Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 7 (2014), 168–179
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

α-ψ-φ-contractive mappings in ordered partial b-metric spaces

Aiman Mukheimer[a,*]

[a] *Department of Mathematics and General Sciences, Prince Sultan University, P.O.Box 66833, Riyadh 11586, Saudi Arabia*

Communicated by Imdad Khan

Abstract

In this paper, we introduce the concept of α-ψ-φ-contractive self mapping in complete ordered partial b-metric space, and we study the existence of fixed points for such mappings under some conditions. Presented theorems in this paper extend and generalize the results derived by Mustafa et al., also some examples are given to illustrate the main results.

Keywords: b-metric space, fixed point theory, contraction, partial metric space.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

Fixed point theory is one of the most popular tool in nonlinear analysis. Most of the generalizations for metric fixed point theorems usually start from Banach contraction principle [8]. It is not easy to point out all the generalizations of this principle. In 1989, Bakhtin [7] introduced the concept of a b-metric space as a generalization of metric spaces. In 1993, Czerwik [9, 10] extended many results related to the $b-$metric spaces. In 1994, Matthews [15] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, O'Neill generalized the concept of partial metric space by admitting negative distances. In 2013, Shukla [21] generalized both the concept of b-metric and partial metric spaces by introducing the partial b-metric spaces. For example, many authors recently studied this principle and its generalizations in different types of metric spaces [12, 23, 1, 2, 18, 20, 5]. Close to our interest in this paper some authors studied some fixed point theorems in the so called $b-$metric space [16, 22, 24]. After then, some authors started to prove α-ψ versions of of certain fixed point theorems in different type metric spaces

*Corresponding author
Email address: mukheimer@psu.edu.sa (Aiman Mukheimer)

Received 2013-12-23

[3, 11, 4]. Mustafa in [17], gave a generalization of Banach's contraction principles in a complete ordered partial b-metric space by introducing a generalized $(\alpha, \psi)_s$-weakly contractive mapping. In this paper, we generalize a result of Mustafa in [17], by introducing the α-ψ-φ-contractive mapping in a complete ordered partial b-metric space.

Definition 1.1. [6] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.
The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

Definition 1.2. [15] Let X be a nonempty set. A function $p : X \times X \to [0, \infty)$ is called a partial metric if for all $x, y, z \in X$ the following conditions are satisfied:
(i) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
(ii) $p(x, x) \leq p(x, y)$;
(iii) $p(x, y) = p(y, x)$;
(iv) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
The pair (X, d) is called a partial metric space.

Remark 1.1. It is clear that the partial metric space need not be a b-metric spaces , since in a partial metric space if $p(x, y) = 0$ implies $p(x, x) = p(x, y) = p(y, y) = 0$ then $x = y$. But in a partial metric space if $x = y$ then $p(x, x) = p(x, y) = p(y, y)$ may not be equal zero. Therefore the partial metric space may not be a b-metric space.

On the other hand, Shukla[21] introduced the notion of a partial b-metric space as follows:

Definition 1.3. [21] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in X$ the following conditions are satisfied:
(i) $x = y$ if and only if $p_b(x, x) = p_b(x, y) = p_b(y, y)$;
(ii) $p_b(x, x) \leq p_b(x, y)$;
(iii) $p_b(x, y) = p_b(y, x)$;
(iv) $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.
The pair (X, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X, p_b).

Remark 1.2. The class of partial b-metric space (X, p_b) is effectively larger than the class of partial metric space, since a partial metric space is a special case of a partial b-metric space (X, p_b) when $s = 1$. Also, the class of partial b-metric space (X, p_b) is effectively larger than the class of b-metric space, since a b-metric space is a special case of a partial b-metric space (X, p_b) when the self distance $p(x, x) = 0$.

The following examples shows that a partial b-metric on X need not be a partial metric, nor a b-metric on X see also [17], [21].

Example 1.1. [21] Let $X = [0, \infty)$. Define a function $p_b : X \times X \to [0, \infty)$ such that $p_b(x, y) = [max\{x, y\}]^2 + |x - y|^2$ For all $x, y \in X$. Then (X, p_b) is a partial b-metric space on X with the coefficient $s = 2 > 1$. But, p_b is not a b-metric nor a partial metric on X.

Proposition 1.1. *[21] Let X be a nonempty set, and let p be a partial metric and d be a b-metric with the coefficient $s \geq 1$ on X. Then the function $p_b : X \times X \to [0, \infty)$ defined by $p_b(x, y) = p(x, y) + d(x, y)$ For all $x, y \in X$, is a partial b-metric on X with the coefficient s.*

Proposition 1.2. *[21] Let (X, p) be a partial metric space and $q \geq 1$. Then (X, p_b) is a partial b-metric space with coefficient $s = 2^{q-1}$, where p_b is defined by $p_b(x, y) = [p(x, y)]^q$.*

Definition 1.4. [14] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:
1. ψ is continuous and nondecreasing;
2. $\psi(t) = 0$ if and only if $t = 0$.

On the other hand, Mustafa[17] modify the Definition 1.3 in order that each partial b-metric p_b generates a b-metric d_{p_b} as follows:

Definition 1.5. [17] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to [0, \infty)$ is a partial b-metric if for all $x, y, z \in X$ the following conditions are satisfied:
(i) $x = y$ if and only if $p_b(x, x) = p_b(x, y) = p_b(y, y)$;
(ii) $p_b(x, x) \leq p_b(x, y)$;
(iii) $p_b(x, y) = p_b(y, x)$;
(iv) $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y) - p_b(z, z)] + (\frac{1-s}{2})(p_b(x, x) + p_b(y, y))$.
The pair (X, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X, p_b).

Example 1.2 (see also[17]). Let $X = \mathbb{R}$ is the set of real numbers. Consider the metric space (X, d) where d is the Euclidean distance metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $p_b(x, y) = (x - y)^2 + 5$ for all $x, y \in X$. Then p_b is a partial b-metric on X with $s = 2$, but it is not a partial metric on X. To see this, Let $x = 1$,$y = 4$ and $z = 2$. Then

$$p_b(1, 4) = (1 - 4)^2 + 5 = 14 \nleq p_b(1, 2) + p_b(2, 4) - p_b(2, 2) = 6 + 9 - 5 = 10.$$

Also, p_b is not a b-metric since $p_b(x, x) \neq 0$ for all $x \in X$.

Proposition 1.3. *[17] Every partial b-metric p_b defines a b-metric d_{p_b}, where*

$$d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, x) - p_b(y, y), \quad \text{for all } x, y \in X \tag{1.1}$$

Definition 1.6. [17] A sequence $\{x_n\}$ in a partial b-metric space (X, p_b) is said to be:
(i) p_b-convergent to a point $x \in X$ if $\lim_{n\to\infty} p_b(x, x_n) = p_b(x, x)$;
(ii) a p_b-Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n, x_m)$exists (and is finite);
(iii) A partial b-metric space (X, p_b) is said to be p_b-complete if every p_b-Cauchy sequence $\{x_n\}$ in X p_b converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n\to\infty} p_b(x_n, x) = p_b(x, x). \tag{1.2}$$

Lemma 1.1. [17] A sequence $\{x_n\}$ is a p_b-Cauchy sequence in a partial $b-$metric space (X, p_b) if and only if it is a b-Cauchy sequence in the b-metric space (X, d_{p_b}).

Lemma 1.2. [17] A partial b-metric space (X, p_b) is p_b-complete if and only if the b-metric space (X, d_{p_b}) is b-complete. Moreover, $\lim_{n\to\infty} d_{p_b}(x_n, x_m) = 0$ if and only if

$$\lim_{n\to\infty} p_b(x, x_n) = \lim_{n,m\to\infty} p_b(x, x_m) = p_b(x, x). \tag{1.3}$$

Lemma 1.3. [17] Let (X, p_b) be a partial b-metric space with the coefficient $s > 1$ and suppose that $\{x_n\}$ and$\{y_n\}$ are convergent to x and y, respectively. Then we have

$$\begin{aligned} \frac{1}{s^2} p_b(x, y) - \frac{1}{s} p_b(x, x) - p_b(y, y) &\leq \liminf_{n\to\infty} p_b(x_n, y_n) \\ &\leq \limsup_{n\to\infty} p_b(x_n, y_n) \\ &\leq s p_b(x, x) + s^2 p_b(y, y) + s^2 p_b(x, y). \end{aligned}$$

Definition 1.7. [13] Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a mapping. We say that T is nondecreasing with respect to $\preceq$ if

$$x, y \in X, \quad x \preceq y \Rightarrow Tx \preceq Ty.$$

Definition 1.8. [13] Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\}$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$.

Definition 1.9. [17] A triple $(X, \preceq, p_b)$ is called an ordered partial b-metric space if $(X, \preceq)$ is a partially ordered set and p_b is a partial b-metric on X.

Definition 1.10. Let (X, p_b) be a partial b-metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is α-admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$. Also we say that T is L_α-admissible (R_α-admissible) if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, y) \geq 1(\alpha(x, Ty) \geq 1)$.

Example 1.3. [19] Let $X = (0, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = lnx$ for all $x \in X$ and

$$\alpha(x, y) = \begin{cases} 2 & if\ x \geq y, \\ 0 & if\ x < y. \end{cases}$$

Then, T is α-admissible.

Example 1.4. [19] Let $X = [0, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = \sqrt{x}$ for all $x \in X$ and

$$\alpha(x, y) = \begin{cases} e^{x-y} & if\ x \geq y, \\ 0 & if\ x < y. \end{cases}$$

Then, T is α-admissible.

2. Main result

We now introduce the α-ψ-φ-contractive self mapping on partial b-metric space.

Definition 2.1. Let (X, p_b) be a partial b-metric space with the coefficient $s \geq 1$. We say that a mapping $T : X \to X$ is an α-ψ-φ-contractive mapping if there exist two altering distance functions ψ, φ and $\alpha : X \times X :\to [0, \infty)$ such that

$$\alpha(x, y)\psi(sp_b(Tx, Ty)) \leq \psi(M_s^T(x, y)) - \varphi(M_s^T(x, y)) \tag{2.1}$$

for all comparable $x, y \in X$.

where

$$M_s^T(x, y) = max\{p_b(x, y), p_b(x, Tx), p_b(y, Ty), \frac{p_b(x, Ty) + p_b(y, Tx)}{2s}\}. \tag{2.2}$$

Theorem 2.1. Let $(X, \preceq, p_b)$ be a p_b-complete ordered partial b-metric space with the coefficient $s \geq 1$. Let $T : X \to X$ be an an α-ψ-φ-contractive mapping. Suppose that the following conditions hold:

(1) T is α-admissible and L_α-admissible (or R_α-admissible);
(2) there exists $x_1 \in X$ such that $x_1 \preceq Tx_1$ and $\alpha(x_1, Tx_1) \geq 1$;
(3) T is continuous, nondecreasing, with respect to $\preceq$ and if $T^n x_1 \to z$ then $\alpha(z, z) \geq 1$.
Then, T has a fixed point.

Proof. Let $x_1 \in X$ such that $x_1 \preceq Tx_1$ and $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 1$. We have $x_2 = Tx_1 \preceq Tx_2 = x_3$ since $x_1 \preceq Tx_1$ and T is nondecreasing. Also, $x_3 = Tx_2 \preceq Tx_3 = x_4$ since $x_2 \preceq Tx_2$ and T is nondecreasing. By induction, we get

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point of T and the proof is finished. So we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is α-admissible, we deduce

$$\alpha(x_1, Tx_1) = \alpha(x_1, x_2) \geq 1 \Rightarrow \alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1.$$

By induction on n we get

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{2.3}$$

for all $n \in \mathbb{N}$.
Hence, by applying the α-ψ-φ-contractive condition and using (2.3) for all $n \in \mathbb{N}$ we get

$$\begin{aligned} \psi(sp_b(x_{n+1}, x_{n+2})) &\leq \alpha(x_n, x_{n+1})\psi(sp_b(Tx_n, Tx_{n+1})) \\ &\leq \psi(M_s^T(x_n, x_{n+1})) - \varphi(M_s^T(x_n, x_{n+1})) \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} M_s^T(x_n, x_{n+1}) &= max\{p_b(x_n, x_{n+1}), p_b(x_n, Tx_n), p_b(x_{n+1}, Tx_{x+1}), \\ &\quad \frac{p_b(x_n, Tx_{n+1}) + p_b(x_{n+1}, Tx_n)}{2s}\} \\ &= max\{p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{x+2}), \\ &\quad \frac{p_b(x_n, x_{n+2}) + p_b(x_{n+1}, x_{n+1})}{2s}\} \\ &\leq max\{p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{x+2}), \\ &\quad \frac{sp_b(x_n, x_{n+1}) + sp_b(x_{n+1}, x_{n+2}) + (1-s)p_b(x_{n+1}, x_{n+1})}{2s}\} \\ &= max\{p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{x+2})\} \end{aligned} \tag{2.5}$$

From (2.4)and (2.5) we get

$$\begin{aligned} \psi(p_b(x_{n+1}, x_{n+2})) &\leq \psi(max\{p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{x+2})\}) \\ &\quad -\varphi(max\{p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{x+2})\}) \end{aligned} \tag{2.6}$$

Assume that

$$max\{p_b(x_n, x_{n+1}), p_b(x_{n+1}, x_{x+2})\} = p_b(x_{n+1}, x_{x+2})$$

then by using properties of φ, we deduce

$$\begin{aligned} \psi(p_b(x_{n+1}, x_{n+2})) &\leq \psi(p_b(x_{n+1}, x_{n+2})) - \varphi(p_b(x_{n+1}, x_{n+2})) \\ &< \psi(p_b(x_{n+1}, x_{n+2})), \end{aligned}$$

which is a contradiction. Thus,

$$\psi(p_b(x_{n+1}, x_{n+2})) \leq \psi(p_b(x_n, x_{n+1})) - \varphi(p_b(x_n, x_{n+1})). \tag{2.7}$$

So the sequence $\{p_b(x_{n+1}, x_{n+2})\}$ is nonnegative and nondecreasing for all $n \in \mathbb{N}$. Hence there exists $r \geq 0$ such that

$$\lim_{n \to \infty} p_b(x_{n+1}, x_{n+2}) = r.$$

Letting $n \to \infty$ in (2.7) , we have

$$\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r).$$

Therefore, $\varphi(r) = 0$, and hence $r = 0$. Thus,

$$\lim_{n\to\infty} p_b(x_{n+1}, x_{n+2}) = 0. \tag{2.8}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in (X, p_b) which is equivalent to show that $\{x_n\}$ is a b-Cauchy sequence in (X, d_{p_b}). Assume not, that is, $\{x_n\}$ is not a b-Cauchy sequence in (X, d_{p_b}). Then there exist $\varepsilon > 0$ such that, for $k > 0$, there exist $n(k) > m(k) > k$ for which we can which we can find two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which

$$d_{p_b}(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \tag{2.9}$$

and

$$d_{p_b}(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.10}$$

Then we have

$$\begin{aligned} \varepsilon \leq d_{p_b}(x_{m(k)}, x_{n(k)}) &\leq sd_{p_b}(x_{m(k)}, x_{n(k)-1}) + sd_{p_b}(x_{n(k)-1}, x_{n(k)}) \\ &< s\varepsilon + sd_{p_b}(x_{n(k)-1}, x_{n(k)}). \end{aligned} \tag{2.11}$$

Taking the upper limit for (2.10) as $k \to \infty$, we have

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)-1}) \leq \limsup_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)-1}) \leq \varepsilon. \tag{2.12}$$

Also, from (2.11)and (2.12), we obtain

$$\varepsilon \leq \limsup_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)}) \leq s\varepsilon.$$

By using the triangular inequality and we deduce,

$$\begin{aligned} d_{p_b}(x_{m(k)+1}, x_{n(k)}) &\leq sd_{p_b}(x_{m(k)+1}, x_{m(k)}) + sd_{p_b}(x_{m(k)}, x_{n(k)}) \\ &\leq sd_{p_b}(x_{m(k)+1}, x_{m(k)}) + s^2 d_{p_b}(x_{m(k)}, x_{n(k)-1}) + s^2 d_{p_b}(x_{n(k)-1}, x_{n(k)}) \\ &\leq sd_{p_b}(x_{m(k)+1}, x_{m(k)}) + s^2\varepsilon + s^2 d_{p_b}(x_{n(k)-1}, x_{n(k)}), \end{aligned}$$

by taking the upper limit as $k \to \infty$ in the above inequality, we get

$$\limsup_{k\to\infty} d_{p_b}(x_{m(k)+1}, x_{n(k)}) \leq s^2\varepsilon.$$

Finally,

$$\begin{aligned} d_{p_b}(x_{m(k)+1}, x_{n(k)-1}) &\leq sd_{p_b}(x_{m(k)+1}, x_{m(k)}) + sd_{p_b}(x_{m(k)}, x_{n(k)-1}) \\ &\leq sd_{p_b}(x_{m(k)+1}, x_{m(k)}) + s\varepsilon. \end{aligned}$$

Also, by taking the upper limit as $k \to \infty$ in the above inequality, we get

$$\limsup_{k\to\infty} d_{p_b}(x_{m(k)+1}, x_{n(k)-1}) \leq s\varepsilon.$$

By using the definition of d_{p_b} and (2.8), we get

$$\limsup_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)-1}) = 2\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}).$$

Hence,

$$\frac{\varepsilon}{2s} \leq \liminf_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}) \leq \limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}) \leq \frac{\varepsilon}{2}, \tag{2.13}$$

Similarly,

$$\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)}) \leq \frac{s\varepsilon}{2}, \tag{2.14}$$

$$\frac{\varepsilon}{2s} \leq \limsup_{k\to\infty} p_b(x_{m(k)+1}, x_{n(k)}), \tag{2.15}$$

$$\limsup_{k\to\infty} p_b(x_{m(k)+1}, x_{n(k)-1}) \leq \frac{s\varepsilon}{2}. \tag{2.16}$$

Since T is L_α-admissible and using (2.3), we obtain $\alpha(x_{m(k)}, x_{n(k)-1}) \geq 1$.
By using (2.1) we get

$$\begin{aligned}\psi(sp_b(x_{m(k)+1}, x_{n(k)})) &\leq \alpha(x_{m(k)}, x_{n(k)-1})\psi(sp_b(Tx_{m(k)}, Tx_{n(k)-1}))\\ &\leq \psi(M_s^T(x_{m(k)}, x_{n(k)-1})) - \varphi(M_s^T(x_{m(k)}, x_{n(k)-1})),\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}M_s^T(x_{m(k)}, x_{n(k)-1}) &= max\{p_b(x_{m(k)}, x_{n(k)-1}), p_b(x_{m(k)}, Tx_{m(k)}), p_b(x_{n(k)-1}, Tx_{n(k)-1}),\\ &\quad \frac{p_b(x_{m(k)}, Tx_{n(k)-1}) + p_b(x_{n(k)-1}, Tx_{m(k)})}{2s}\}\\ &= max\{p_b(x_{m(k)}, x_{n(k)-1}), p_b(x_{m(k)}, x_{m(k)+1}), p_b(x_{n(k)-1}, x_{n(k)}),\\ &\quad \frac{p_b(x_{m(k)}, x_{n(k)}) + p_b(x_{n(k)-1}, x_{m(k)+1})}{2s}\}\end{aligned} \tag{2.18}$$

Taking the upper limit as $k \to \infty$ in the above inequality using (2.8),(2.13),(2.14)and (2.16) we get

$$\begin{aligned}\limsup_{k\to\infty} M_s^T(x_{m(k)}, x_{n(k)-1}) &= max\{\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}), \limsup_{k\to\infty} p_b(x_{m(k)}, x_{m(k)+1}),\\ &\quad \limsup_{k\to\infty} p_b(x_{n(k)-1}, x_{n(k)}),\\ &\quad \frac{\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)}) + \limsup_{k\to\infty} p_b(x_{n(k)-1}, x_{m(k)+1})}{2s}\}\\ &= max\{\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}), 0, 0,\\ &\quad \frac{\limsup_{k\to\infty} p_b(x_{m(k)}, x_{n(k)}) + \limsup_{k\to\infty} p_b(x_{n(k)-1}, x_{m(k)+1})}{2s}\}\\ &\leq max\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\}\\ &= \frac{\varepsilon}{2}.\end{aligned} \tag{2.19}$$

Next, by taking the upper limit in (2.17) as $k \to \infty$ and using(2.15) and (2.19) we obtain

$$\begin{aligned}\psi(s\frac{\varepsilon}{2s}) &\leq \psi(\limsup_{k\to\infty} sp_b(x_{m(k)+1}, x_{n(k)}))\\ &\leq \psi(\limsup_{k\to\infty} M_s^T(x_{m(k)}, x_{n(k)-1})) - \liminf_{k\to\infty} \varphi(M_s^T(x_{m(k)}, x_{n(k)-1})),\\ &\leq \psi(\frac{\varepsilon}{2}) - \varphi(\liminf_{k\to\infty} M_s^T(x_{m(k)}, x_{n(k)-1})),\end{aligned}$$

which implies that

$$\varphi(\liminf_{k\to\infty} M_s^T(x_{m(k)}, x_{n(k)-1})) = 0,$$

so

$$\liminf_{k\to\infty} M_s^T(x_{m(k)}, x_{n(k)-1})) = 0,$$

and by using (2.17) we obtain,

$$\liminf_{k\to\infty} p_b(x_{m(k)}, x_{n(k)-1}) = 0.$$

Therefore,

$$\liminf_{k\to\infty} d_{p_b}(x_{m(k)}, x_{n(k)-1}) = 0,$$

which is a contradiction with (2.13). Thus, the sequence is a b-Cauchy in the b-metric space (X, d_{p_b}). Since (X, p_b) is p_b-complete, then (X, d_{p_b}) is a b-complete b-metric space. So, it follows from the completeness that there exist $z \in X$ such that,

$$\lim_{n\to\infty} d_{p_b}(x_n, z) = 0.$$

Therefore, by using (2.8), the condition $p_b(x_n, x_n) \leq p_b(z, x_n)$ and $\lim_{n\to\infty} p_b(x_n, x_n) = 0$ we get

$$\lim_{n\to\infty} p_b(x_n, z) = \lim_{n\to\infty} p_b(x_n, x_n) = p_b(z, z) = 0.$$

By using the triangular inequality, we obtain

$$p_b(z, Tz) \leq sp_b(z, Tx_n) + sp_b(Tx_n, Tz).$$

So by taking limit as $n \to \infty$ in the above inequality and using the continuity of T we get

$$p_b(z, Tz) \leq s \lim_{n\to\infty} p_b(z, x_{n+1}) + s \lim_{n\to\infty} p_b(Tx_n, Tz) = sp_b(Tz, Tz). \tag{2.20}$$

Since $\alpha(z, z) \geq 1$ and using (2.1) we get

$$\psi(sp_b(Tz, Tz)) \leq \alpha(z, z)\psi(sp_b(Tz, Tz)) \leq \psi(M_s^T(z, z)) - \varphi(M_s^T(z, z)).$$

where

$$M_s^T(z, z) = max\{p_b(z, z), p_b(z, Tz), p_b(z, Tz), \frac{p_b(z, Tz) + p_b(z, Tz)}{2s}\} = p_b(z, Tz).$$

So

$$\psi(sp_b(Tz, Tz)) \leq \alpha(z, z)\psi(sp_b(Tz, Tz)) \leq \psi(p_b(z, Tz)) - \varphi(p_b(z, Tz)). \tag{2.21}$$

Since ψ is nondecreasing $sp_b(Tz, Tz) \leq p_b(z, Tz)$ and $sp_b(Tz, Tz) = p_b(z, Tz)$, we deduce $\varphi(p_b(z, Tz)) = 0.$, Hence we have $p_b(Tz, z) = p_b(Tz, Tz) = p_b(z, z) = 0$ and $Tz = z$. Thus, z is a fixed point of T. This completes the proof. □

In our next theorem we omit the condition of continuity in Theorem 2.1.

Theorem 2.2. Let $(X, \preceq, p_b)$ be a p_b-complete ordered partial b-metric space with the coefficient $s \geq 1$. Let $T : X \to X$ be an an α-ψ-φ-contractive mapping. Suppose that the following conditions hold:

(1) T is α-admissible and L_α-admissible (or R_α-admissible);
(2) there exists $x_1 \in X$ such that $x_1 \preceq Tx_1$ and $\alpha(x_1, Tx_1) \geq 1$;
(3) T is nondecreasing, with respect to $\preceq$;
(4) If $\{x_n\}$ is a sequence in X such that $x_n \preceq x$ for all $n \in \mathbb{N}$, $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in X$, as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$;
Then, T has a fixed point.

Proof. Following the proof of Theorem 2.1, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, is an increasing p_b-Cauchy sequence in the p_b-complete b-metric space (X, p_b). It follows from the completeness of (X, p_b) that there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Using the assumption on X, we deduce $x_n \preceq z$ for all $n \in \mathbb{N}$. So it is enough to show $fz = z$. Now, by using (2.1) and $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \psi(sp_b(x_{n+1}, Tz)) &\leq \alpha(x_n, z)\psi(sp_b(Tx_n, Tz)) \\ &\leq \psi(M_s^T(x_n, z)) - \varphi(M_s^f(x_n, z)), \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} M_s^T(x_n, z) &= max\{p_b(x_n, z), p_b(x_n, Tx_n), p_b(z, Tz), \tfrac{p_b(x_n,Tz)+p_b(Tx_n,z)}{2s}\} \\ &\leq max\{p_b(x_n, z), p_b(x_n, x_{n+1}), p_b(z, Tz), \tfrac{p_b(x_n,Tz)+p_b(x_{n+1},z)}{2s}\}. \end{aligned} \tag{2.23}$$

By taking the limit as $n \to \infty$ in above inequality and using Lemma 1.3, we get

$$\begin{aligned} \frac{p_b(z, Tz)}{2s^2} &= min\{p_b(z, Tz), \frac{\frac{p_b(z,Tz)}{s}}{2s}\} \\ &\leq \liminf_{n\to\infty} M_s^T(x_n, z) \\ &\leq \limsup_{n\to\infty} M_s^T(x_n, z) \\ &\leq max\{p_b(z, Tz), \frac{sp_b(z, Tz)}{2s}\} = p_b(z, Tz). \end{aligned} \tag{2.24}$$

Again, by using (2.22) and taking the upper limit as $n \to \infty$

$$\begin{aligned} \psi(sp_b(x_{n+1}, Tz)) &\leq \alpha(x_n, z)\psi(sp_b(Tx_n, Tz)) \\ &\leq \psi(M_s^f(x_n, z)) - \varphi(M_s^f(x_n, z)), \end{aligned}$$

and using Lemma 1.3, we get

$$\begin{aligned} \psi(p_b(z, Tz)) &= \psi(s\frac{1}{s}p_b(x_{n+1}, Tz)) \\ &\leq \psi(s \limsup_{n\to\infty} p_b(x_{n+1}, Tz)) \\ &\leq \psi(\limsup_{n\to\infty} M_s^f(x_n, z)) - \liminf_{n\to\infty} \varphi(M_s^T(x_n, z)) \\ &\leq \psi(p_b(z, Tz)) - \varphi(\liminf_{n\to\infty} M_s^T(x_n, z)). \end{aligned}$$

Therefore, $\varphi(\liminf_{n\to\infty} M_s^T(x_n, z)) \leq 0$, it means that $\liminf_{n\to\infty} M_s^T(x_n, z)) = 0$. Thus, from (2.24)we get $z = Tz$, and hence z is a fixed point of T. This completes the proof. □

Example 2.1. Let $X = [1, \infty)$ be equipped with the partial order $\preceq$ defined by

$$x \preceq y \Longleftrightarrow x \leq y$$

and with the functional $p_b : X \times X \to [0, \infty)$ defined by $p_b(x, y) = |x - y|^2 + 2$ For all $x, y \in X$. Clearly, (X, p_b) is a partial complete b-metric space with $s = 2$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x+6}{4} & \text{if } 1 \leq x \leq 2, \\ \frac{x^2}{2} & \text{if } x > 2, \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & if\ x, y \in [1, 2], \\ 0 & otherwise. \end{cases}$$

and taking the altering distance functions $\psi(t) = t$ and

$$\varphi(t) = \begin{cases} \frac{(t-1)^2}{2} & if\ x, y \in [1, 2], \\ \frac{t}{4} & t > 2. \end{cases}$$

Then T is continuous and increasing, $1 \preceq T1$. For Checking the contraction condition (2.1) for all comparable $x, y \in X$. Let $x = 1$ and $y = 4$, we get

$$\begin{aligned} \psi(2p_b(T1, T4)) = \psi(2p_b(\frac{7}{4}, 8)) &= \frac{641}{8} \not\leq \frac{108}{8} \\ &= \psi(18) - \varphi(18) = 18 - \frac{18}{4} = \psi(M_s^T(1, 4)) - \varphi(M_s^T(1, 4)). \end{aligned}$$

We will prove the following:
i) $T : X \to X$ is an α-ψ-φ-contractive mapping, with $\psi(t) = t$ for all $t \geq 0$;
ii) T is $\alpha-$admissible;
iii) there exists $x_1 = 1 \in X$ and $x_1 \preceq Tx_1$, such that $\alpha(x_1, Tx_1) \geq 1$;
iv) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$;

Proof. *i*) Clearly T is α-ψ-φ-contractive mapping with $\psi(t) = t$ for all $t \geq 0$, since for all $x, y \in X$,

$$\alpha(x, y)\psi(sp_b(Tx, Ty)) = \psi(2p_b(\frac{x+6}{4}, \frac{y+6}{4})) = 2|\frac{x+6}{4} - \frac{y+6}{4}|^2 + 2 = \frac{1}{8}|x - y|^2 + 2,$$

while without loss of generality if $1 \leq y \leq x \leq 2$,then

$$\begin{aligned} \alpha(x, y)\psi(sp_b(Tx, Ty)) &= \psi(2p_b(\frac{x+6}{4}, \frac{y+6}{4})) \\ &= 2|\frac{x+6}{4} - \frac{y+6}{4}|^2 + 2 = \frac{1}{8}|x - y|^2 + 2 \\ &\leq \frac{9}{4} = 3 - \frac{3}{4} = \psi(3) - \varphi(3) \\ &= \psi(M_s^T(x, y)) - \varphi(M_s^T(x, y)). \end{aligned}$$

ii) Let $(x, y) \in X \times X$ such that $\alpha(x, y) \geq 1$. From the definition of T and α we have both $Tx = \frac{x+6}{4}$,and $Ty = \frac{y+6}{4}$ are in $[1, 2]$, so we have $\alpha(Tx, Ty) = 1 \geq 1$. Then T is an α-admissible.
iii) Taking $x_1 = 1 \in X$, we have

$$\alpha(x_1, Tx_1) = \alpha(1, T1) = \alpha(1, \frac{7}{4}) = 1 \geq 1.$$

iv) let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and by the definition of α, we have $x_n \in$[1,2] for all $n \in \mathbb{N}$ and $x \in [1, 2]$. Then $\alpha(x_n, x) = 1 \geq 1$. Now, all the hypothesis of Theorem 2.1 are satisfied. Therefore, T has a fixed point. □

Example 2.2. Let $X = [0, \infty)$ be equipped with the partial order $\preceq$ defined by

$$x \preceq y \Longleftrightarrow x \leq y$$

and with the functional $p_b : X \times X \to [0, \infty)$ defined by $p_b(x, y) = [max\{x, y\}]^2$ For all $x, y \in X$. Clearly, (X, p_b) is a partial ordered complete b-metric space with $s = 2$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{\sqrt{2}\sqrt{1+x}} & \text{if } 0 \leq x \leq 1, \\ \frac{x}{2} & \text{if } x > 1, \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & if\ x, y \in [0,1], \\ 0 & otherwise. \end{cases}$$

and taking the altering distance functions $\psi(t) = t$ and

$$\varphi(t) = \begin{cases} \frac{t\sqrt{t}}{1+\sqrt{t}} & if\ t \in [0,1], \\ \frac{t}{2} & t > 1. \end{cases}$$

Then T is continuous and increasing, $0 \preceq T0$. For Checking the contraction condition (2.1) for all comparable $x, y \in X$. Let $x = 0$ and $y = 4$, we get

$$\begin{aligned} \psi(2p_b(T0, T2)) = \psi(2p_b((0,2)) &= 8 \nleq 2 = 4 - 2 \\ &= \psi(4) - \varphi(4) = \psi(M_s^T(0,2)) - \varphi(M_s^T(0,2)). \end{aligned}$$

We will prove the following:
i) $T : X \to X$ is an α-ψ-φ-contractive mapping, with $\psi(t) = t$ for all $t \geq 0$;
ii) T is $\alpha-$admissible;
iii) there exists $x_1 = 0 \in X$ and $x_1 \preceq Tx_1$, such that $\alpha(x_1, Tx_1) \geq 1$;
iv) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$;

Proof. *i*) Clearly T is α-ψ-φ-contractive mapping with $\psi(t) = t$ for all $t \geq 0$, since for all $x, y \in X$,

$$\alpha(x,y)\psi(sp_b(Tx,Ty)) \leq \psi(M_s^T(x,y)) - \varphi(M_s^T(x,y))$$

Since,

$$\begin{aligned} \alpha(x,y)\psi(sp_b(Tx,Ty)) &= \psi(2p_b(\frac{x}{\sqrt{2}\sqrt{1+x}}, \frac{y}{\sqrt{2}\sqrt{1+y}})) \\ &= 2[max\{\frac{x}{\sqrt{2}\sqrt{1+x}}, \frac{y}{\sqrt{2}\sqrt{1+y}}\}]^2 \\ &= \frac{x^2}{(1+x)} \end{aligned}$$

and

$$M_s^T(x,y) = max\{x^2, x^2, y^2, \frac{x^2 + [max\{y, \frac{x}{\sqrt{2}\sqrt{1+x}}\}]^2}{4}\} = x^2$$

Thus

$$\alpha(x,y)\psi(sp_b(Tx,Ty)) = \frac{x^2}{(1+x)} \leq x^2 - \frac{x^3}{1+x} = \psi(x^2) - \varphi(x^2) = \psi(M_s^T(x,y)) - \varphi(M_s^T(x,y)).$$

ii) Let $(x, y) \in X \times X$ such that $\alpha(x, y) \geq 1$. From the definition of T and α we have both $Tx = \frac{x}{\sqrt{2}\sqrt{1+x}}$,and $Ty = \frac{y}{\sqrt{2}\sqrt{1+y}}$ are in $[0, 1]$, so we have $\alpha(Tx, Ty) = 1 \geq 1$. Then T is an α-admissible.
iii) Taking $x_1 = 0 \in X$, we have

$$\alpha(x_1, Tx_1) = \alpha(0, T0) = \alpha(0, 0) = 1 \geq 1.$$

iv) let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and by the definition of α, we have $x_n \in$[0,1] for all $n \in \mathbb{N}$ and $x \in [0, 1]$. Then $\alpha(x_n, x) = 1 \geq 1$.
Now, all the hypothesis of Theorem 2.1 are satisfied. Therefore, T has a fixed point □

References

[1] T. Abdeljawad, J. Alzabut, A. Mukheimer, Y. Zaidan, *Banach contraction principle for cyclical mappings on partial metric spaces*, Fixed Point Theory and Applications, **2012**, 2012:154, 7 pp. 1

[2] T. Abdeljawad, J. Alzabut, A. Mukheimer, Y. Zaidan, *Best Proximity Points For Cyclical Contraction Mappings With 0-Boundedly Compact Decompositions*, Journal of Computational Analysis and Applications, **15** (2013), 678–685. 1

[3] T. Abdeljawad, *Meir-Keeler α-contractive fixed and common fixed point theorems*, Fixed Point Theory and Applications, **2013**, 2013:19, 10 pp. 1

[4] J. Asl, S. Rezapour, N. Shahzad, *On fixed points of $\alpha - -\psi$-contractive multi functions*, Fixed Point Theory and Applications, **2012**, 2012:212, 6 pp. 1

[5] H. Aydi, *Some fixed point results in ordered partial metric spaces*, Journal of Nonlinear Sciences and Applications, **3** (2011), 210–217. 1

[6] H. Aydi, M. Bota, E. Karapinar, S. Mitrovic, *A fixed point theorem for set valued quasi-contractions in b-metric spaces*, Fixed Point Theory and its Applications, **2012**, 2012:88, 8 pp. 1.1

[7] I. A. Bakhtin, *The contraction principle in quasimetric spaces*, It. Funct. Anal., **30** (1989), 26–37. 1

[8] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrals*, Fundamenta Mathematicae, **3** (1922), 133–181. 1

[9] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena., **46** (1998), 263–276. 1

[10] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. inform. Univ. Osrav., **1** (1993), 5–11. 1

[11] E. Karapinar, R. P. Agarwal, *A note on Coupled fixed point theorems for $\alpha - -\psi$-contractive-type mappings in partially ordered metric spaces*, Fixed Point Theory and Applications, **2013**, 2013:216, 16 pp. 1

[12] A. Kaewcharoen, T. Yuying, *Unique common fixed point theorems on partial metric spaces*, Journal of Nonlinear Sciences and Applications, **7** (2014), 90–101. 1

[13] E. Karapinar, B. Samet, *Generalized α-ψ contractive type mappings and related fixed point theorems with applications*, Abstract and Applied Analysis, (2012), Art. ID 793486, 17 pp. 1.7, 1.8

[14] M. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc., **30** (1984), 1–9. 1.4

[15] S. G. Matthews, *Partal metric topology Proc. 8th Summer Conference on General Topology and Applications*, Ann. N.Y. Acad. Sci., **728** (1994), 183–197. 1, 1.2

[16] K. Mehmet, H. Kiziltunc, *On Some Well Known Fixed Point Theorems in b-Metric Spaces*, Turkish Journal of Analysis and Number Theory, **1** (2013), 13–16. 1

[17] Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg, *Some common fixed point result in ordered partal b-metric spaces*, Journal of Inequalities and Applications, (2013), 2013:562. 1, 1, 1, 1.5, 1.2, 1.3, 1.6, 1.1, 1.2, 1.3, 1.9

[18] H. Nashinea, M. Imdadb, M. Hasanc, *Common fixed point theorems under rational contractions in complex valued metric spaces*, Journal of Nonlinear Sciences and Applications, **7** (2014), 42–50. 1

[19] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha - \psi$-contactive type mappings*, Nonlinear Analysis, **75** (2012), 2154–2165. 1.3, 1.4

[20] W. Shatanawia, H. Nashineb, *A generalization of Banach's contraction principle for nonlinear contraction in a partial metric space*, Journal of Nonlinear Sciences and Applications, **5** (2012), 37–43. 1

[21] S. Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterranean Journal of Mathematics, doi:10.1007/s00009-013-0327-4, (2013). 1, 1, 1.3, 1, 1.1, 1.1, 1.2

[22] S. Singh, B. Chamola, *Quasi-contractions and approximate fixed points*, J. Natur. Phys. Sci., **16** (2002), 105–107. 1

[23] C. Vetroa, F. Vetrob, *Common fixed points of mappings satisfying implicit relations in partial metric spaces*, Journal of Nonlinear Sciences and Applications, **6** (2013), 152–161. 1

[24] H. Yingtaweesittikul, *Suzuki type fixed point for generalized multi-valued mappings in b-metric spaces*, Fixed Point Theory and Applications, **2013**, 2013:215, 9 pp. 1