Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 7 (2014), 246–254
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

# Existence and uniqueness of solutions of differential equations of fractional order with integral boundary conditions

J. A. Nanware[a,*], D. B. Dhaigude[b]

[a]*Department of Mathematics, Shrikrishna Mahavidyalaya, Gunjoti - 413 606, Dist. Osmanabad (M.S), India.*
[b]*Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad - 431 004, India.*

Communicated by C. Cuevas

**Abstract**

Recently, Wang and Xie [T. Wang, F. Xie, J. Nonlinear Sci. Appl., **1** (2009), 206–212] developed monotone iterative method for Riemann-Liouville fractional differential equations with integral boundary conditions with the strong hypothesis of locally Hölder continuity and obtained existence and uniqueness of a solution for the problem. In this paper, we apply the comparison result without locally Hölder continuity due to Vasundhara Devi to develop monotone iterative method for the problem and obtain existence and uniqueness of a solution of the problem. 

*Keywords:* Fractional differential equations, existence and uniqueness, lower and upper solutions, integral boundary conditions.
*2010 MSC:* 34A12, 34C60, 34A45.

## 1. Introduction

The fractional calculus was developed during nineteenth century [13, 22, 28]. The study of theory of differential equations of fractional order [17, 20] parallel to the well-known theory of ordinary differential equations [15, 19] has been growing independently since last three decades. Lakshmikantham and Vatsala [16, 18] obtained local and global existence of solutions of Riemann-Liouville fractional differential equations and uniqueness of solutions. Monotone method for Riemann-Liouville fractional differential equations with

*Corresponding author
*Email addresses:* `jag_skmg91@rediffmail.com` (J. A. Nanware), (D. B. Dhaigude)

*Received 2012-10-31*

initial conditions is developed by McRae [22] involving study of qualitative properties of solutions of initial value problem. Jankwoski [12] formulated some comparison results and obtained existence and uniqueness of solutions of differential equations with integral boundary conditions . Recently, Wang and Xie [29] developed monotone method and obtained existence and uniqueness of solution of fractional differential equation with integral boundary condition. Vasundhara Devi developed [6] the general monotone method for periodic boundary value problem of Caputo fractional differential equation when the function is sum of nondecreasing and nonincreasing function. The Caputo fractional differential equations with periodic boundary conditions have been studied by present authors [8, 9, 23] and developed monotone method for the problem. Existence and uniqueness of solution of Riemann-Liouville fractional differential equations with integral boundary conditions is also obtained by Nanware and Dhaigude in [23, 24, 25, 26, 27]. The qualitative properties such as existence, periodicity, ergodicity, almost periodic, pseudo-almost periodic etc. of solutions of fractional differential equations and fractional integro-differential equations was studied by many researchers. For more details see [1, 2, 3, 4, 5, 10, 11, 14, 21].

In this paper, we consider the Riemann-Liouville fractional differential equations with integral boundary conditions and develop monotone iterative method for Riemann-Liouville fractional differential equations with integral boundary conditions without locally Hölder continuity and obtained existence and uniqueness of solution of the problem.

The paper is organized in the following manner: In section 2, we consider some definitions and lemmas required in next section. In section 3, monotone iterative method is developed for the problem. As an application of the method existence and uniqueness results for Riemann-Liouville fractional differential equations with integral boundary conditions are obtained.

## 2. Preliminaries

In 2009, Wang and Xie [29] developed monotone iterative method for the following fractional differential equations with integral boundary conditions with Hölder continuity and obtained existence and uniqueness of solution of the problem

$$\begin{aligned} D^q u(t) &= f(t,u), \quad t \in J = [0,T], \quad T \geq 0, \\ u(0) &= \lambda \int_0^T u(s)ds + d, \quad d \in \mathbb{R} \end{aligned} \tag{2.1}$$

where $0 < q < 1, \lambda$ is 1 or $-1$ and $f \in C[J \times \mathbb{R}, \mathbb{R}], D^q$ is Riemann-Liouville fractional derivative of order $q$.

**Lemma 2.1.** ([7]) *Let $m \in C_p([t_0,T],\mathbb{R})$ and for any $t_1 \in (t_0,T]$ we have $m(t_1) = 0$ and $m(t) < 0$ for $t_0 \leq t \leq t_1$. Then it follows that $D^q m(t_1) \geq 0$.*

**Lemma 2.2.** ([16]) *Let $\{u_\epsilon(t)\}$ be a family of continuous functions on $[t_0,T]$, for each $\epsilon > 0$ where $D^q u_\epsilon(t) = f(t, u_\epsilon(t)), \quad u_\epsilon(t_0) = u_\epsilon(t)(t-t_0)^{1-q}\}_{t=t_0}$ and $|f(t,u_\epsilon(t))| \leq M$ for $t_0 \leq t \leq T$. Then the family $\{u_\epsilon(t)\}$ is equicontinuous on $[t_0,T]$.*

**Theorem 2.3.** ([29]) *Assume that:*

**(i)** *$v(t)$ and $w(t)$ in $C_p(J,\mathbb{R})$ are lower and upper solutions of* (2.1)

**(ii)** *$f(t,u(t))$ satisfy one-sided Lipschitz condition:*

$$f(t,u) - f(t,v) \leq L(u-v), \qquad L \geq 0$$

*Then $v(0) \leq w(0)$ implies that $v(t) \leq w(t), \quad 0 \leq t \leq T$.*

In this paper, we consider the problem (2.1) and develop monotone iterative method without assuming locally Hölder continuity.

**Definition 2.4.** A pair of functions $v(t)$ and $w(t)$ in $C_p(J,\mathbb{R})$ are said to be lower and upper solutions of the problem (2.1) if

$$D^q v(t) \leq f(t, v(t)), \qquad v(0) \leq \int_0^T v(s)ds$$
$$D^q w(t) \geq f(t, w(t)), \qquad w(0) \geq \int_0^T w(s)ds.$$

## 3. Monotone Iterative Method

In this section we develop monotone iterative method for the problem (2.1) and obtain the existence and uniqueness of solution of the problem (2.1).

**CASE-I** ($\lambda = 1$)

**Theorem 3.1.** *Assume that:*

**(i)** *$f(t, u(t))$ is nondecreasing in $u$ for each $t$.*

**(ii)** *$v_0(t)$ and $w_0(t)$ in $C_p(J,\mathbb{R})$ are lower and upper solutions of (2.1) such that $v_0(t) \leq w_0(t)$ on $J = [0, T]$*

**(iii)** *$f(t, u)$ satisfies one-sided Lipschitz condition,*

$$f(t, u) - f(t, v) \leq -L(u - v), \quad L \geq 0$$

*Then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_p(J,\mathbb{R})$ such that*

$$\{v_n(t)\} \to v(t) \quad \text{and} \quad \{w_n(t)\} \to w(t) \quad \text{as} \quad n \to \infty$$

*where $v(t)$ and $w(t)$ are minimal and maximal solutions of (2.1) respectively that satisfy*

$$D^q v(t) = f(t, v(t))$$
$$D^q w(t) = f(t, w(t))$$

*on $J$.*

*Proof.* For any $\eta$ and $\mu$ in $C_p(J,\mathbb{R})$ such that for $v_0(0) \leq \eta$ and $w_0(0) \leq \mu$ on $J$, consider the following linear fractional differential equation

$$D^q u(t) + Mu(t) = f(t, \eta) + M\eta, \quad u(0) = \int_0^T u(s)ds + d. \tag{3.1}$$

Firstly, prove the uniqueness of solution of linear fractional differential equation (3.1). For this, let $u_1(t)$ and $u_2(t)$ be two solutions of (3.1). Then we have

$$D^q u_1(t) + Mu_1(t) = f(t, \eta) + M\eta, \quad u_1(0) = \int_0^T u_1(s)ds + d$$
$$D^q u_2(t) + Mu_2(t) = f(t, \eta) + M\eta, \quad u_2(0) = \int_0^T u_2(s)ds + d.$$

where $\eta \in C_p[J,\mathbb{R}]$. Hence

$$D^q(u_1(t) - u_2(t)) = -M(u_1(t) - u_2(t))$$

and $u_1(0) - u_2(0) = 0$. This implies $u_1(t) = u_2(t)$.
Define the sequences as follows:

$$D^q v_{n+1}(t) = f(t, v_n) - M(v_{n+1} - v_n), \qquad v_{n+1}(0) = \int_0^T v_n(s)ds + d$$
$$D^q w_{n+1}(t) = f(t, w_n) - M(w_{n+1} - w_n), \qquad w_{n+1}(0) = \int_0^T w_n(s)ds + d$$

Now

$$\begin{aligned} D^q v_{n+1}(t) + M v_{n+1}(t) &= f(t, v_n) + M v_n(t), \quad v_{n+1}(0) = \int_0^T v_n(s)ds + d \\ D^q w_{n+1}(t) + M w_{n+1}(t) &= f(t, w_n) + M w_n(t), \quad w_{n+1}(0) = \int_0^T w_n(s)ds + d \end{aligned} \tag{3.2}$$

It follows that there exist unique solutions $v_{n+1}(t)$ and $w_{n+1}(t)$ for above equation. Putting $n = 0$ in (3.2), the existence of solutions of $v_1(t)$ and $w_1(t)$ is clear. Next show that $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. Setting $p(t) = v_0(t) - v_1(t)$ we have

$$\begin{aligned} D^q p(t) &= D^q v_0(t) - D^q v_1(t) \\ &\leq -Mp(t) \\ \text{and} \quad p(0) &\leq 0. \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t) &\leq 0. \end{aligned}$$

Hence $v_0(t) \leq v_1(t)$. Similarly, we prove $w_0(t) \geq w_1(t)$ and $v_1(t) \leq w_1(t)$. Thus $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. Assume that for some $k > 1$, $v_{k-1}(t) \leq v_k(t) \leq w_k(t) \leq w_{k-1}(t)$. We claim that $v_k(t) \leq v_{k+1}(t) \leq w_{k+1}(t) \leq w_k(t)$ on $J$. To prove this, set $p(t) = v_k(t) - v_{k+1}(t)$. Since $f(t, u) + Mu$ is nondecreasing in $u$, we have

$$\begin{aligned} D^q p(t) &= D^q v_k(t) - D^q v_{k+1}(t) \\ &\leq -M(v_k - v_{k+1}) \\ &\leq -Mp(t) \\ \text{and} \quad p(0) &\leq 0. \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t) &\leq 0. \end{aligned}$$

Hence $p(t) \leq 0$ which implies $v_k \leq v_{k+1}$. Similarly we prove that $v_{k+1}(t) \leq w_{k+1}(t)$.
Using corresponding fractional Volterra integral equations

$$v_{n+1}(t) = v_0 + \frac{1}{\Gamma(q)} \int_{t_0}^T (t-s)^{q-1} \{f(s, v_{n+1}(s)) - M(v_{n+1} - v_n)\} ds$$
$$w_{n+1}(t) = w_0 + \frac{1}{\Gamma(q)} \int_{t_0}^T (t-s)^{q-1} \{f(s, w_{n+1}(s)) - M(w_{n+1} - w_n)\} ds$$

it follows that $v(t)$ and $w(t)$ are solutions of (3.1).
Next claim that $v(t)$ and $w(t)$ are the minimal and maximal solution of (2.1). Let $u(t)$ be any solution of (2.1) different from $v(t)$ and $w(t)$, so that there exists $k$ such that $v_k(t) \leq u_k(t) \leq w_k(t)$ on $J$ and set

$p(t) = v_{k+1}(t) - u_k(t)$ so that

$$\begin{aligned} D^q p(t) &= D^q v_{k+1}(t) - D^q u_k(t) \\ &\leq -M(v_{k+1}(t) - u_k(t)) \\ &\leq -Mp(t) \\ \text{and} \quad p(0) &\leq 0. \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t) &\leq 0. \end{aligned}$$

Thus $v_{k+1}(t) \leq u_k(t)$ on $J$. Since $v_0(t) \leq u_0(t)$ on $J$, by induction it follows that $v_k(t) \leq u_k(t)$ for all k. Similarly we can prove $u_k(t) \leq w_k(t)$ for all k on $J$. Thus $v_k(t) \leq u_k(t) \leq w_k(t)$ on $J$ . Taking limit as $n \to \infty$, it follows that $v(t) \leq u(t) \leq w(t)$ on $J$. □

Next we obtain the uniqueness of solutions of (2.1) in the following

**Theorem 3.2.** *Assume that:*

**(i)** *$f(t, u(t))$ in $C[J \times \mathbb{R}, \mathbb{R}]$, is nondecreasing in $u$ for each $t$.*

**(ii)** *$v_0(t)$ and $w_0(t)$ in $C(J, \mathbb{R})$ are lower and upper solutions of* (2.1) *such that $v_0(t) \leq w_0(t)$ on $J$*

**(iii)** *functions $f(t, u)$ satisfy Lipschitz condition,*

$$|f(t, u) - f(t, v)| \leq L|u - v|, \qquad L \geq 0$$

**(iv)** $\lim_{n\to\infty} ||w_n(t) - v_n(t)|| = 0$, *where the norm is defined by* $||f|| = \int_0^T |f(s)|ds$

*then the solution of* (2.1) *is unique.*

*Proof.* Since $v(t) \leq w(t)$, it is sufficient to prove $v(t) \geq w(t)$. Consider $p(t) = w(t) - v(t)$ we find that

$$\begin{aligned} D^q p(t) &= D^q w(t) - D^q v(t) \\ &\leq M(w(t) - v(t)) \\ &\leq -Mp(t) \\ \text{and} \quad p(0) &\leq 0. \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t) &\leq 0. \end{aligned}$$

Thus, $p(t) \leq 0$ implies $w(t) \leq v(t)$. Hence $v(t) = w(t)$ is the unique solution of (2.1) on $J$. □

**CASE-II ($\lambda = -1$)**

**Definition 3.3.** A pair of functions $v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are said to be weakly coupled lower and upper solutions of the problem (2.1) if

$$D^q v(t) \leq f(t, v(t)), \qquad v(0) \leq -\int_0^T w(s)ds$$
$$D^q w(t) \geq f(t, w(t)), \qquad w(0) \geq -\int_0^T v(s)ds.$$

**Theorem 3.4.** *Assume that:*

**(i)** *$f(t, u(t))$ is nondecreasing in $u$ for each $t$.*

**(ii)** *$v_0(t)$ and $w_0(t)$ in $C_p(J,\mathbb{R})$ are weakly coupled lower and upper solutions of* (2.1) *such that $v_0(t) \leq w_0(t)$ on $J = [0,T]$*

**(iii)** *$f(t,u)$ satisfies one-sided Lipschitz condition,*

$$f(t,u) - f(t,v) \leq -L(u-v), \quad L \geq 0$$

*Then there exist monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_p(J,\mathbb{R})$ such that*

$$\{v_n(t)\} \to v(t) \quad and \quad \{w_n(t)\} \to w(t) \quad as \quad n \to \infty$$

*where $v(t)$ and $w(t)$ are minimal and maximal solutions of* (2.1) *respectively.*

*Proof.* For any $\eta(t)$ and $\mu(t)$ in $C_p(J,\mathbb{R})$ such that for $v_0(0) \leq \eta(t)$ and $w_0(0) \leq \mu(t)$ on $J$, consider the following linear fractional differential equation

$$D^q u(t) + Mu(t) = f(t,\eta) + M\eta, \quad u(0) = \int_0^T u(s)ds + d. \tag{3.3}$$

Uniqueness of solution of linear fractional differential equation (3.3) can be proved as in Theorem 3.1. Define the sequences as follows:

$$D^q v_{n+1}(t) = f(t,v_n) - M(v_{n+1} - v_n), \qquad v_{n+1}(0) = \int_0^T w_n(s)ds + d$$

$$D^q w_{n+1}(t) = f(t,w_n) - M(w_{n+1} - w_n), \qquad w_{n+1}(0) = \int_0^T v_n(s)ds + d$$

Now

$$\begin{aligned} D^q v_{n+1}(t) + Mv_{n+1}(t) = f(t,v_n) + Mv_n(t), \quad v_{n+1}(0) = \int_0^T w_n(s)ds + d \\ D^q w_{n+1}(t) + Mw_{n+1}(t) = f(t,w_n) + Mw_n(t), \quad w_{n+1}(0) = \int_0^T v_n(s)ds + d \end{aligned} \tag{3.4}$$

It follows that there exist unique solutions $v_{n+1}(t)$ and $w_{n+1}(t)$ for above equation. Putting $n = 0$ in (3.4), we get the existence of solutions of $v_1(t)$ and $w_1(t)$. Next we show that $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. Setting $p(t) = v_0(t) - v_1(t)$ we have

$$\begin{aligned} D^q p(t) &= D^q v_0(t) - D^q v_1(t) \\ &\leq -Mp(t) \\ \text{and} \quad p(0) &\leq 0. \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t) &\leq 0. \end{aligned}$$

Hence $v_0(t) \leq v_1(t)$. Similarly, we prove $w_0(t) \geq w_1(t)$ and $v_1(t) \leq w_1(t)$. Thus $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. Assume that for some $k > 1$, $v_{k-1}(t) \leq v_k(t) \leq w_k(t) \leq w_{k-1}(t)$. We claim that $v_k(t) \leq v_{k+1}(t) \leq w_{k+1}(t) \leq w_k(t)$ on $J$. To prove this, set $p(t) = v_k(t) - v_{k+1}(t)$. Since $f(t,u) + Mu$ is nondecreasing in $u$, we have

$$\begin{aligned} D^q p(t) &= D^q v_k(t) - D^q v_{k+1}(t) \\ &\leq -M(v_k - v_{k+1}) \\ &\leq -Mp(t) \\ \text{and} \quad p(0) &\leq 0. \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t) &\leq 0. \end{aligned}$$

Hence $p(t) \leq 0$ which implies $v_k \leq v_{k+1}$. Similarly we prove that $v_{k+1}(t) \leq w_{k+1}(t)$.
Using corresponding fractional Volterra integral equations

$$v_{n+1}(t) = v_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{T} (t-s)^{q-1}\{f(s, v_{n+1}(s)) - M(v_{n+1} - v_n)\}ds$$
$$w_{n+1}(t) = w_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{T} (t-s)^{q-1}\{f(s, w_{n+1}(s)) - M(w_{n+1} - w_n)\}ds$$

it follows that $v(t)$ and $w(t)$ are solutions of (3.3).
Next we claim that $v(t)$ and $w(t)$ are the minimal and maximal solution of (2.1). Let $u(t)$ be any solution of (2.1) different from $v(t)$ and $w(t)$, so that there exists $k$ such that $v_k(t) \leq u_k(t) \leq w_k(t)$ on $J$ and set $p(t) = v_{k+1} - u_k$ so that

$$\begin{aligned} D^q p(t) &= D^q v_{k+1} - D^q u_k \\ &\leq -M(v_{k+1} - u_k) \\ &\leq -Mp(t) \\ \text{and} \quad p(t) &\leq 0. \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t) &\leq 0. \end{aligned}$$

Thus $v_{k+1}(t) \leq u_k(t)$ on $J$. Since $v_0(t) \leq u_0(t)$ on $J$, by induction it follows that $v_k(t) \leq u_k(t)$ for all k. Similarly we can prove $u_k(t) \leq w_k(t)$ for all k on $J$. Thus $v_k(t) \leq u_k(t) \leq w_k(t)$ on $J$ . Taking limit as $n \to \infty$, it follows that $v(t) \leq u(t) \leq w(t)$ on $J$. □

Next we obtain the uniqueness of solutions of (2.1) in the following

**Theorem 3.5.** *Assume that:*

**(i)** $f(t, u(t))$ *in* $C[J \times \mathbb{R}, \mathbb{R}]$, *is nondecreasing in* $u$ *for each* $t$.

**(ii)** $v_0(t)$ *and* $w_0(t)$ *in* $C_p(J, \mathbb{R})$ *are weakly coupled lower and upper solutions of* (2.1) *such that* $v_0(t) \leq w_0(t)$ *on* $J$

**(iii)** *functions* $f(t, u)$ *satisfy Lipschitz condition,*

$$|f(t, u) - f(t, v)| \leq L|u - v|, \qquad L \geq 0$$

**(iv)** $\lim_{n\to\infty} ||w_n(t) - v_n(t)|| = 0$, *where the norm is defined by* $||f|| = \int_0^T |f(s)|ds$

*then the solution of* (2.1) *is unique.*

*Proof.* This can be proved as in Theorem 3.2. □

## Acknowledgements

Authors are thankful to the referee for remarkable comments on the paper which helps us to improve the paper.

## References


[1] R. P. Agarwal, B. de Andrade, G. Siracusa, *On Fractional Integro-Differential Equations with State-dependent Delay,* Comp. Math. Appl., **62** (2011), 1143–1149. 1

[2] R. P. Agarwal, M.Benchohra, S.Mamani, *A Survey on Existence Results for Boundary Value Problems of Nonlinear Fractional Differential Equations and Inclusions,* Acta Appl. Math., **109** (2010), 973–1033. 1

[3] R. P. Agarwal, B. de Andrade, C. Cuevas, *On type of periodicity and Ergodicity to a class of Fractional Order Differential Equations,* Adv. Differen. Eqs., Hindawi Publl.Corp., NY , USA, **Article ID 179750** (2010), 25 pages. 1

[4] E. Cuesta, *Asymptotic Behaviour of the Solutions of Fractional Integro-Differential Equations and Some Time Discretizations,* Dis. Cont. Dyn. Sys., Series A, (2007), 277–285. 1

[5] C. Cuevas, H. Soto and A. Sepulveda, *Almost Periodic and Pseudo-almost Periodic Solutions to Fractional Differential and Integro-Differential Equations,* Appl. Math. Comput., **218** (2011), 1735–1745. 1

[6] J. V. Devi, *Generalized Monotone Method for Periodic Boundary Value Problems of Caputo Fractional Differential Equations,* Commun. Appl. Anal., **12** (2008), 399–406. 1

[7] J. V. Devi, F. A. McRae, Z. Drici, *Variational Lyapunov Method for Fractional Differential Equations,* Comp. Math. Appl., **64**, (2012), 2982–2989. 2.1

[8] D. B. Dhaigude, J. A. Nanware, V. R. Nikam, *Monotone Technique for System of Caputo Fractional Differential Equations with Periodic Boundary Conditions,* Dyn. Conti. Dis. Impul. Sys., Series-A :Mathematical Analysis, **19** (2012), 575–584. 1

[9] D. B. Dhaigude, J. A. Nanware, *Monotone Technique for Finite System of Caputo Fractional Differential Equations with Periodic Boundary Conditions*, (To appear). 1

[10] T. Diagana, G. M. Mophou, G. M. N'Gue're'kata, *On the Existence of Mild Solutions to Some Semilinear Fractional Integro-Differential Equations*, Electron. J. Qual. Theory Differ. Equ., **58** (2010), 1–17. 1

[11] G. M. N'Gue're'kata, *A Cauchy Problem for Some Fractional Abstract Differential Equations with Non-local Conditions,* Nonlinear Anal., **70** (2009), 1873–1876. 1

[12] T. Jankwoski, *Differential Equations with Integral Boundary Conditions,* J. Comput. Appl. Math., **147** (2002), 1–8. 1

[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations,* North Holland Mathematical Studies, Vol.204. Elsevier(North-Holland) Sciences Publishers, Amsterdam, (2006). 1

[14] P. Kumar, D. N. Pandey, D Bahuguna, *On a new class of abstract impulsive functional differential equations of fractional order*, J. Nonlinear Sci. Appl., **7** (2014), 102–114. 1

[15] G. S. Ladde, V. Lakshmikantham, A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations,* Pitman Advanced Publishing Program, London, (1985). 1

[16] V. Lakshmikantham, A. S. Vatsala, *Theory of Fractional Differential Equations and Applications*, Communications in Applied Analysis, **11** (2007), 395–402. 1, 2.2

[17] V. Lakshmikantham, A. S. Vatsala, *Basic Theory of Fractional Differential Equations and Applications,* Nonlinear Anal., **69** (2008), 2677–2682. 1

[18] V. Lakshmikantham, A. S. Vatsala, *General Uniqueness and Monotone Iterative Technique for Fractional Differential Equations*, Appl. Math. Letters, **21** (2008), 828–834. 1

[19] V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities Vol.I.*, Academic Press, Newyork, (1969). 1

[20] V. Lakshmikantham, S. Leela, J. V. Devi, *Theory and Applications of Fractional Dynamic Systems*, Cambridge Scientific Publishers Ltd. (2009). 1

[21] Y. Liu, H. Shi, *Existence of unbounded positive solutions for BVPs of singular fractional differential equations*, J. Nonlinear Sci. Appl., **5** (2012), 281–293. 1

[22] F. A. Mc Rae, *Monotone Iterative Technique and Existence Results for Fractional Differential Equations*, Nonlinear Anal., **71** (2009), 6093–6096. 1

[23] J. A. Nanware, *Monotone Method In Fractional Differential Equations and Applications,* Ph.D Thesis, Dr. Babasaheb Ambedkar Marathwada University, 2013.

[24] J. A. Nanware, D. B. Dhaigude, *Boundary Value Problems for Differential Equations of Noninteger Order Involving Caputo Fractional Derivative,* Proceedings of Jangjeon Mathematical Society, South Korea (To appear). 1

[25] J. A. Nanware, *Existence and Uniqueness Results for Fractional Differential Equations Via Monotone Method,* Bull. Marathwada Math. Soc., **14**, (2013), 39–56. 1
1

[26] J. A. Nanware, D. B. Dhaigude, *Existence and Uniqueness of solution of Riemann-Liouville Fractional Differential Equations with Integral Boundary Conditions,* Int. J. Nonlinear Sci., **14** (2012), 410–415. 1

[27] J. A. Nanware, D. B. Dhaigude, *Monotone Iterative Scheme for System of Riemann-Liouville Fractional Differential Equations with Integral Boundary Conditions,* Math. Modelling Sci. Computation, Springer-Verlag, **283** (2012), 395–402. 1

[28] I. Podlubny, *Fractional Differential Equations,* Academic Press, San Diego, (1999). 1

[29] T. Wang, F. Xie, *Existence and Uniqueness of Fractional Differential Equations with Integral Boundary Conditions,* J. Nonlinear Sci. Appl., **1** (2009), 206–212. 1, 2, 2.3