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Fixed point theorems in E-b-metric spaces

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Abstract

In this paper we introduce the notion of E-b-metric space and we present a singlevalued and multivalued nonlinear fixed point theorem in an E-b-metric space using the Picard and weak Picard operators technique. The proofs are based on the concept of strict positivity in a Riesz space introduced by Páles and Petre.

Keywords: Contraction Principle, fixed point, iterative method, multivalued operator, φ-contraction, Riesz space, vector lattice, vector b-metric space.
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1. Introduction, Notations and Terminology

In this work we extend the results obtained by Páles and Petre in [12]. Thus, the aim of this paper is to develop some new fixed point theorems for operators defined on a vector b-metric space into itself, which satisfies a nonlinear φ-contraction condition.

The workspace is based on the concept of strict order unit elements, which generalizes the concept of order unit elements under mild assumptions. Hence, we are able to use a kind of "ε–δ" formalism for the vector metric space setting to prove our main results in Section 3.

The proofs incorporate several recent improvements and use the Picard and weak Picard operators technique. For instance, the invalidity of an inequality $a \leq b$ in a Riesz space E does not mean that $b < a$ must be valid. Therefore, for the vector-comparison operator φ, the property $\varphi(t) < t$ for $t \in E_+$ cannot be obtained as the consequence of the iteration property $\varphi^n(t) \underset{n\to\infty}{\longrightarrow} 0$ contrary to the real case.

In general, we follow the notation and terminology of Aliprantis and Border [1]. Briefly, we recall the basic concepts and notations introduced therein and Çevik and Altun [3] (see also [2, 4, 6]–[11] and [13]–[21]).

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Given a partially ordered set $(E, \leq)$, the notation $x < y$ means $x \leq y$ and $x \neq y$. An order interval $[x, y]$ is the set $\{z \in E : x \leq z \leq y\}$.

A partially ordered set $(E, \leq)$ is a *lattice* if each pair of elements has a supremum and an infimum. A real linear space E with an order relation "$\leq$" on E which is compatible with the algebraic structure of E is called an *ordered linear space*. An ordered linear space E for which $(E, \leq)$ is a lattice is called a *Riesz space* or linear lattice. The cone of nonnegative elements in a Riesz space E is denoted by E_+. For an element $x \in E$, the absolute value $|x|$ of x is defined as $|x| := x \vee (-x)$. Many familiar spaces are Riesz spaces, see [1].

The notation $x_n \downarrow x$ means that (x_n) is a decreasing sequence and x is the greatest lower bound (i.e., infimum) for the set $\{x_n : n \in \mathbb{N}\}$. A Riesz space E is said to be *Archimedean* if $\frac{1}{n}x \downarrow 0$ holds for every $x \in E_+$. A Riesz space E is called *order complete* or Dedekind complete if every nonempty subset of E which is bounded from above (below) has a supremum (infimum). Any order complete Riesz space is Archimedean. The converse is false in general (e.g. $C[0, 1]$). We recall now some useful definitions.

Definition 1.1 ([3])**.** Let E be a Riesz space. A sequence (b_n) in E is called *order-convergent* (or *o*-convergent) to b (we write $b_n \xrightarrow{o} b$) if, there exists a sequence (a_n) in E_+ satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$.

Definition 1.2 ([3])**.** Let X be a nonempty set and E be a Riesz space. The function $d : X \times X \to E_+$ is said to be a *vector metric* or E-metric if, for any $x, y, z \in X$ the following conditions are satisfied:

(a) $d(x, y) = 0$ if and only if, $x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, z) \leq d(x, y) + d(y, z)$.

In this case, the triple (X, d, E) is said to be a *vector metric space* or an E-metric space.

A Riesz space E is always an E-metric space with the E-metric $d : E \times E \to E_+$, defined by $d(x, y) = |x - y|$. This E-metric is called the absolute valued metric on E. For more examples of E-metric spaces, see [3].

Definition 1.3 ([3])**.** Let (X, d, E) be an E-metric space. A sequence (x_n) in X, *E-converges* to some $x \in E$, written $x_n \xrightarrow{d,E} x$, if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$ for all $n \in \mathbb{N}$. A subset $Y \subset X$ is said to be *E-closed* if, $(x_n) \subset Y$ and $x_n \xrightarrow{d,E} x$ implies $x \in Y$.

A sequence (x_n) in X is called *E-Cauchy*, if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$, for all $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$. An E-metric space X is called E-complete if each E-Cauchy sequence in X E-converges to a limit in X.

For a nonempty set X, we denote $P(X) = \{Y \mid \emptyset \neq Y \subseteq X\}$ and in the context of an E-metric space (X, d, E) we use the notation $P_{cl}(X) = \{Y \in P(X) \mid Y \text{ is } E\text{-closed}\}$.

Given a multivalued operator $F : X \to P(X)$, the fixed point set and the graph of F are defined by

$$\mathrm{Fix}(F) := \{x \in X \mid x \in F(x)\},$$
$$\mathrm{Graph}(F) := \{(x, y) \mid x \in X \text{ and } y \in F(x)\}.$$

A sequence $(x_n)_{n \in \mathbb{N}} \subset X$ is said to be a *successive approximation* of F with initial pair (x, y) if it satisfies

$$\begin{cases} x_0 = x, \ x_1 = y, \\ x_{n+1} \in F(x_n) \qquad \text{for all } n \in \mathbb{N}. \end{cases}$$

2. *E*-*b*-metric spaces and strict order unit elements

First, we describe the notion of an E-b-metric space, which we will use in the proofs of Section 3. Notice that d is defined as in Definition 1.2.

Definition 2.1. Let X be a nonempty set and $s \geq 1$ be a real number. A functional $d : X \times X \to E_+$ is called an *E-b-metric* if, for any $x, y, z \in X$ the following conditions are satisfied:

(a) $d(x, y) = 0$ if and only if, $x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The triple (X, d, E) is called an *E-b-metric space.*

For several examples of b-metric spaces, see [5]. Next, we recall from [1] that an element $e \in E_+$ in a Riesz space E is called an *order unit element* if, for any $x \in E$, there exists $\lambda \in \mathbb{R}_+$ such that $|x| \leq \lambda e$. However, this notion of strict positiveness is insufficient for our purposes. Therefore, we recall the following concept and some auxiliary results, which we will use in Section 3.

Definition 2.2 ([12])**.** We say that $e \in E_+$ is a *strict order unit element*, written $e \gg 0$ if, for any subset $H \subset E_+$ with $\inf H = 0$, there exist $h_1, \ldots, h_n \in H$ such that $\min(h_1, \ldots, h_n) \leq e$.

For example, if $E = \mathbb{R}^2$, $E_+ = \mathbb{R}^2_+$, then $e = (e_1, e_2) \gg 0$ if and only if $e_1 > 0$ and $e_2 > 0$. Thus, in this case, we can see that order unit elements are strict order unit elements as well.

Proposition 2.3 ([12])**.** *If E is Archimedean and e is a strict order unit element, then e is an order unit element.*

The reverse implication in the above proposition is not true, in general, as is shown by the next proposition.

Proposition 2.4 ([12])**.** *In the space $E = \ell^\infty$ with the positive cone*

$$E_+ = \{(e_1, e_2, \ldots) : e_i \geq 0\} \subset \ell^\infty,$$

$e \in E_+$ is an order unit element if and only if $\inf\{e_1, e_2, \ldots\} > 0$. However, there is no strict order unit element in E.

Let us denote by E_{++} the set of strict order unit elements in E.

Proposition 2.5 ([12])**.** *E_{++} is a convex cone.*

Lemma 2.6 ([12])**.** *Let E be order complete and assume that E_{++} is nonempty. Then $h_n \xrightarrow{o} 0$ if and only if, for all $e \in E_{++}$, there exists $n_0 \in \mathbb{N}$ such that*

$$|h_n| \leq e, \text{ for all } n \geq n_0.$$

3. Main Results

First, in the context of E-b-metric spaces, we prove an auxiliary result which characterize the convergence of sequences in terms of strict order unit elements.

Lemma 3.1. *Let (X, d, E) be an E-b-metric space, where E is order complete and E_{++} is nonempty. Then $(x_n)_{n\in\mathbb{N}}$ is an E-Cauchy sequence if and only if, for any $e \in E_{++}$, there exists $n_0 \in \mathbb{N}$ such that*

$$d(x_n, x_m) \leq e, \text{ for all } m > n \geq n_0.$$

Proof. Assume that $(x_n)_{n\in\mathbb{N}}$ is an E-Cauchy sequence. This means that there exists a decreasing null-sequence $h_n \downarrow 0$ such that $d(x_n, x_m) \leq h_n$ for $m \geq n$. Let $e \in E_{++}$ be arbitrary. Then, by Lemma 2.6, there exists $n_0 \in \mathbb{N}$ such that $h_{n_0} \leq e$. Hence, for $m > n \geq n_0$, we get $d(x_n, x_m) \leq h_n \leq h_{n_0} \leq e$, which completes the proof of the necessity.

For the sufficiency, let $e \in E_{++}$. Then, there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_m) \leq e$ for all $m > n_0$. This implies that the sequence $d(x_{n_0}, x_m)_{m\in\mathbb{N}}$ is E-bounded, i.e., there is an element $h \in E_+$ such that $d(x_{n_0}, x_m) \leq h$ for all $m \in \mathbb{N}$. Hence, $d(x_n, x_m) \leq s[d(x_n, x_{n_0}) + d(x_{n_0}, x_m)] \leq 2sh$ for all $m, n \in \mathbb{N}$. In other words, the set $\{d(x_n, x_m) : m, n \in \mathbb{N}\}$ is E-bounded. Thus, by taking advantage of the order completeness of E, we may define the sequence (h_k) by

$$h_k := \sup\{d(x_n, x_m) : m > n \geq k\}.$$

Obviously, (h_k) is a decreasing sequence. By the assumption, there exist indices $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$d(x_n, x_m) \leq \frac{1}{k}e, \text{ for all } m > n \geq n_k.$$

By this inequality, we get that $h_{n_k} \leq \frac{1}{k}e$. Hence (h_{n_k}) is a null-sequence which implies that (h_k) is also a null-sequence. On the other hand, by the definition of (h_k), we have that $d(x_k, x_m) \leq h_k$ for all $k \in \mathbb{N}$. Thus, the proof of the sufficiency is complete. □

Definition 3.2. An increasing map $\varphi : E_+ \to E_+$ is called an *o-comparison* operator if, for all $t \in E_+$, $\varphi^n(t) \xrightarrow{o} 0$.

Now, we present a nonlinear fixed point principle for singlevalued and multivalued operators in E-b-metric spaces using the Picard and weak Picard operators technique, which extends the Banach type fixed point theorem given by several authors: J. Matkowski [11], C. Çevik and I. Altun [3], and Zs. Páles and I.-R. Petre [12]. Moreover, in the following results we do not need to impose the condition $\varphi(t) < t$ on the o-comparison operator φ, which is usually necessary. We start with the singlevalued case.

Definition 3.3. Let (X, d, E) be an E-b-metric space and $\varphi : E_+ \to E_+$ be an o-comparison operator. We say that the operator $f : X \to X$ is a *nonlinear φ-contraction*, if and only if

$$d[f(x), f(y)] \leq \varphi[d(x, y)], \text{ for all } x, y \in X.$$

Definition 3.4. Let (X, d, E) be an E-b-metric space and let $f : X \to X$ be a vector Picard operator (i.e. x^* is a unique fixed point of f and for any $x \in X$, the sequence $x_n = f^n(x)$ E-converges to x^* as $n \to \infty$). Then, the operator f is called a *vector ψ-Picard operator* iff, $\psi : E_+ \to E_+$ has the properties: for any decreasing sequence $(t_n) \subset E_+$ with $t_n \downarrow t$, we have $\psi(t_n) \downarrow \psi(t)$, for any $t \in E_+$ with $t > 0$ and $d(x, x^*) \leq \psi[d(x, f(x))]$, for any $x \in X$.

Theorem 3.5. *Let (X, d, E) be a complete E-b-metric space with E order complete and $s \geq 1$. We assume that E_{++} is nonempty and let $f : X \to X$ be a nonlinear φ-contraction. If for any decreasing sequence $(t_n) \subset E_+$ with $t_n \downarrow t$, we have $\varphi(t_n) \downarrow \varphi(t)$, and the operator $\psi : E_+ \to E_+$ defined by $\psi(t) = \frac{1}{s}t - \varphi(t)$ is invertible, then f is a vector ψ^{-1}-Picard operator.*

Proof. Let $x_0 \in X$. Inductively, we have

$$d(x_n, x_{n+p}) \leq \varphi^n[d(x_0, x_p)], \text{ for any } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*.$$

Let $e \in E_{++}$. As E_{++} is a cone, so $\frac{1}{2}e \in E_{++}$. The operator φ is an o-comparison operator, so we have $\varphi^n(e) \xrightarrow{o} 0$ as $n \to \infty$. In view of Lemma 2.6, we have that there exists $\ell \in \mathbb{N}$ such that

$$\varphi^n(e) \leq \eta_n \leq \frac{1}{2}e, \text{ for any } n \geq \ell.$$

Thus,

$$e - \varphi^{\ell}(e) \geq \frac{1}{2}e \gg 0.$$

Let $e^* := \max\{d(x_1, x_0), d(x_2, x_0), \ldots, d(x_\ell, x_0)\}$. Since $\varphi^n(e^*) \xrightarrow{o} 0$ as $n \to \infty$, again by Lemma 2.6, there exists $k \in \mathbb{N}$ such that

$$\varphi^n(e^*) \leq e - \varphi^{\ell}(e), \text{ for any } n \geq k.$$

Using induction on $j \in \mathbb{N}$, we prove that $d(x_m, x_n) \leq \tilde{e} \in E_{++}$ for $n \geq k$ and $n + j\ell < m \leq n + (j+1)\ell$. To show this inequality for $j = 0$, let $k \leq n < m \leq n + \ell$. Then $0 < m - n \leq \ell$ and consequently,

$$d(x_m, x_n) \leq \varphi^n[d(x_{m-n}, x_0)] \leq \varphi^n(e^*) \leq e - \varphi^{\ell}(e) \leq e.$$

Assume now that the statements holds for $j - 1$ and let $n + j\ell < m \leq n + (j+1)\ell$. Then

$$\begin{aligned} d(x_m, x_n) &\leq s[d(x_m, x_{n+\ell}) + d(x_{n+\ell}, x_n)] \\ &\leq s\varphi^{\ell}[d(x_{m-\ell}, x_n)] + \varphi^n[d(x_\ell, x_0)] \\ &\leq s[\varphi^{\ell}(e) + \varphi^n(e^*)] \\ &\leq s[\varphi^{\ell}(e) + e - \varphi^{\ell}(e)] \\ &= se := \tilde{e} \in E_{++}. \end{aligned}$$

Thus, we obtain that the sequence (x_n) is E-Cauchy in X. By E-completeness of X, it follows that there exists $x^* \in X$ such that $x_n \xrightarrow{d,E} x^*$.

Let $\varepsilon \in E_{++}$. Applying Lemma 2.6 to the sequence $h_n := d(x^*, x_n)$, there exists $k_0 \in \mathbb{N}$ such that

$$d(x^*, x_n) \leq \varepsilon, \text{ for any } n \geq k_0.$$

We have

$$\begin{aligned} d[x^*, f(x^*)] &\leq s\{d(x^*, x_{k_0+1}) + d[x_{k_0+1}, f(x^*)]\} \\ &\leq s\{\varepsilon + \varphi[d(x_{k_0}, x^*)]\} \\ &\leq s[\varepsilon + \varphi(\varepsilon)]. \end{aligned}$$

Thus, x^* is a fixed point of f in X.

For the uniqueness, we suppose that $y^* \in X$ is another fixed point of f with $y^* \neq x^*$. Then

$$\begin{aligned} d(x^*, y^*) &\leq s[d(x^*, x_n) + d(x_n, y^*)] \\ &\leq s\{d(x^*, x_n) + d[f^n(x_0), f^n(y^*)]\} \\ &\leq s\{d(x^*, x_n) + \varphi^n[d(x_0, y^*)]\}. \end{aligned}$$

Letting $n \to \infty$, we get that $y^* = x^*$.

For any $n \in \mathbb{N}$, we have the following error estimate for the fixed point x^*.

$$\begin{aligned} d(x_n, x^*) &= d[f^n(x_0), f^n(x^*)] \\ &\leq \varphi^n[d(x_0, x^*)]. \end{aligned}$$

We have

$$\begin{aligned} d(x_0, x^*) &\leq s[d(x_0, x_1) + d(x_1, x^*)] \\ &\leq sd(x_0, x_1) + s\varphi[d(x_0, x^*)] \end{aligned}$$

and thus,

$$d(x_0, x^*) \leq \psi^{-1}[d(x_0, x_1)].$$

Hence, f is a vector ψ^{-1}-Picard operator. □

We now discuss the multivalued case.

Definition 3.6. Let (X, d, E) be an E-b-metric space and let $\varphi : E_+ \to E_+$ be an o-comparison operator. We say that the operator $F : X \to P_{cl}(X)$ is a *multivalued nonlinear φ-contraction* if, for any $x, y \in X$ and any $u \in F(x)$, there exists $v \in F(y)$ such that

$$d(u, v) \leq \varphi[d(x, y)].$$

Definition 3.7. Let (X, d, E) be an E-b-metric space and let $F : X \to P(X)$ be a multivalued vector weak Picard operator (i.e. the fixed point set of F is nonempty and for any $(x, y) \in \mathrm{Graph}(F)$, the successive approximation sequence $(x_n)_{n\in\mathbb{N}^*} \subset X$ E-converges to a fixed point of F). Then, the operator F is called a *vector ψ-weak Picard operator* iff, $\psi : E_+ \to E_+$ has the properties: for any decreasing sequence $(t_n) \subset E_+$ with $t_n \downarrow t$, we have $\psi(t_n) \downarrow \psi(t)$, for any $t \in E_+$ with $t > 0$ and there exists a selection f^∞ for F^∞ (where $F^\infty : \mathrm{Graph}(F) \to P(\mathrm{Fix}(F))$ is given by the formula $F^\infty(x, y) = z \in \mathrm{Fix}(F)$: there exists a sequence of successive approximations of F starting from (x, y) which E-converges to z) such that $d[x, f^\infty(x, y)] \leq \psi[d(x, y)]$, for any $(x, y) \in \mathrm{Graph}(F)$.

Theorem 3.8. *Let (X, d, E) be a complete E-b-metric space with E order complete and $s \geq 1$. We assume that E_{++} is nonempty and let $F : X \to P_{cl}(X)$ be a multivalued nonlinear φ-contraction. If for any decreasing sequence $(t_n) \subset E_+$ with $t_n \downarrow t$, we have $\varphi(t_n) \downarrow \varphi(t)$ and $\varphi(st) \leq s\varphi(t)$, for any $t \in E_+$ with $t > 0$, and the operator $\psi : E_+ \to E_+$ defined by $\psi(t) = \frac{1}{s}t - s^2\varphi(t)$ is invertible, then F is a vector ψ^{-1}-weak Picard operator.*

Proof. Let $x_0 \in X$ and $x_1 \in F(x_0)$. Then, by the φ-contraction hypothesis, there exists $x_2 \in F(x_1)$ such that

$$d(x_1, x_2) \leq \varphi[d(x_0, x_1)].$$

Inductively, we can define the sequence $(x_n) \in X$ such that $x_{n+1} \in F(x_n)$ and

$$d(x_n, x_{n+1}) \leq \varphi[d(x_{n-1}, x_n)], \text{ for any } n \in \mathbb{N}^*.$$

Iterating this inequality, we obtain that

$$d(x_n, x_{n+p}) \leq \varphi^n[d(x_0, x_p)], \text{ for any } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*.$$

To show that (x_n) is an E-Cauchy sequence in X, it is suffices by Lemma 3.1, to prove that, for any $e \in E_{++}$, there exists $k \in \mathbb{N}$ such that $d(x_m, x_n) \leq e$ for all $m > n \geq k$.

Let $e \in E_{++}$. As E_{++} is a cone, so $\frac{1}{2}e \in E_{++}$. The operator φ is an o-comparison operator, so we have $\varphi^n(e) \xrightarrow{o} 0$ as $n \to \infty$. In view of Lemma 2.6, there exists $\ell \in \mathbb{N}$ such that

$$\varphi^n(e) \leq \frac{1}{2}e, \text{ for any } n \geq \ell.$$

Thus,

$$e - \varphi^\ell(e) \geq \frac{1}{2}e \gg 0.$$

Let $e^* := \max\{d(x_1, x_0), d(x_2, x_0), \ldots, d(x_\ell, x_0)\}$. Since $\varphi^n(e^*) \xrightarrow{o} 0$ as $n \to \infty$, again by Lemma 2.6, there exists $k \in \mathbb{N}$ such that

$$\varphi^n(e^*) \leq e - \varphi^\ell(e), \text{ for any } n \geq k.$$

Using induction on $j \in \mathbb{N}$, we prove that $d(x_m, x_n) \leq e$ for $n \geq k$ and $n + j\ell < m \leq n + (j+1)\ell$. To show this inequality for $j = 0$, let $k \leq n < m \leq n + \ell$. Then $0 < m - n \leq \ell$ and consequently,

$$d(x_m, x_n) \leq \varphi^n[d(x_{m-n}, x_0)] \leq \varphi^n(e^*) \leq e - \varphi^\ell(e) \leq e.$$

Assume now that the statements hold for $j-1$ and let $n+j\ell < m \leq n+(j+1)\ell$. Then

$$\begin{aligned}
d(x_m, x_n) &\leq s[d(x_m, x_{n+\ell}) + d(x_{n+\ell}, x_n)] \\
&\leq s\varphi^{\ell}[d(x_{m-\ell}, x_n)] + \varphi^n[d(x_\ell, x_0)] \\
&\leq s[\varphi^{\ell}(e) + \varphi^n(e^*)] \\
&\leq s[\varphi^{\ell}(e) + e - \varphi^{\ell}(e)] \\
&= se := \tilde{e} \in E_{++}.
\end{aligned}$$

Thus, we obtain that the sequence (x_n) is E-Cauchy in X. By E-completeness of X, it follows that there exists $x^* \in X$ such that $x_n \stackrel{d,E}{\longrightarrow} x^*$.

By the φ-contraction hypothesis of F and the construction of sequence (x_n) for any $n \in \mathbb{N}$, there exists $u_n \in F(x^*)$ such that

$$d(x_{n+1}, u_n) \leq \varphi[d(x_n, x^*)].$$

Therefore,

$$\begin{aligned}
d(x^*, u_n) &\leq s[d(x^*, x_{n+1}) + d(x_{n+1}, u_n)] \\
&\leq s\{d(x^*, x_{n+1}) + \varphi[d(x_n, x^*)]\} := a_n.
\end{aligned}$$

Since $x_n \stackrel{d,E}{\longrightarrow} x^*$ and using the theorem assumptions, it follows that the sequence (a_n) o-converges to zero, which implies that $u_n \stackrel{d,E}{\longrightarrow} x^*$. By the closedness of $F(x^*)$, we have that $x^* \in F(x^*)$, i.e. x^* is a fixed point of F.

For any $n, m \in \mathbb{N}$, we have the following error estimate for a fixed point x^*.

$$\begin{aligned}
d(x^*, x_n) &\leq s[d(x^*, x_m) + d(x_m, x_n)] \\
&\leq s\{d(x^*, x_m) + \varphi[d(x_{m-1}, x_{n-1})]\} \\
&\leq s\{d(x^*, x_m) + s\varphi[d(x_{m-1}, x^*) + d(x^*, x_{n-1})]\}.
\end{aligned}$$

Letting $m \to \infty$ and using again assumptions on the o-comparison operator φ, inductively, we get

$$d(x^*, x_n) \leq s^{2n}\varphi^n[d(x^*, x_0)], \text{ for any } n \in \mathbb{N}.$$

So, we have

$$\begin{aligned}
d(x_0, x^*) &\leq s[d(x_0, x_1) + d(x_1, x^*)] \\
&\leq s\left\{d(x_0, x_1) + s^2\varphi[d(x_0, x^*)]\right\}
\end{aligned}$$

and thus,

$$d(x_0, x^*) \leq \psi^{-1}[d(x_0, x_1)].$$

Hence, F is a vector ψ^{-1}-weak Picard operator. □

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