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Certain sufficient conditions on $|N, p_n, q_n|_k$ summability of orthogonal series

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Abstract

In this paper we obtain some sufficient conditions on $|N, p_n, q_n|_k$ summability of an orthogonal series. These conditions are expressed in terms of the coefficients of the orthogonal series. Also, several known and new results are deduced as corollaries of the main results.

Keywords: Orthogonal series, generalized Nörlund summability.
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1. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums $\{s_n\}$. Then, let p denotes the sequence $\{p_n\}$. For two given sequences p and q, the convolution $(p * q)_n$ is defined by

$$(p * q)_n = \sum_{m=0}^{n} p_m q_{n-m} = \sum_{m=0}^{n} p_{n-m} q_m.$$

We write

$$R_n := (p * q)_n, \ R_n^j := \sum_{m=j}^{n} p_{n-m} q_m$$

and

$$R_n^{n+1} = 0, \ R_n^0 = R_n.$$

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Also we put

$$P_n := (p * 1)_n = \sum_{m=0}^{n} p_m \quad \text{and} \quad Q_n := (1 * q)_n = \sum_{m=0}^{n} q_m.$$

When $R_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{R_n} \sum_{m=0}^{n} p_{n-m} q_m s_m.$$

The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely summable $(N, p_n, q_n)_k$ with index k, if for $k \geq 1$ the series

$$\sum_{n=0}^{\infty} \left(\frac{R_n}{q_n}\right)^{k-1} |t_n^{p,q} - t_{n-1}^{p,q}|^k$$

converges [8], and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |N, p_n, q_n|_k.$$

We note that for $k = 1$, $|N, p_n, q_n|_k$ summability is the same as $|N, p_n, q_n| \equiv |N, p_n, q_n|_1$ summability introduced by Tanaka [7].

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval (a, b). We assume that $f(x)$ belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x), \tag{1.1}$$

where $a_n = \int_a^b f(x)\varphi_n(x)dx$, $(n = 0, 1, 2, \dots)$.

Our main purpose of the present paper is to study the $|N, p_n, q_n|_k$ summability of the orthogonal series (1.1), for $1 \leq k \leq 2$, and to deduce as corollaries all results of Y. Okuyama [6].

Throughout this paper K denotes a positive constant that it may depends only on k, and be different in different relations.

The following lemma due to Beppo Levi (see, for example [3]) is often used in the theory of functions in which are involved the series and integrals and which are involved in [1] and [2] too. It will need us to prove main results.

Lemma 1.1. *If $f_n(t) \in L(E)$ are non-negative functions and*

$$\sum_{n=1}^{\infty} \int_E f_n(t)dt < \infty, \tag{1.2}$$

then the series

$$\sum_{n=1}^{\infty} f_n(t)$$

converges almost everywhere on E to a function $f(t) \in L(E)$. Moreover, the series (1.2) is also convergent to f in the norm of $L(E)$.

2. Main results

We prove the following theorem.

Theorem 2.1. *If for $1 \leq k \leq 2$ the series*

$$\sum_{n=0}^{\infty}\left\{\left(\frac{R_n}{q_n}\right)^{2-\frac{2}{k}}\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2|a_j|^2\right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n\varphi_n(x)$$

is summable $|N, p_n, q_n|_k$ almost everywhere.

Proof. For the generalized Nörlund transform $t_n^{p,q}(x)$ of the partial sums of the orthogonal series $\sum_{n=0}^{\infty} a_n\varphi_n(x)$ we have that

$$\begin{aligned} t_n^{p,q}(x) &= \frac{1}{R_n}\sum_{m=0}^{n} p_{n-m}q_m\sum_{j=0}^{m} a_j\varphi_j(x) \\ &= \frac{1}{R_n}\sum_{j=0}^{n} a_j\varphi_j(x)\sum_{m=j}^{n} p_{n-m}q_m \\ &= \frac{1}{R_n}\sum_{j=0}^{n} R_n^j a_j\varphi_j(x) \end{aligned}$$

where $\sum_{j=0}^{m} a_j\varphi_j(x)$ are partial sums of order k of the series (1.1).

As in [6] page 163 one can find that

$$\triangle t_n^{p,q}(x) := t_n^{p,q}(x) - t_{n-1}^{p,q}(x) = \sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)a_j\varphi_j(x).$$

Using the Hölder's inequality and orthogonality to the latter equality, we have that

$$\begin{aligned} \int_a^b |\triangle t_n^{p,q}(x)|^k dx &\leq (b-a)^{1-\frac{k}{2}}\left(\int_a^b |t_n^{p,q}(x) - t_{n-1}^{p,q}(x)|^2 dx\right)^{\frac{k}{2}} \\ &= (b-a)^{1-\frac{k}{2}}\left[\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2|a_j|^2\right]^{\frac{k}{2}}. \end{aligned}$$

Hence, the series

$$\sum_{n=1}^{\infty}\left(\frac{R_n}{q_n}\right)^{k-1}\int_a^b |\triangle t_n^{p,q}(x)|^k dx \leq K\sum_{n=1}^{\infty}\left(\frac{R_n}{q_n}\right)^{k-1}\left[\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2|a_j|^2\right]^{\frac{k}{2}} \tag{2.1}$$

converges by the assumption. From this fact and since the functions $|\triangle t_n^{p,q}(x)|$ are non-negative, then by the Lemma 1.1 the series

$$\sum_{n=1}^{\infty}\left(\frac{R_n}{q_n}\right)^{k-1}|\triangle t_n^{p,q}(x)|^k$$

converges almost everywhere. This completes the proof of the theorem. □

For $k = 1$ in Theorem 2.1 we have the following result.

Corollary 2.2 ([6]). *If the series*

$$\sum_{n=0}^{\infty}\left\{\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2|a_j|^2\right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is summable $|N,p_n,q_n|$ almost everywhere.

Let us prove now another two corollaries of the Theorem 2.1.

Corollary 2.3. *If for $1\le k\le 2$ the series*

$$\sum_{n=0}^{\infty}\left(\frac{p_n}{P_n^{1/k}P_{n-1}}\right)^k\left\{\sum_{j=1}^{n}p_{n-j}^2\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^2|a_j|^2\right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is summable $|N,p_n|_k\equiv|N,p_n,1|_k$ almost everywhere.

Proof. After some elementary calculations one can show that

$$\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}=\frac{p_n}{P_nP_{n-1}}\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)p_{n-j}$$

for all $q_n=1$, and the proof follows immediately from Theorem 2.1. □

Remark 2.4. We note that:

1. If $p_n=1$ for all values of n then $|N,p_n|_k$ summability reduces to $|C,1|_k$ summability
2. If $k=1$ and $p_n=1/(n+1)$ then $|N,p_n|_k$ is equivalent to $|R,\log n,1|$ summability.

These facts show us that Theorem 2.1 includes also sufficient conditions under which the series (1.1) is $|C,1|_k$ summable, respectively $|R,\log n,1|$ summable.

Corollary 2.5. *If for $1\le k\le 2$ the series*

$$\sum_{n=0}^{\infty}\left(\frac{q_n^{1/k}}{Q_n^{1/k}Q_{n-1}}\right)^k\left\{\sum_{j=1}^{n}Q_{j-1}^2|a_j|^2\right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is summable $|\overline{N},q_n|_k\equiv|\overline{N},1,q_n|_k$ almost everywhere.

Proof. Since

$$\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}=-\frac{q_nQ_{j-1}}{Q_nQ_{n-1}}$$

for all $p_n=1$, then the proof follows directly from Theorem 2.1. □

Also, putting $k=1$ in Corollaries 2.3 and 2.5 we obtain

Corollary 2.6 ([4]). *If the series*

$$\sum_{n=0}^{\infty}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{j=1}^{n}p_{n-j}^2\left(\frac{P_n}{p_n}-\frac{P_{n-j}}{p_{n-j}}\right)^2|a_j|^2\right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is summable $|N,p_n|$ almost everywhere.

Corollary 2.7 ([5]). *If the series*

$$\sum_{n=0}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\left\{\sum_{j=1}^{n}Q_{j-1}^2|a_j|^2\right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is summable $|\overline{N},q_n|$ almost everywhere.

Now we shall prove a general theorem concerning $|N,p_n,q_n|_k$ summability of an orthogonal series which involves a positive sequence with certain additional conditions.

For this reason first we put

$$\Lambda^{(k)}(j):=\frac{1}{j^{\frac{2}{k}-1}}\sum_{n=j}^{\infty}n^{\frac{4}{k}-2}\left(\frac{R_n}{q_n}\right)^{2-\frac{2}{k}}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2 \tag{2.2}$$

then the following theorem holds true.

Theorem 2.8. *Let $1\le k\le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series*

$$\sum_{n=1}^{\infty}|a_n|^2\Omega^{\frac{2}{k}-1}(n)\Lambda^{(k)}(n)$$

converges, then the orthogonal series $\sum_{n=0}^{\infty}a_n\varphi_n(x)\in|N,p_n,q_n|_k$ almost everywhere, where $\Lambda^{(k)}(n)$ is defined by (2.2).

Proof. Applying Hölder's inequality to the inequality (2.1) we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\frac{R_n}{q_n}\right)^{k-1}\int_a^b|\triangle t_n^{p,q}(x)|^kdx\le\\
&\le K\sum_{n=1}^{\infty}\left(\frac{R_n}{q_n}\right)^{k-1}\left[\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2|a_j|^2\right]^{\frac{k}{2}}\\
&=K\sum_{n=1}^{\infty}\frac{1}{(n\Omega(n))^{\frac{2-k}{2}}}\left[(n\Omega(n))^{\frac{2}{k}-1}\left(\frac{R_n}{q_n}\right)^{2-\frac{2}{k}}\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2|a_j|^2\right]^{\frac{k}{2}}\\
&\le K\left(\sum_{n=1}^{\infty}\frac{1}{(n\Omega(n))}\right)^{\frac{2-k}{2}}\left[\sum_{n=1}^{\infty}(n\Omega(n))^{\frac{2}{k}-1}\left(\frac{R_n}{q_n}\right)^{2-\frac{2}{k}}\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2|a_j|^2\right]^{\frac{k}{2}}
\end{aligned}$$

$$\le K\left\{\sum_{j=1}^{\infty}|a_j|^2\sum_{n=j}^{\infty}(n\Omega(n))^{\frac{2}{k}-1}\left(\frac{R_n}{q_n}\right)^{2-\frac{2}{k}}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2\right\}^{\frac{k}{2}}$$

$$\le K\left\{\sum_{j=1}^{\infty}|a_j|^2\left(\frac{\Omega(j)}{j}\right)^{\frac{2}{k}-1}\sum_{n=j}^{\infty}n^{\frac{4}{k}-2}\left(\frac{R_n}{q_n}\right)^{2-\frac{2}{k}}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2\right\}^{\frac{k}{2}}$$

$$= K\left\{\sum_{j=1}^{\infty}|a_j|^2\Omega^{\frac{2}{k}-1}(j)\Lambda^{(k)}(j)\right\}^{\frac{k}{2}},$$

which is finite by assumption, and this completes the proof. □

A direct consequence of the theorem 2.8 is the following ($k = 1$).

Corollary 2.9 ([6]). *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum_{n=1}^{\infty}|a_n|^2\Omega(n)\Lambda^{(1)}(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty}a_n\varphi_n(x)\in|N,p_n,q_n|$ almost everywhere, where $\Lambda^{(1)}(n)$ is defined by $\Lambda^{(1)}(j) := \frac{1}{j}\sum_{n=j}^{\infty}n^2\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2$.*

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