \in X$.*

*Corresponding author
Email addresses: drhknashine@gmail.com, hemantnashine@gmail.com (H. K. Nashine), mhimdad@yahoo.co.in (M. Imdad), hasan352000@gmail.com (M. Hasan)

Received 2012-5-14

The existing literature of fixed point theory contains a great number of generalizations of Banach contraction principle by using different form of contraction condition in various spaces. But majority of such generalizations are obtained by improving underlying contraction conditions which also includes contraction conditions dscribed by rational expressions. Recently, Azam et al. [1] introduced the notion of complex valued metric spaces and established some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. Indeed the definition of a cone metric space banks on the underlying Banach space which is not a division Ring. However, in complex valued metric spaces, one can study improvements of a host of results of analysis involving divisions. One can refer related results in [6, 7].

In this paper, proceeding on the lines of Azam et al. [1], we prove results on common fixed point for a pair of mappings satisfying relatively more general contraction conditions described by rational expressions in complex valued metric spaces. Some illustrative examples are also furnished to support the usability of our results.

2. Preliminaries

In what follows, we recall some notations and definitions that will be utilized in our subsequent discussion.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$, (ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$, (iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \preceq z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

The following definition is recently introduced by Azam et al. [1].

Definition 2.1. Let X be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies the following conditions:

(d_1): $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d_2): $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3): $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Definition 2.2. Let (X, d) be a complex valued metric space and $B \subseteq X$.

(i) $b \in B$ is called an interior point of a set B whenever there is $0 \prec r \in \mathbb{C}$ such that $N(b, r) \subseteq B$ where $N(b, r) = \{y \in X : d(b, y) \prec r\}$.

(ii) A point $x \in X$ is called a limit point of B whenever for every $0 \prec r \in \mathbb{C}$, $N(x, r) \cap (B \backslash \{X\}) \neq \emptyset$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A whereas a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{N(x, r) : x \in X, 0 \prec r\}.$$

is a sub-basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 2.3. Let (X, d) be a complex valued metric space and $\{x_n\}_{n\geq 1}$ be a sequence in X and $x \in X$. We say that

(i) the sequence $\{x_n\}_{n\geq 1}$ converges to x if for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$,

(ii) the sequence$\{x_n\}_{n\geq 1}$ is Cauchy sequence if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$,

(iii) the metric space (X, d) is a complete complex valued metric space if every Cauchy sequence is convergent.

Definition 2.4. (cf.[4]) Two families of self-mappings $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ are said to be pairwise commuting if: (i) $T_iT_j = T_jT_i$, $i, j \in \{1, 2, ...m\}$; (ii) $S_kS_l = S_lS_k$, $k, l \in \{1, 2, ...n\}$; (iii) $T_iS_k = S_kT_i$, $i \in \{1, 2, ...m\}$, $k \in \{1, 2, ...n\}$.

In [1], Azam et al. established the following two lemmas.

Lemma 2.5. *(cf. [1]) Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.6. *(cf. [1]) Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

3. Main results

In this section, we prove some common fixed point theorems for contraction conditions described by rational expressions (e.g. Azam et al. [1]). Our main result runs as follows.

Theorem 3.1. *Let (X, d) be a complete complex valued metric space and the mappings $S, T : X \to X$ satisfy:*

$$d(Sx, Ty) \precsim \alpha d(x, y) + \frac{\beta d(x, Sx)d(y, Ty)}{d(x, Ty) + d(y, Sx) + d(x, y)} \tag{3.1.1}$$

for all $x, y \in X$ such that $x \neq y$, $d(x, Ty) + d(y, Sx) + d(x, y) \neq 0$ where α, β are nonnegative reals with $\alpha + \beta < 1$ or $d(Sx, Ty) = 0$ if $d(x, Ty) + d(y, Sx) + d(x, y) = 0$. Then S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X and define $x_{2k+1} = Sx_{2k}$, $x_{2k+2} = Tx_{2k+1}, k = 0, 1, 2, \cdots$. Then,

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \precsim \alpha d(x_{2k}, x_{2k+1}) + \frac{\beta d(x_{2k}, Sx_{2k})d(x_{2k+1}, Tx_{2k+1})}{d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k}) + d(x_{2k}, x_{2k+1})} \\ &\precsim \alpha d(x_{2k}, x_{2k+1}) + \frac{\beta d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1}) + d(x_{2k}, x_{2k+1})} \end{aligned}$$

so that

$$|d(x_{2k+1}, x_{2k+2})| \leq \alpha |d(x_{2k}, x_{2k+1})| + \frac{\beta |d(x_{2k}, x_{2k+1})||d(x_{2k+1}, x_{2k+2})|}{|d(x_{2k}, x_{2k+2}) + d(x_{2k}, x_{2k+1})|}.$$

As (owing to triangular inequality) $|d(x_{2k+1}, x_{2k+2})| \leq |d(x_{2k+1}, x_{2k}) + d(x_{2k}, x_{2k+2})|$, therefore

$$|d(x_{2k+1}, x_{2k+2})| \leq \alpha |d(x_{2k}, x_{2k+1})| + \beta |d(x_{2k}, x_{2k+1})| = (\alpha + \beta)|d(x_{2k}, x_{2k+1})|.$$

Similarly,

$$\begin{aligned} d(x_{2k+3}, x_{2k+2}) &= d(Sx_{2k+2}, Tx_{2k+1}) \\ &\precsim \alpha d(x_{2k+2}, x_{2k+1}) + \frac{\beta d(x_{2k+2}, Sx_{2k+2})d(x_{2k+1}, Tx_{2k+1})}{d(x_{2k+1}, Sx_{2k+2}) + d(x_{2k+2}, Tx_{2k+1}) + d(x_{2k+2}, x_{2k+1})} \\ &\precsim \alpha d(x_{2k+2}, x_{2k+1}) + \frac{\beta d(x_{2k+2}, x_{2k+3})d(x_{2k+1}, x_{2k+2})}{d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+2}) + d(x_{2k+2}, x_{2k+1})} \end{aligned}$$

so that

$$|d(x_{2k+3}, x_{2k+2})| \leq \alpha |d(x_{2k+2}, x_{2k+1})| + \frac{\beta |d(x_{2k+1}, x_{2k+2})||d(x_{2k+2}, x_{2k+3})|}{|d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+1})|}.$$

As earlier, by the triangular inequality $|d(x_{2k+2}, x_{2k+3})| \le |d(x_{2k+2}, x_{2k+1}) + d(x_{2k+1}, x_{2k+3})|$ so that

$$|d(x_{2k+2}, x_{2k+3})| \le \alpha|(x_{2k+2}, x_{2k+1})| + \beta|d(x_{2k+1}, x_{2k+2})| = (\alpha+\beta)|d(x_{2k+1}, x_{2k+2})|.$$

If $\delta = \alpha + \beta < 1$, then $|d(x_{n+1}, x_{n+2})| \le \delta|d(x_n, x_{n+1})| \le \cdots \le \delta^{n+1}|d(x_0, x_1)|$ so that for any $m > n$,

$$\begin{aligned} |d(x_n, x_m)| &\le |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + \cdots + |d(x_{m-1}, x_m)| \\ &\le [\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}]|d(x_0, x_1)| \le \frac{\delta^n}{1-\delta}|d(x_0, x_1)| \end{aligned}$$

and hence $|d(x_m, x_n)| \le \frac{\delta^n}{1-\delta}|d(x_0, x_1)| \to 0, \ \ as \ m, n \to \infty$. which amounts to say that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Let on contrary that $u \ne Su$ so that $d(u, Su) = z > 0$. Now, we can write

$$\begin{aligned} z &\precsim d(u, x_{2k+2}) + d(x_{2k+2}, Su) \precsim d(u, x_{2k+2}) + d(Tx_{2k+1}, Su) \\ &\precsim d(u, x_{2k+2}) + \alpha d(x_{2k+1}, u) + \frac{\beta d(u, Su) d(x_{2k+1}, Tx_{2k+1})}{d(u, Tx_{2k+1}) + d(x_{2k+1}, Su) + d(u, x_{2k+1})} \\ &\precsim d(u, x_{2k+2}) + \alpha d(x_{2k+1}, u) + \frac{\beta z d(x_{2k+1}, x_{2k+2})}{d(u, x_{2k+2}) + d(x_{2k+1}, Su) + d(u, x_{2k+1})}, \end{aligned}$$

so that

$$|z| \le |d(u, x_{2k+2})| + \alpha|d(x_{2k+1}, u)| + \frac{\beta|z||d(x_{2k+1}, x_{2k+2})|}{|d(u, x_{2k+2}) + d(x_{2k+1}, Su) + d(u, x_{2k+1})|}$$

which on making $n \to \infty$, gives rise $|d(u, Su)| = 0$ which is a contradiction so that $u = Su$. Similarly, one can show that $u = Tu$.

To prove the uniqueness of common fixed point of S and T, let u^* in X be another common fixed point of S and T. Then

$$d(u, u^*) = d(Su, Tu^*) \precsim \alpha d(u, u^*) + \frac{\beta d(u, Su) d(u^*, Tu^*)}{d(u, Tu^*) + d(u^*, Su) + d(u, u^*)}$$

so that $|d(u, u^*)| \le \alpha|d(u, u^*)| + \frac{\beta|d(u,Su)||d(u^*,Tu^*)|}{|d(u,Tu^*)+d(u^*,Su)+d(u,u^*)|} \le \alpha|d(u, u^*)|$, so that $u^* = u$ which proves the uniqueness of common fixed point.

Secondly, we consider the case: $d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k}) + d(x_{2k}, x_{2k+1}) = 0$ (for any k) implies $d(Sx_{2k}, Tx_{2k+1}) = 0$, so that $x_{2k} = Sx_{2k} = x_{2k+1} = Tx_{2k+1} = x_{2k+2}$. Thus, we have $x_{2k+1} = Sx_{2k} = x_{2k}$, so there exist n_1 and m_1 such that $n_1 = Sm_1 = m_1$. Using foregoing arguments, one can also show that there exist n_2 and m_2 such that $n_2 = Tm_2 = m_2$. As $d(m_1, Tm_2) + d(m_2, Sm_1) + d(m_1, m_2) = 0$, (due to definition) implies $d(Sm_1, Tm_2) = 0$, so that $n_1 = Sm_1 = Tm_2 = n_2$ which in turn yields that $n_1 = Sm_1 = Sn_1$. Similarly, one can also have $n_2 = Tn_2$. As $n_1 = n_2$, implies $Sn_1 = Tn_1 = n_1$, therefore $n_1 = n_2$, is common fixed point of S and T.

We now prove that S and T have unique common fixed point. For this, assume that n_1^* in X is another common fixed point of S and T. Then we have $Sn_1^* = Tn_1^* = n_1^*$. As $d(n_1, Tn_1^*) + d(n_1^*, Sn_1) + d(n_1, n_1^*) = 0$, therefore $d(n_1, n_1^*) = d(Sn_1, Tn_1^*) = 0$. This implies that $n_1^* = n_1$. This completes the proof of the theorem. □

Corollary 3.2. *Let (X, d) be a complete complex valued metric space and let the mapping $T : X \to X$ satisfy:*

$$d(Tx, Ty) \precsim \alpha d(x, y) + \frac{\beta d(x, Tx) d(y, Ty)}{d(x, Ty) + d(y, Tx) + d(x, y)}$$

for all $x, y \in X$ such that $x \ne y$, $d(x, Ty) + d(y, Tx) + d(x, y) \ne 0$ where α, β are nonnegative reals with $\alpha + \beta < 1$ or $d(Tx, Ty) = 0$ if $d(x, Ty) + d(y, Tx) + d(x, y) = 0$. Then T has a unique fixed point.

As an application of Theorem 3.1, we prove the following theorem for two finite families of mappings.

Theorem 3.3. *If $\{T_i\}_1^m$ and $\{S_i\}_1^n$ are two finite pairwise commuting finite families of self-mappings defined on a complete complex valued metric space (X, d) such that the mappings S and T (with $T = T_1T_2...T_m$ and $S = S_1S_2...S_n$) satisfy the condition (3.1.1), then the component maps of the two families $\{T_i\}_1^m$ and $\{S_i\}_1^n$ have a unique common fixed point.*

Proof. In view of Theorem 3.1, one can infer that T and S have a unique common fixed point l i.e. $Tl = Sl = l$. Now we are required to show that l is common fixed point of all the components maps of both the families. In view of pairwise commutativity of the families $\{T_i\}_1^m$ and $\{S_i\}_1^n$, (for every $1 \le k \le m$) we can write

$$T_kl = T_kSl = ST_kl \ \ and \ \ T_kl = T_kTl = TT_kl$$

which show that T_kl (for every k) is also a common fixed point of T and S. By using the uniqueness of common fixed point, we can write $T_kl = l$ (for every k) which shows that l is a common fixed point of the family $\{T_i\}_1^m$. Using the foregoing arguments, one can also show that (for every $1 \le k \le n$) $S_kl = l$. This completes the proof of the theorem. □

By setting $T_1 = T_2 = ... = T_m = F$ and $S_1 = S_2 = ... = S_n = G$, in Theorem 3.3, we derive the following common fixed point theorem involving iterates of mappings.

Corollary 3.4. *If F and G are two commuting self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d(F^mx, G^ny) \precsim \alpha d(x, y) + \frac{\beta d(x, F^mx)d(y, G^ny)}{d(x, G^ny) + d(y, F^mx) + d(x, y)}$$

for all $x, y \in X$, where α, β are nonnegative reals with $\alpha + \beta < 1$ or $d(F^mx, G^ny) = 0$ if $d(x, G^ny) + d(y, F^mx) + d(x, y) = 0$, then F and G have a unique common fixed point.

By setting $m = n$ and $F = G = T$ in Corollary 3.3, we deduce the following corollary.

Corollary 3.5. *Let (X, d) be a complete complex valued metric space and let the mapping $T : X \to X$ satisfies (for some fixed n):*

$$d(T^nx, T^ny) \precsim \alpha d(x, y) + \frac{\beta d(x, T^nx)d(y, T^ny)}{d(x, T^ny) + d(y, T^nx) + d(x, y)}$$

for all $x, y \in X$ such that $x \neq y$, $d(x, Ty) + d(y, Tx) + d(x, y) \neq 0$ where α, β are nonnegative reals with $\alpha + \beta < 1$ or $d(T^nx, T^ny) = 0$ if $d(x, T^ny) + d(y, T^nx) + d(x, y) = 0$. Then T has a unique fixed point.

Proof. By Corollary 3.1, we obtain $v \in X$ such that $T^nv = v$. The result then follows from the observation

$$\begin{aligned} d(Tv, v) &= d(TT^nv, T^nv) = d(T^nTv, T^nv) \precsim \alpha d(Tv, v) + \frac{\beta d(Tv, T^nTv)d(v, T^nv)}{d(Tv, T^nv) + d(v, T^nTv) + d(Tv, v)} \\ &\precsim \alpha d(Tv, v) + \frac{\beta d(Tv, TT^nv)d(v, v)}{d(Tv, v) + d(v, TT^nv) + d(Tv, v)} = \alpha d(Tv, v). \end{aligned}$$

□

By setting $\beta = 0$, we draw following corollary which can be viewed as an extension of Bryant [3] theorem to complex valued metric spaces.

Corollary 3.6. *If $T : X \to X$ is a mapping defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d(T^nx, T^ny) \precsim \alpha d(x, y)$$

for all $x, y \in X$, where α is nonnegative real with $\alpha < 1$, then T has a unique fixed point.

The following example demonstrates the superiority of Bryant theorem over Banach contraction theorem.

Example 3.7. Let $X = \mathbb{C}$ be set of complex number. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then $(\mathbb{C}, d)$, is a complete complex valued metric space. Define $T : \mathbb{C} \to \mathbb{C}$ as

$$T(x+iy) = \begin{cases} 0\ , & x, y \in Q \\ i, & x, y \in Q^c \\ 1, & x \in Q^c, y \in Q \\ 1+i, & x \in Q, y \in Q^c \end{cases}$$

Now for $x = \frac{1}{\sqrt{2}}$ and $y = 0$ we get $d(T(\frac{1}{\sqrt{2}}), T(0)) = d(1, 0) = 1 \precsim \lambda d(\frac{1}{\sqrt{2}}, 0) = \lambda \frac{1}{\sqrt{2}}$. Thus $\lambda \geq \sqrt{2}$, which is a contradiction as $0 \leq \lambda < 1$. However, notice that $T^2 z = 0$, so that $0 = d(T^2 z_1, T^2 z_2) \precsim \lambda d(z_1, z_2)$, which shows that T^2, satisfies the requirement of Bryant Theorem and $z = 0$ is the unique fixed point of T.

In what follows, we prove similar type of results for a different rational expression studied in Imdad and Khan [5].

Theorem 3.8. *Let (X, d) be a complete complex valued metric space and let the mappings $S, T : X \to X$ satisfy:*

$$d(Sx, Ty) \precsim \alpha d(x, y) + \frac{\beta[d^2(x, Ty) + d^2(y, Sx)]}{d(x, Ty) + d(y, Sx)} + \gamma[d(x, Sx) + d(y, Ty)] \tag{3.3.1}$$

for all $x, y \in X$ such that $x \neq y$, where α, β and γ are nonnegative reals with $\alpha+2\beta+2\gamma < 1$ or $d(Sx, Ty) = 0$ if $d(x, Ty) + d(y, Sx) = 0$. Then pair (S, T) have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X and define $x_{2k+1} = Sx_{2k},\ \ x_{2k+2} = Tx_{2k+1}, k = 0, 1, 2, \cdots$. Then,

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \precsim \alpha d(x_{2k}, x_{2k+1}) + \frac{\beta[d^2(x_{2k}, Tx_{2k+1}) + d^2(x_{2k+1}, Sx_{2k})]}{d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k})} \\ &+ \gamma[d(x_{2k}, Sx_{2k}) + d(x_{2k+1}, Tx_{2k+1})] \\ &\precsim \alpha d(x_{2k}, x_{2k+1}) + \frac{\beta[d^2(x_{2k}, x_{2k+2}) + d^2(x_{2k+1}, x_{2k+1})]}{d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})} \\ &+ \gamma[d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})] \end{aligned}$$

$$\text{so that } |d(x_{2k+1}, x_{2k+2})| \leq \alpha |d(x_{2k}, x_{2k+1})| + \frac{\beta |d^2(x_{2k}, x_{2k+2})|}{|d(x_{2k}, x_{2k+2})|} + \gamma[|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|]$$

$$\text{or } |d(x_{2k+1}, x_{2k+2})| \leq \alpha |d(x_{2k}, x_{2k+1})| + \beta |d(x_{2k}, x_{2k+2})| + \gamma[|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|].$$

As $|d(x_{2k}, x_{2k+2})| < |d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|$, therefore

$$\begin{aligned} |d(x_{2k+1}, x_{2k+2})| \ \ \leq \alpha \ \ & |d(x_{2k}, x_{2k+1})| + \beta[|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|] \\ & + \gamma[|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|]. \end{aligned}$$

or

$$|d(x_{2k+1}, x_{2k+2})| \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \right) |d(x_{2k}, x_{2k+1})|.$$

Also,

$$\begin{aligned} d(x_{2k+3}, x_{2k+2}) &= d(Sx_{2k+2}, Tx_{2k+1}) \precsim \alpha d(x_{2k+2}, x_{2k+1}) + \frac{\beta[d^2(x_{2k+2}, Tx_{2k+1}) + d^2(x_{2k+1}, Sx_{2k+2})]}{d(x_{2k+2}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k+2})} \\ &+ \gamma[d(x_{2k+2}, Sx_{2k+2}) + d(x_{2k+1}, Tx_{2k+1})] \\ &\precsim \alpha d(x_{2k+2}, x_{2k+1}) + \frac{\beta[d^2(x_{2k+2}, x_{2k+2}) + d^2(x_{2k+1}, x_{2k+3})]}{d(x_{2k+2}, x_{2k+2}) + d(x_{2k+1}, x_{2k+3})} \\ &+ \gamma[d(x_{2k+2}, x_{2k+3}) + d(x_{2k+1}, x_{2k+2})] \end{aligned}$$

so that $|d(x_{2k+3}, x_{2k+2})| \leq \alpha|d(x_{2k+2}, x_{2k+1})| + \dfrac{\beta|d^2(x_{2k+1}, x_{2k+3})|}{|d(x_{2k+1}, x_{2k+3})|} + \gamma[|d(x_{2k+2}, x_{2k+3})| + |d(x_{2k+2}, x_{2k+1})|]$

or $|d(x_{2k+3}, x_{2k+2})| \leq \alpha|d(x_{2k+2}, x_{2k+1})| + \beta|d(x_{2k+1}, x_{2k+3})| + \gamma[|d(x_{2k+3}, x_{2k+2})| + |d(x_{2k+2}, x_{2k+1})|]$.

As $|d(x_{2k+1}, x_{2k+3})| < |d(x_{2k+1}, x_{2k+2})| + |d(x_{2k+2}, x_{2k+3})|$, therefore

$$\begin{aligned} |d(x_{2k+3}, x_{2k+2})| \leq \alpha|d(x_{2k+2}, x_{2k+1})| &+ \beta[|d(x_{2k+3}, x_{2k+2})| + |d(x_{2k+2}, x_{2k+1})|] \\ &+ \gamma[|d(x_{2k+3}, x_{2k+2})| + |d(x_{2k+2}, x_{2k+1})|]. \end{aligned}$$

$$\text{or } |d(x_{2k+3}, x_{2k+2})| \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)|d(x_{2k+2}, x_{2k+1})|.$$

If $\delta = \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right) < 1$, we have $|d(x_{n+1}, x_{n+2})| \leq \delta|d(x_n, x_{n+1})| \leq \cdots \leq \delta^{n+1}|d(x_0, x_1)|$ so that for any $m > n$,

$$\begin{aligned} |d(x_n, x_m)| &\leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + \cdots + |d(x_{m-1}, x_m)| \\ &\leq [\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}]|d(x_0, x_1)| \leq \frac{\delta^n}{1-\delta}|d(x_0, x_1)| \to 0, \ \ as \ \ m, n \to \infty, \end{aligned}$$

which shows that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$. Let on contrary $u \neq Su$, so that $d(u, Su) = z > 0$ and we can have

$$\begin{aligned} z &\precsim d(u, x_{2k+2}) + d(x_{2k+2}, Su) \precsim d(u, x_{2k+2}) + d(Tx_{2k+1}, Su) \\ &\precsim d(u, x_{2k+2}) + \alpha d(u, x_{2k+1}) + \frac{\beta[d^2(u, Tx_{2k+1}) + d^2(x_{2k+1}, Su)]}{d(u, Tx_{2k+1}) + d(x_{2k+1}, Su)} \\ &+ \gamma[d(u, Su) + d(x_{2k+1}, Tx_{2k+1})] \\ &\precsim d(u, x_{2k+2}) + \alpha d(u, x_{2k+1}) + \frac{\beta[d^2(u, x_{2k+2}) + d^2(x_{2k+1}, Su)]}{d(u, x_{2k+2}) + d(x_{2k+1}, Su)} + \gamma[z + d(x_{2k+1}, x_{2k+2})], \end{aligned}$$

$$\text{or } |z| \leq |d(u, x_{2k+2})| + \alpha|d(x_{2k+1}, u)| + \frac{\beta|[d^2(u, x_{2k+2}) + d^2(x_{2k+1}, Su)]|}{|d(u, x_{2k+2}) + d(x_{2k+1}, Su)|} + \gamma[|z| + |d(x_{2k+1}, x_{2k+2})|]$$

which on making $n \to \infty$, gives rise $|d(u, Su)| = 0$ a contradiction so that $u = Su$. Similarly, one can show that $u = Tu$. As in Theorem 3.1. the uniqueness of common fixed point remains a consequence of contraction condition (3.3.1). The proof can be completed in the line of Theorem 3.1. This completes the proof of the theorem. □

By setting $S = T$, we get the following:

Corollary 3.9. *Let (X, d) be a complete complex valued metric space and let the mapping $T : X \to X$ satisfy:*

$$d(Tx, Ty) \precsim \alpha d(x, y) + \frac{\beta[d^2(x, Ty) + d^2(y, Tx)]}{d(x, Ty) + d(y, Tx)} + \gamma[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ such that $x \neq y$, where α, β and γ are nonnegative reals with $\alpha+2\beta+2\gamma < 1$ or $d(Tx, Ty) = 0$ if $d(x, Ty) + d(y, Tx) = 0$. Then T has a unique fixed point.

As an application of Theorem 3.8, we prove the following theorem for two finite families of mappings.

Theorem 3.10. *If $\{T_i\}_1^m$ and $\{S_i\}_1^n$ are two finite pairwise commuting finite families of self-mapping defined on a complete complex valued metric space (X, d) such that the mappings S and T (with $T = T_1T_2...T_m$ and $S = S_1S_2...S_n$) satisfy the condition (3.3.1), then the component maps of the two families $\{T_i\}_1^m$ and $\{S_i\}_1^n$ have a unique common fixed point.*

Proof. The proof of this theorem is identical to that of Theorem 3.8. □

By setting $T_1 = T_2 = ... = T_m = G$ and $S_1 = S_2 = ... = S_n = F$, in Theorem 3.10, we derive the following common fixed point theorem involving iterates of mappings.

Corollary 3.11. *Let (X, d) be a complete complex valued metric space and let the mappings $F, G : X \to X$ satisfy:*

$$d(F^m x, G^n y) \precsim \alpha d(x, y) + \frac{\beta[d^2(x, G^n y) + d^2(y, F^m x)]}{d(x, G^n y) + d(y, F^m x)} + \gamma[d(x, F^m x) + d(y, G^n y)]$$

for all $x, y \in X$ such that $x \neq y$, where α, β and γ are nonnegative reals with $\alpha + 2\beta + 2\gamma < 1$ or $d(F^m x, G^n y) = 0$ if $d(x, G^n y) + d(y, F^m x) = 0$. Then pair (F, G) have unique common fixed point.

By setting $m = n$ and $F = G = T$ in Corollary 3.11, we deduce the following corollary.

Corollary 3.12. *Let (X, d) be a complete complex valued metric space and let the mapping $T : X \to X$ satisfy (for some fixed n):*

$$d(T^n x, T^n y) \precsim \alpha d(x, y) + \frac{\beta[d^2(x, T^n y) + d^2(y, T^n x)]}{d(x, T^n y) + d(y, T^n x)} + \gamma[d(x, T^n x) + d(y, T^n y)]$$

for all $x, y \in X$ such that $x \neq y$, where α, β and γ are nonnegative reals with $\alpha + 2\beta + 2\gamma < 1$ or $d(T^n x, T^n y) = 0$ if $d(x, T^n y) + d(y, T^n x) = 0$. Then T has a unique fixed point.

By setting $\beta = \gamma = 0$ we draw following corollary which can be viewed as an extension of Bryant [3] theorem to complex valued metric spaces.

Corollary 3.13. *If $T : X \to X$ is a mapping defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d(T^n x, T^n y) \precsim \alpha d(x, y)$$

for all $x, y \in X$, where α is a nonnegative real with $\alpha < 1$, then T has a unique fixed point.

We conclude this paper with an illustrative example which demonstrates Theorem 3.1.

Example 3.14. Consider

$$X_1 = \{z \in \mathbb{C} : Re(z) \geq 0, Im(z) = 0\} \text{ and } X_2 = \{z \in \mathbb{C} : Im(z) \geq 0, Re(z) = 0\}$$

and write $X = X_1 \cup X_2$. Define a mapping $d : X \times X \to \mathbb{C}$ as :

$$d(z_1, z_2) = \begin{cases} \max\{x_1, x_2\} + i \max\{x_1, x_2\} , & z_1, z_2 \in X_1 \\ \max\{y_1, y_2\} + i \max\{y_1, y_2\}, & z_1, z_2 \in X_2 \\ (x_1 + y_2) + i(x_1 + y_2), & z_1 \in X_1, z_2 \in X_2 \\ (x_2 + y_1) + i(x_2 + y_1), & z_1 \in X_2, z_2 \in X_1 \end{cases}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. By a routine calculation, one can easily verify that (X, d) is a complete complex valued metric space.
Set $T = S$ and define a self-mapping T on X (with $z = (x, y)$) as

$$Tz = \begin{cases} (\frac{x}{2}, 0) , & z \in X_1 \\ (0, \dfrac{y}{2}), & z \in X_2. \end{cases}$$

Now, we show that $T(= S)$ satisfies condition (3.1.1). We distinguish the following cases: Before discussing diffrent cases, one needs to notice that

$$0 \precsim d(Sz_1, Tz_2), d(z_1, z_2), \frac{d(z_1, Sz_1)d(z_2, Tz_2)}{d(z_1, Tz_2) + d(z_2, Sz_1) + d(z_1, z_2)}.$$

Firstly, if $z_1, z_2 \in X_1$, then we have

$$d(Sz_1, Tz_2) = d((\frac{x_1}{2}, 0), (\frac{x_2}{2}, 0)) = \max\{\frac{x_1}{2}, \frac{x_2}{2}\} + i \max\{\frac{x_1}{2}, \frac{x_2}{2}\}$$

$$= \max\{\frac{x_1}{2}, \frac{x_2}{2}\}(1 + i) = \frac{1}{2} \max\{x_1, x_2\}(1 + i) \precsim \frac{1}{2} d(z_1, z_2).$$

Seconly,if $z_1, z_2 \in X_2$, then we have

$$d(Sz_1, Tz_2) = d((0, \frac{y_1}{2}), (0, \frac{y_2}{2})) = \max\{\frac{y_1}{2}, \frac{y_2}{2}\} + i \max\{\frac{y_1}{2}, \frac{y_2}{2}\}$$

$$= \max\{\frac{y_1}{2}, \frac{y_2}{2}\}(1 + i) = \frac{1}{2} \max\{y_1, y_2\}(1 + i) \precsim \frac{1}{2} d(z_1, z_2).$$

Thirdly, if $z_1 \in X_1, z_2 \in X_2$, then we have

$$d(Sz_1, Tz_2) = d((\frac{x_1}{2}, 0), (0, \frac{y_2}{2})) = [\frac{x_1}{2} + \frac{y_2}{2}](1 + i) = \frac{1}{2}[x_1 + y_2](1 + i) = \frac{1}{2}[x_1 + y_2](1 + i) \precsim \frac{1}{2} d(z_1, z_2).$$

Finally, if $z_2 \in X_1, z_1 \in X_2$, then we have

$$d(Sz_1, Tz_2) = d((0, \frac{y_1}{2}), (\frac{x_2}{2}, 0)) = [\frac{y_1}{2} + \frac{x_2}{2}](1 + i) = \frac{1}{2}[y_1 + x_2](1 + i) = \frac{1}{2}[y_1 + x_2](1 + i) \precsim \frac{1}{2} d(z_1, z_2).$$

Thus, condition (3.1.1) is satisfied with $\alpha = \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$ and, in all, conditions of Theorem 3.1 are satisfied. Notice that the point $0 \in X$ remains fixed under T and is indeed unique. Thus, in all, this example substantiates the genuineness of our results proved in this paper.

Acknowledgements:

The authors are thankful to the referee for his/her valuable comments and suggestions.

References

[1] A. Azam, B. Fisher, M. Khan, *Common Fixed Point Theorems in Complex Valued Metric Spaces*, Num. Func. Anal. Opt. **32** (2011), 243–253. 1, 2, 2, 2.5, 2.6, 3

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181. 1, 1.1

[3] V. W. Bryant, *A remark on a fixed-point theorem for iterated mappings*, Amer. Math. Monthly **75** (1968), 399–400. 3, 3

[4] M. Imdad, J. Ali, M. Tanveer, *Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces*, Chaos Solitons Fractals **42** (2009), 3121–3129. 2.4

[5] M. Imdad, Q. H. Khan, *Six mapping satisfying a rational inequality*, Radovi Math(Presently Sarajevo Journal), **9** (1999), 251–260. 3

[6] F. Rouzkard, M. Imdad, *Some common fixed point theorems on complex valued metric spaces*, Comp. Math. Appls. **64** (2012), 1866–1874. 1

[7] W. Sintunavarat, P. Kumam, *Generalized common fixed point theorems in complex valued metric spaces and applications*, J. Inequalities Appl. **2012** (2012), 11 pages. 1