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Some new generalizations of Ostrowski type inequalities on time scales involving combination of Δ-integral means

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Abstract

In this paper we obtain some new generalizations of Ostrowski type inequalities on time scales involving combination of Δ-integral means, i.e., a new Ostrowski type inequality on time scales involving combination of Δ-integral means, two Ostrowski type inequalities for two functions on time scales, and some new perturbed Ostrowski type inequalities on time scales. We also give some other interesting inequalities as special cases.

Keywords: Ostrowski inequality, perturbed Ostrowski inequality, Δ-integral means, time scales.
2010 MSC: 26D15, 26E70, 58C05, 65D30.

1. Introduction

In 1988, Hilger introduced the time scale theory in order to unify continuous and discrete analysis [18]. Such theory has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in population dynamics [4], economics [3], physics [38], space weather [25] and so on.

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Recently, many authors studied the theory of certain integral inequalities on time scales (see [7, 8, 9, 10, 12, 19, 20, 22, 23, 26, 27, 28, 29, 30, 31, 32, 33, 34, 36, 37, 40, 41]).

In 1938, Ostrowski derived a formula to estimate the absolute deviation of a differentiable function from its integral mean [35]. The result is nowadays known as the Ostrowski inequality [2, 13, 14, 15, 16, 17, 39], which can be obtained by using the Montgomery identity. The Ostrowski inequality and the Montgomery identity were generalized by Bohner and Matthews to an arbitrary time scale [8], unifying the discrete, the continuous, and the quantum cases:

Theorem A (Ostrowski's inequality on time scales [8]). Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Then

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(\sigma(s)) \Delta s \right| \le \frac{M}{b-a} [h_2(t,a) + h_2(t,b)], \tag{1.1}$$

where $h_2(\cdot, \cdot)$ is defined by Definition 2.5 below and $M = \sup_{a<t<b} \left| f^{\Delta}(t) \right| < \infty$. This inequality is sharp in the sense that the right-hand side of (1.1) cannot be replaced by a smaller one.

The purpose of this paper is to obtain some new generalizations of Ostrowski type inequalities on time scales using the kernel given in [11]. We first establish a new Ostrowski type inequality on time scales involving combination of Δ-integral means. Then we derive two Ostrowski type inequalities for two functions on time scales. Finally, four new perturbed Ostrowski type inequalities on time scales are obtained. We also give some other interesting inequalities as special cases.

This paper is organized as follows. In Section 2, we briefly present the general definitions and theorems related to the time scales calculus. Some new generalizations of Ostrowski type inequalities on time scales involving combination of Δ-integral means are derived in Section 3.

2. Time Scales Essentials

In this section we briefly introduce the time scales theory. For further details and proofs we refer the reader to Hilger's Ph.D. thesis [18], the books [5, 6, 24], and the survey [1].

Definition 2.1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$. The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ as follows. If $\sigma(t) > t$, *then we say that* t is right-scattered, if $\rho(t) < t$ then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, the t is called right-dense, and if $\rho(t) = t$ then t is called left-dense, Points that both right-dense and left-dense are called dense. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$ defined by $\mu(t) = \sigma(t) - t$ is called the *graininess function.* The set $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 2.2. Let $f : \mathbb{T} \to \mathbb{R}$. f is called differentiable at $t \in \mathbb{T}^k$, with (delta) derivative $f^{\Delta}(t) \in \mathbb{R}$, if for any given $\varepsilon > 0$ there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \le \varepsilon \left| \sigma(t) - s \right|, \quad \forall \ s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = \frac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = f(t+1) - f(t)$.

Theorem B. Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t) = f(t) g^{\Delta}(t) + f^{\Delta}(t) g(\sigma(t)).$$

Definition 2.3. The function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous (denote* $f \in C_{rd}(\mathbb{T}, \mathbb{R})$), if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$.

It follows from [5, Theorem 1.74] that every rd-continuous function has an anti-derivative.

Definition 2.4. Let $f \in C_{rd}(\mathbb{T},\mathbb{R})$. Then $F : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it satisfies $F^{\Delta}(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we define the Δ-integral of f as

$$\int_a^t f(s)\Delta s = F(t) - F(a), \quad t \in \mathbb{T}.$$

Theorem C. Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then
(1) $\int_a^b [\alpha f(t) + \beta g(t)]\, \Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$,
(2) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,
(3) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,
(4) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$,

Theorem D. If f is Δ-integrable on $[a, b]$, then so is $|f|$,and

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b |f(t)|\, \Delta t.$$

Definition 2.5. Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$h_0(t, s) = 1 \quad \text{for all} \quad s, t \in \mathbb{T}$$

and then recursively by

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\Delta\tau \quad \text{for all} \quad s, t \in \mathbb{T}.$$

3. Main Results

3.1. A new Ostrowski type inequality on time scales

Lemma 3.1. *Let $a, b, x, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Then for all $x \in [a, b]$, we have*

$$\int_a^b P(x,t)f^{\Delta}(t)\Delta t = f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t + \frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right], \tag{3.1}$$

where

$$P(x,t) = \begin{cases} \frac{\alpha}{\alpha+\beta}\left(\frac{t-a}{x-a}\right), & a \leq t < x, \\ \frac{-\beta}{\alpha+\beta}\left(\frac{b-t}{b-x}\right), & x \leq t < b \end{cases} \tag{3.2}$$

which is firstly given in [11].

Proof. Using Part (4) of Theorem C, we have

$$\int_a^x \frac{\alpha}{\alpha+\beta}\left(\frac{t-a}{x-a}\right) f^{\Delta}(t)\Delta t = \frac{\alpha}{\alpha+\beta} f(x) - \frac{\alpha}{(\alpha+\beta)(x-a)}\int_a^x f(\sigma(t))\Delta t \tag{3.3}$$

and

$$\int_x^b \frac{-\beta}{\alpha+\beta}\left(\frac{b-t}{b-x}\right) f^{\Delta}(t)\Delta t = \frac{\beta}{\alpha+\beta} f(x) - \frac{\beta}{(\alpha+\beta)(b-x)}\int_x^b f(\sigma(t))\Delta t. \tag{3.4}$$

Therefore, the identity (3.1) is obtained by combining the identities (3.3) and (3.4). □

Corollary 3.2. *In the case of $\mathbb{T}=\mathbb{R}$ in Lemma 3.1, we have*

$$\int_a^b P(x,t)f'(t)dt = f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)dt + \frac{\beta}{b-x}\int_x^b f(t)dt\right],$$

where

$$P(x,t) = \begin{cases} \frac{\alpha}{\alpha+\beta}\left(\frac{t-a}{x-a}\right), & a \le t < x, \\ \frac{-\beta}{\alpha+\beta}\left(\frac{b-t}{b-x}\right), & x \le t < b. \end{cases}$$

This is the result given in Lemma 1 of [11].

Corollary 3.3. *In the case of $\mathbb{T}=\mathbb{Z}$ in Lemma 3.1, we have*

$$\sum_{t=a}^{b-1} P(x,t)\Delta f(t) = f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1} f(t+1) + \frac{\beta}{b-x}\sum_{t=x}^{b-1} f(t+1)\right],$$

where

$$P(x,t) = \begin{cases} \frac{\alpha}{\alpha+\beta}\left(\frac{t-a}{x-a}\right), & a \le t < x-1, \\ \frac{-\beta}{\alpha+\beta}\left(\frac{b-t}{b-x}\right), & x \le t < b-1. \end{cases}$$

Corollary 3.4. *In the case of $\mathbb{T}=q^{\mathbb{Z}}\cup\{0\}$ $(q>1)$ in Lemma 3.1, we have*

$$\int_a^b P(x,t)\mathrm{D}_q f(t)\mathrm{d}_q t = f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(qt)\mathrm{d}_q t + \frac{\beta}{b-x}\int_x^b f(qt)\mathrm{d}_q t\right],$$

where

$$P(x,t) = \begin{cases} \frac{\alpha}{\alpha+\beta}\left(\frac{t-a}{x-a}\right), & a \le t < x, \\ \frac{-\beta}{\alpha+\beta}\left(\frac{b-t}{b-x}\right), & x \le t < b. \end{cases}$$

Here, for $s,t \in q^{\mathbb{Z}}\cup\{0\}$ with $t \ge s$, we use the definitions

$$(\mathrm{D}_q f)(t) = \frac{f(qt)-f(t)}{(q-1)t} \quad \textit{and} \quad \int_s^t f(\eta)\mathrm{d}_q\eta = (q-1)\sum_{\ell=\log_q(s)}^{\log_q(t/q)} f(q^\ell)q^\ell,$$

by adopting the convention that $\log_q(0) := -\infty$ and $\log_q(\infty) := \infty$ (see [21]).

Theorem 3.5. *Let $a,b,x,t \in \mathbb{T}$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be differentiable. Then for all $t\in[a,b]$, we have*

$$\left|f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t + \frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right| \le \frac{M}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a) + \frac{\beta}{b-x}h_2(x,b)\right],$$

where

$$M = \sup_{a<t<b}\left|f^\Delta(t)\right| < \infty.$$

Proof. This is easily obtained from Lemma 3.1 by using the properties of modulus and the definition of $h_2(\cdot,\cdot)$. □

Remark 3.6. In the case of $\alpha = x-a$ and $\beta = b-x$, Theorem 3.5 is reduced to Theorem A.

Corollary 3.7. *Theorem 3.5 is reduced in the case* $\mathbb{T}=\mathbb{R}$ *to*

$$\left|f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)dt+\frac{\beta}{b-x}\int_x^b f(t)dt\right]\right|\leq\frac{M}{2(\alpha+\beta)}\left[\alpha(x-a)+\beta(b-x)\right],$$

where

$$M=\sup_{a<t<b}\left|f'(t)\right|<\infty,$$

which corresponds to Theorem 2 of [11].

Corollary 3.8. *Theorem 3.5 is reduced in the case* $\mathbb{T}=\mathbb{Z}$ *to*

$$\left|f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1}f(t+1)+\frac{\beta}{b-x}\sum_{t=x}^{b-1}f(t+1)\right]\right|\leq\frac{M}{2(\alpha+\beta)}\left[\alpha(x-a-1)+\beta(b-x+1)\right],$$

where

$$M=\sup_{a<t<b}\left|\Delta f(t)\right|<\infty.$$

Corollary 3.9. *Theorem 3.5 is reduced in the case* $\mathbb{T}=q^{\mathbb{Z}}\cup\{0\}$ $(q>1)$ *to*

$$\left|f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(qt)\mathrm{d}_qt+\frac{\beta}{b-x}\int_x^b f(qt)\mathrm{d}_qt\right]\right|\leq\frac{M}{(\alpha+\beta)(1+q)}\left[\alpha(x-qa)+\beta(qb-x)\right],$$

where

$$M=\sup_{a<t<b}\left|(\mathrm{D}_qf)(t)\right|<\infty.$$

3.2. Ostrowski type inequalities for two functions on time scales

Theorem 3.10. *Let* $a,b,x,t\in\mathbb{T}$, $a<b$ *and* $f,g:[a,b]\to\mathbb{R}$ *be differentiable. Then for all* $x\in[a,b]$, *we have*

$$\begin{aligned}&\left|f(x)g(x)-\frac{1}{2(\alpha+\beta)}\left\{g(x)\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right.\right.\\&\left.\left.+f(x)\left[\frac{\alpha}{x-a}\int_a^x g(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b g(\sigma(t))\Delta t\right]\right\}\right|\\&\leq\frac{M_1\left|g(x)\right|+M_2\left|f(x)\right|}{2(\alpha+\beta)}\left[\frac{\alpha}{x-a}h_2(x,a)+\frac{\beta}{b-x}h_2(x,b)\right]\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}&\left|f(x)g(x)-\frac{1}{\alpha+\beta}\left\{f(x)\left[\frac{\alpha}{x-a}\int_a^x g(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b g(\sigma(t))\Delta t\right]\right.\right.\\&\left.+g(x)\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right\}\\&+\frac{1}{(\alpha+\beta)^2}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\\&\left.\times\left[\frac{\alpha}{x-a}\int_a^x g(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b g(\sigma(t))\Delta t\right]\right|\\&\leq\frac{M_1M_2}{(\alpha+\beta)^2}\left[\frac{\alpha}{x-a}h_2(x,a)+\frac{\beta}{b-x}h_2(x,b)\right]^2,\end{aligned}\tag{3.6}$$

where

$$M_1=\sup_{a<t<b}\left|f^{\Delta}(t)\right|<\infty\quad\textit{and}\quad M_2=\sup_{a<t<b}\left|g^{\Delta}(t)\right|<\infty.$$

Proof. We have

$$f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]=\int_a^b P(x,t)f^{\Delta}(t)\Delta t \tag{3.7}$$

and

$$g(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x g(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b g(\sigma(t))\Delta t\right]=\int_a^b P(x,t)g^{\Delta}(t)\Delta t. \tag{3.8}$$

Multiplying (3.7) by $g(x)$ and (3.8) by $f(x)$, adding the resultant identities, we have

$$\begin{aligned}&f(x)g(x)-\frac{1}{2(\alpha+\beta)}\left\{g(x)\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right.\\&\left.+f(x)\left[\frac{\alpha}{x-a}\int_a^x g(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b g(\sigma(t))\Delta t\right]\right\}\\=&\frac{1}{2}\left[g(x)\int_a^b P(x,t)f^{\Delta}(t)\Delta t+f(x)\int_a^b P(x,t)g^{\Delta}(t)\Delta t\right].\end{aligned}$$

Using the properties of modulus, we get

$$\begin{aligned}&\left|f(x)g(x)-\frac{1}{2(\alpha+\beta)}\left\{g(x)\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right.\right.\\&\left.\left.+f(x)\left[\frac{\alpha}{x-a}\int_a^x g(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b g(\sigma(t))\Delta t\right]\right\}\right|\\\le&\frac{1}{2}\left[|f(x)|\int_a^b |P(x,t)|\left|f^{\Delta}(t)\right|\Delta t+|f(x)|\int_a^b |P(x,t)|\left|g^{\Delta}(t)\right|\Delta t\right]\\\le&\frac{M_1|g(x)|+M_2|f(x)|}{2(\alpha+\beta)}\left[\frac{\alpha}{x-a}h_2(x,a)+\frac{\beta}{b-x}h_2(x,b)\right].\end{aligned}$$

This completes the proof of the inequality (3.5).

Multiplying the left sides and right sides of (3.7) and (3.8), we get

$$\begin{aligned}&f(x)g(x)-\frac{1}{\alpha+\beta}\left\{f(x)\left[\frac{\alpha}{x-a}\int_a^x g(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b g(\sigma(t))\Delta t\right]\right.\\&\left.+g(x)\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right\}\\&+\frac{1}{(\alpha+\beta)^2}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\\&\times\left[\frac{\alpha}{x-a}\int_a^x g(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b g(\sigma(t))\Delta t\right]\\=&\left(\int_a^b P(x,t)f^{\Delta}(t)\Delta t\right)\left(\int_a^b P(x,t)g^{\Delta}(t)\Delta t\right).\end{aligned}$$

Using the properties of modulus, we can easily obtain (3.6). □

Corollary 3.11. *Theorem 3.10 is reduced in the case* $\mathbb{T}=\mathbb{R}$ *to*

$$\begin{aligned}&\left|f(x)g(x)-\frac{1}{2(\alpha+\beta)}\right.\\&\left.\times\left\{g(x)\left[\frac{\alpha}{x-a}\int_a^x f(t)dt+\frac{\beta}{b-x}\int_x^b f(t)dt\right]+f(x)\left[\frac{\alpha}{x-a}\int_a^x g(t)dt+\frac{\beta}{b-x}\int_x^b g(t)dt\right]\right\}\right|\\\le&\frac{M_1|g(x)|+M_2|f(x)|}{4(\alpha+\beta)}\left[\alpha(x-a)+\beta(b-x)\right]\end{aligned}$$

and

$$\begin{aligned}&\Bigg|f(x)g(x)-\frac{1}{\alpha+\beta}\\&\times\left\{f(x)\left[\frac{\alpha}{x-a}\int_a^x g(t)dt+\frac{\beta}{b-x}\int_x^b g(t)dt\right]+g(x)\left[\frac{\alpha}{x-a}\int_a^x f(t)dt+\frac{\beta}{b-x}\int_x^b f(t)dt\right]\right\}\\&+\frac{1}{(\alpha+\beta)^2}\left[\frac{\alpha}{x-a}\int_a^x f(t)dt+\frac{\beta}{b-x}\int_x^b f(t)dt\right]\left[\frac{\alpha}{x-a}\int_a^x g(t)dt+\frac{\beta}{b-x}\int_x^b g(t)dt\right]\Bigg|\\\leq&\frac{M_1M_2}{4(\alpha+\beta)^2}\left[\alpha(x-a)+\beta(b-x)\right]^2,\end{aligned}$$

where

$$M_1=\sup_{a<t<b}\left|f'(t)\right|<\infty \quad and \quad M_2=\sup_{a<t<b}\left|g'(t)\right|<\infty.$$

Corollary 3.12. *Theorem 3.10 is reduced in the case* $\mathbb{T}=\mathbb{Z}$ *to*

$$\begin{aligned}&\Bigg|f(x)g(x)-\frac{1}{2(\alpha+\beta)}\Bigg\{g(x)\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1}f(t+1)+\frac{\beta}{b-x}\sum_{t=x}^{b-1}f(t+1)\right]\\&+f(x)\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1}g(t+1)+\frac{\beta}{b-x}\sum_{t=x}^{b-1}g(t+1)\right]\Bigg\}\Bigg|\\\leq&\frac{M_1|g(x)|+M_2|f(x)|}{4(\alpha+\beta)}\left[\alpha(x-a-1)+\beta(b-x+1)\right]\end{aligned}$$

and

$$\begin{aligned}&\Bigg|f(x)g(x)-\frac{1}{\alpha+\beta}\Bigg\{f(x)\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1}g(t+1)+\frac{\beta}{b-x}\sum_{t=x}^{b-1}g(t+1)\right]\\&+g(x)\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1}f(t+1)+\frac{\beta}{b-x}\sum_{t=x}^{b-1}f(t+1)\right]\Bigg\}\\&+\frac{1}{(\alpha+\beta)^2}\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1}f(t+1)+\frac{\beta}{b-x}\sum_{t=x}^{b-1}f(t+1)\right]\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1}g(t+1)+\frac{\beta}{b-x}\sum_{t=x}^{b-1}g(t+1)\right]\Bigg|\\\leq&\frac{M_1M_2}{4(\alpha+\beta)^2}\left[\alpha(x-a-1)+\beta(b-x+1)\right]^2,\end{aligned}$$

where

$$M_1=\sup_{a<t<b}|\Delta f(t)|<\infty \quad and \quad M_2=\sup_{a<t<b}|\Delta g(t)|<\infty.$$

Corollary 3.13. *Theorem 3.10 is reduced in the case* $\mathbb{T}=q^{\mathbb{Z}}\cup\{0\}$ $(q>1)$ *to*

$$\begin{aligned}&\Bigg|f(x)g(x)-\frac{1}{2(\alpha+\beta)}\Bigg\{g(x)\left[\frac{\alpha}{x-a}\int_a^x f(qt)\mathrm{d}_qt+\frac{\beta}{b-x}\int_x^b f(qt)\mathrm{d}_qt\right]\\&+f(x)\left[\frac{\alpha}{x-a}\int_a^x g(qt)\mathrm{d}_qt+\frac{\beta}{b-x}\int_x^b g(qt)\mathrm{d}_qt\right]\Bigg\}\Bigg|\\\leq&\frac{M_1|g(x)|+M_2|f(x)|}{2(\alpha+\beta)(1+q)}\left[\alpha(x-qa)+\beta(qb-x)\right]\end{aligned}$$

and

$$\begin{aligned}
&\left| f(x)g(x) - \frac{1}{\alpha+\beta}\left\{ f(x)\left[\frac{\alpha}{x-a}\int_a^x g(qt)\mathrm{d}_q t + \frac{\beta}{b-x}\int_x^b g(qt)\mathrm{d}_q t\right]\right.\right.\\
&\left.+g(x)\left[\frac{\alpha}{x-a}\int_a^x f(qt)\mathrm{d}_q t + \frac{\beta}{b-x}\int_x^b f(qt)\mathrm{d}_q t\right]\right\}\\
&\left.+\frac{1}{(\alpha+\beta)^2}\left[\frac{\alpha}{x-a}\int_a^x f(qt)\mathrm{d}_q t + \frac{\beta}{b-x}\int_x^b f(qt)\mathrm{d}_q t\right]\left[\frac{\alpha}{x-a}\int_a^x g(qt)\mathrm{d}_q t + \frac{\beta}{b-x}\int_x^b g(qt)\mathrm{d}_q t\right]\right|\\
\le&\frac{M_1 M_2}{(\alpha+\beta)^2(1+q)^2}\left[\alpha(x-qa)+\beta(qb-x)\right]^2,
\end{aligned}$$

where

$$M_1 = \sup_{a<t<b}|(\mathrm{D}_q f)(t)| < \infty \quad \text{and} \quad M_2 = \sup_{a<t<b}|(\mathrm{D}_q g)(t)| < \infty.$$

3.3. New perturbed Ostrowski type inequalities on time scales

Theorem 3.14. *Let $a, b, x, t \in \mathbb{T}$, $a < b$ and $f, g : [a, b] \to \mathbb{R}$ be differentiable. Then for all $t \in [a, b]$, we have*

$$\begin{aligned}
&\left| f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t + \frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right.\\
&\left.-\frac{f(b)-f(a)}{b-a}\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a) - \frac{\beta}{b-x}h_2(x,b)\right]\right|\\
\le&\left\{\frac{1}{(b-a)(\alpha+\beta)^2}\left[\frac{\alpha^2}{(x-a)^2}\int_a^x (t-a)^2\Delta t + \frac{\beta^2}{(b-x)^2}\int_x^b (b-t)^2\Delta t\right]\right.\\
&\left.-\frac{1}{(b-a)^2(\alpha+\beta)^2}\left[\frac{\alpha}{x-a}h_2(x,a) - \frac{\beta}{b-x}h_2(x,b)\right]^2\right\}^{\frac{1}{2}}\\
&\times\left[(b-a)\int_a^b \left(f^\Delta(t)\right)^2\Delta t - (f(b)-f(a))^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{3.9}$$

Proof. We have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b P(x,t)f^\Delta(t)\Delta t - \left(\frac{1}{b-a}\int_a^b P(x,t)\Delta t\right)\left(\frac{1}{b-a}\int_a^b f^\Delta(t)\Delta t\right)\\
=&\frac{1}{2(b-a)^2}\int_a^b\int_a^b (P(x,t)-P(x,s))\left(f^\Delta(t)-f^\Delta(s)\right)\Delta t\Delta s.
\end{aligned} \tag{3.10}$$

From (3.1), we also have

$$\int_a^b P(x,t)f^\Delta(t)\Delta t = f(x) - \frac{1}{\alpha+\beta}\left\{\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t + \frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right\} \tag{3.11}$$

and

$$\frac{1}{b-a}\int_a^b f^\Delta(t)\Delta t = \frac{f(b)-f(a)}{b-a}. \tag{3.12}$$

Using the Cauchy-Schwartz inequality, we may write

$$\left|\frac{1}{2(b-a)^2}\int_a^b\int_a^b (P(x,t)-P(x,s))\left(f^\Delta(t)-f^\Delta(s)\right)\Delta t\Delta s\right|$$
$$\leq\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b (P(x,t)-P(x,s))^2\Delta t\Delta s\right)^{\frac{1}{2}}\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f^\Delta(t)-f^\Delta(s)\right)^2\Delta t\Delta s\right)^{\frac{1}{2}}. \quad (3.13)$$

However

$$\frac{1}{2(b-a)^2}\int_a^b (P(x,t)-P(x,s))^2\Delta t\Delta s=\frac{1}{b-a}\int_a^b P^2(x,t)\Delta t-\left(\frac{1}{b-a}\int_a^b P(x,t)\Delta t\right)^2$$
$$=\left[\frac{1}{b-a}\left(\frac{\alpha^2}{(\alpha+\beta)^2(x-a)^2}\int_a^x (t-a)^2\Delta t+\frac{\beta^2}{(\alpha+\beta)^2(b-x)^2}\int_x^b (b-t)^2\Delta t\right)\right.$$
$$\left.-\frac{1}{(b-a)^2}\left(\frac{\alpha}{(\alpha+\beta)(x-a)}h_2(x,a)-\frac{\beta}{(\alpha+\beta)(b-x)}h_2(x,b)\right)^2\right] \quad (3.14)$$

and

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f^\Delta(t)-f^\Delta(s)\right)^2\Delta t\Delta s=\frac{1}{b-a}\int_a^b \left(f^\Delta(t)\right)^2\Delta t-\left(\frac{1}{b-a}\int_a^b f^\Delta(t)\Delta t\right)^2. \quad (3.15)$$

Using (3.10)-(3.15), we can easily obtain the inequality (3.9). □

Corollary 3.15. *Theorem 3.14 is reduced in the case* $\mathbb{T}=\mathbb{R}$ *to*

$$\left|f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)dt+\frac{\beta}{b-x}\int_x^b f(t)dt\right]-\frac{f(b)-f(a)}{b-a}\frac{1}{2(\alpha+\beta)}[\alpha(x-a)-\beta(b-x)]\right|$$
$$\leq\left[\frac{1}{3(b-a)(\alpha+\beta)^2}\left(\alpha^2(x-a)+\beta^2(b-x)\right)-\frac{1}{4(b-a)^2(\alpha+\beta)^2}((\alpha+\beta)x-(\alpha a+\beta b))^2\right]^{\frac{1}{2}}$$
$$\times\left[(b-a)\int_a^b \left(f'(t)\right)^2dt-(f(b)-f(a))^2\right]^{\frac{1}{2}}.$$

Corollary 3.16. *Theorem 3.14 is reduced in the case* $\mathbb{T}=\mathbb{Z}$ *to*

$$\left|f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1}f(t+1)+\frac{\beta}{b-x}\sum_{t=x}^{b-1}f(t+1)\right]\right.$$
$$\left.-\frac{f(b)-f(a)}{b-a}\frac{1}{2(\alpha+\beta)}[\alpha(x-a-1)-\beta(b-x+1)]\right|$$
$$\leq\left\{\frac{1}{(b-a)(\alpha+\beta)^2}\left[\frac{\alpha^2}{(x-a)^2}\left(\frac{1}{6}x\left(6a^2-6ax+6a+2x^2-3x+1\right)-\frac{1}{6}a(2a+1)(a+1)\right)\right.\right.$$
$$\left.-\frac{\beta^2}{(b-x)^2}\left(\frac{1}{6}x\left(6b^2-6bx+6b+2x^2-3x+1\right)-\frac{1}{6}b(2b+1)(b+1)\right)\right]$$
$$\left.-\frac{1}{4(b-a)^2(\alpha+\beta)^2}((\alpha(x-a-1)-\beta(b-x+1)))^2\right\}^{\frac{1}{2}}\left[(b-a)\sum_{t=a}^{b-1}(\Delta f(t))^2-(f(b)-f(a))^2\right]^{\frac{1}{2}}.$$

Corollary 3.17. *Theorem 3.14 is reduced in the case* $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ $(q > 1)$ *to*

$$\begin{aligned}&\left|f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(qt)\mathrm{d}_q t + \frac{\beta}{b-x}\int_x^b f(qt)\mathrm{d}_q t\right]\right.\\ &\left.-\frac{1}{(\alpha+\beta)(1+q)}\left[\alpha(x-qa) - \beta(qb-x)\right]\frac{f(b)-f(a)}{b-a}\right|\\ \leq&\left[\frac{1}{(b-a)(\alpha+\beta)^2}\left(\frac{\alpha^2(x-qa)((x-a)+q(x-qa))}{(x-a)(1+2q+2q^2+q^3)} + \frac{\beta^2(x-qb)((x-b)+q(x-qb))}{(b-x)(1+2q+2q^2+q^3)}\right)\right.\\ &\left.-\frac{1}{(\alpha+\beta)^2(b-a)^2(1+q)^2}\left(\alpha(x-qa) - \beta(qb-x)\right)^2\right]^{\frac{1}{2}}\left[(b-a)\int_a^b (\mathrm{D}_q f(t))^2\,\mathrm{d}_q t - (f(b)-f(a))^2\right]^{\frac{1}{2}}.\end{aligned}$$

Theorem 3.18. *Let* $a, b, x, t \in \mathbb{T}$, $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *be differentiable function such that there exist constants* $\gamma, \Gamma \in \mathbb{R}$, *with* $\gamma \leq f^{\Delta}(x) \leq \Gamma$, $x \in [a, b]$. *Then for all* $x \in [a, b]$, *we have*

$$\begin{aligned}&\left|f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t + \frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right.\\ &\left.-\frac{\gamma+\Gamma}{2}\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a) - \frac{\beta}{b-x}h_2(x,b)\right]\right|\\ \leq&\frac{\Gamma-\gamma}{2}\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a) + \frac{\beta}{b-x}h_2(x,b)\right].\end{aligned} \tag{3.16}$$

Proof. From (3.1), we may write

$$f(x) = \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t + \frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right] + \int_a^b P(x,t)f^{\Delta}(t)\Delta t. \tag{3.17}$$

We also have

$$\int_a^b P(x,t)\Delta t = \frac{\alpha}{(\alpha+\beta)(x-a)}h_2(x,a) - \frac{\beta}{(\alpha+\beta)(b-x)}h_2(x,b). \tag{3.18}$$

Let $C = \frac{\gamma+\Gamma}{2}$. From (3.17) and (3.18), we get

$$\begin{aligned}\int_a^b P(x,t)\left(f^{\Delta}(t) - C\right)\Delta t =& f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t + \frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\\ &-\frac{\gamma+\Gamma}{2}\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a) - \frac{\beta}{b-x}h_2(x,b)\right]\end{aligned} \tag{3.19}$$

Using the properties of modulus, we get

$$\left|\int_a^b P(x,t)\left(f^{\Delta}(t) - C\right)\Delta t\right| \leq \frac{\Gamma-\gamma}{2}\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a) + \frac{\beta}{b-x}h_2(x,b)\right]. \tag{3.20}$$

From (3.19)-(3.20), we can easily get (3.16). □

Corollary 3.19. *Theorem 3.18 is reduced in the case* $\mathbb{T} = \mathbb{R}$ *to*

$$\begin{aligned}&\left|f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)dt + \frac{\beta}{b-x}\int_x^b f(t)dt\right] - \frac{\gamma+\Gamma}{4(\alpha+\beta)}\left[\alpha(x-a) - \beta(b-x)\right]\right|\\ \leq&\frac{\Gamma-\gamma}{4(\alpha+\beta)}\left[\alpha(x-a) + \beta(b-x)\right].\end{aligned}$$

Corollary 3.20. *Theorem 3.18 is reduced in the case* $\mathbb{T}=\mathbb{Z}$ *to*

$$\left|f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1}f(t+1)+\frac{\beta}{b-x}\sum_{t=x}^{b-1}f(t+1)\right]-\frac{\gamma+\Gamma}{4(\alpha+\beta)}\left[\alpha(x-a-1)-\beta(b-x+1)\right]\right|$$
$$\leq\frac{\Gamma-\gamma}{4(\alpha+\beta)}\left[\alpha(x-a-1)+\beta(b-x+1)\right].$$

Corollary 3.21. *Theorem 3.18 is reduced in the case* $\mathbb{T}=q^{\mathbb{Z}}\cup\{0\}$ $(q>1)$ *to*

$$\left|f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(qt)\mathrm{d}_qt+\frac{\beta}{b-x}\int_x^b f(qt)\mathrm{d}_qt\right]-\frac{\gamma+\Gamma}{2(\alpha+\beta)(1+q)}\left[\alpha(x-qa)-\beta(qb-x)\right]\right|$$
$$\leq\frac{\Gamma-\gamma}{2(\alpha+\beta)(1+q)}\left[\alpha(x-qa)+\beta(qb-x)\right].$$

Theorem 3.22. *Let* $a,b,x,t\in\mathbb{T}$, $a<b$ *and* $f:[a,b]\to\mathbb{R}$ *be differentiable function such that there exist constants* $\gamma,\Gamma\in\mathbb{R}$ *with* $\gamma\leq f^{\Delta}(t)\leq\Gamma$, $t\in\mathbb{T}$. *Then for all* $t\in[a,b]$, *we have*

$$\begin{aligned}&\left|f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right.\\&\left.-\frac{\gamma}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a)-\frac{\beta}{b-x}h_2(x,b)\right]\right|\\&\leq\left(\frac{1}{2}+\frac{|\alpha-\beta|}{2(\alpha+\beta)}\right)(S-\gamma)(b-a)\end{aligned}\tag{3.21}$$

and

$$\begin{aligned}&\left|f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right.\\&\left.-\frac{\Gamma}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a)-\frac{\beta}{b-x}h_2(x,b)\right]\right|\\&\leq\left(\frac{1}{2}+\frac{|\alpha-\beta|}{2(\alpha+\beta)}\right)(\Gamma-S)(b-a),\end{aligned}\tag{3.22}$$

where $S=(f(b)-f(a))/(b-a)$.

Proof. From (3.1), we may write

$$f(x)=\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]+\int_a^b P(x,t)f^{\Delta}(t)\Delta t.\tag{3.23}$$

We also have

$$\int_a^b P(x,t)\Delta t=\frac{\alpha}{(\alpha+\beta)(x-a)}h_2(x,a)-\frac{\beta}{(\alpha+\beta)(b-x)}h_2(x,b).\tag{3.24}$$

Let $C\in\mathbb{R}$ be a constant. From (3.23) and (3.24), it follows that

$$\begin{aligned}\int_a^b P(x,t)\left[f^{\Delta}(t)-C\right]\Delta t=&f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\\&-\frac{C}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a)-\frac{\beta}{b-x}h_2(x,b)\right].\end{aligned}\tag{3.25}$$

In case of $C = \gamma$ in (3.25), we have

$$\int_a^b P(x,t)\left[f^{\Delta}(t)-\gamma\right]\Delta t = f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t + \frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right] - \frac{\gamma}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a) - \frac{\beta}{b-x}h_2(x,b)\right]. \tag{3.26}$$

On the other hand, we have

$$\left|\int_a^b P(x,t)\left[f^{\Delta}(t)-\gamma\right]\Delta t\right| \le \max_{a<t<b}|P(x,t)|\int_a^b \left|f^{\Delta}(t)-\gamma\right|\Delta t \tag{3.27}$$

We also have (see [11, Theorem 2])

$$\max_{a\le t\le b}|P(x,t)| \le \frac{1}{2} + \frac{|\alpha-\beta|}{2(\alpha+\beta)} \tag{3.28}$$

and

$$\int_a^b \left|f^{\Delta}(t)-\gamma\right|\Delta t = (S-\gamma)(b-a). \tag{3.29}$$

From (3.26)-(3.29), it follows that (3.22) holds.

In case of $C = \Gamma$ in (3.25), we can get (3.22) similarly. □

Corollary 3.23. *Theorem 3.22 is reduced in the case $\mathbb{T} = \mathbb{R}$ to*

$$\left|f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)dt + \frac{\beta}{b-x}\int_x^b f(t)dt\right] - \frac{\gamma}{2(\alpha+\beta)}\left[\alpha(x-a)-\beta(b-x)\right]\right| \le \left(\frac{1}{2} + \frac{|\alpha-\beta|}{2(\alpha+\beta)}\right)(S-\gamma)(b-a)$$

and

$$\left|f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)dt + \frac{\beta}{b-x}\int_x^b f(t)dt\right] - \frac{\Gamma}{2(\alpha+\beta)}\left[\alpha(x-a)-\beta(b-x)\right]\right| \le \left(\frac{1}{2} + \frac{|\alpha-\beta|}{2(\alpha+\beta)}\right)(\Gamma-S)(b-a).$$

Corollary 3.24. *Theorem 3.22 is reduced in the case $\mathbb{T} = \mathbb{Z}$ to*

$$\left|f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1} f(t+1) + \frac{\beta}{b-x}\sum_{t=x}^{b-1} f(t+1)\right] - \frac{\gamma}{2(\alpha+\beta)}\left[\alpha(x-a-1)-\beta(b-x+1)\right]\right| \le \left(\frac{1}{2} + \frac{|\alpha-\beta|}{2(\alpha+\beta)}\right)(S-\gamma)(b-a)$$

and

$$\left|f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\sum_{t=a}^{x-1} f(t+1) + \frac{\beta}{b-x}\sum_{t=x}^{b-1} f(t+1)\right] - \frac{\Gamma}{2(\alpha+\beta)}\left[\alpha(x-a-1)-\beta(b-x+1)\right]\right| \le \left(\frac{1}{2} + \frac{|\alpha-\beta|}{2(\alpha+\beta)}\right)(\Gamma-S)(b-a).$$

Corollary 3.25. *Theorem 3.22 is reduced in the case* $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ $(q > 1)$ *to*

$$\left|f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(qt)\mathrm{d}_q t + \frac{\beta}{b-x}\int_x^b f(qt)\mathrm{d}_q t\right] - \frac{\gamma}{(\alpha+\beta)(1+q)}\left[\alpha(x-qa) - \beta(qb-x)\right]\right|$$
$$\leq \left(\frac{1}{2} + \frac{|\alpha-\beta|}{2(\alpha+\beta)}\right)(S-\gamma)(b-a)$$

and

$$\left|f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(qt)\mathrm{d}_q t + \frac{\beta}{b-x}\int_x^b f(qt)\mathrm{d}_q t\right] - \frac{\Gamma}{(\alpha+\beta)(1+q)}\left[\alpha(x-qa) - \beta(qb-x)\right]\right|$$
$$\leq \left(\frac{1}{2} + \frac{|\alpha-\beta|}{2(\alpha+\beta)}\right)(\Gamma-S)(b-a).$$

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