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Some inequalities of Hermite-Hadamard type for n–times differentiable (ρ, m)–geometrically convex functions

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Abstract

In this paper, some generalized Hermite–Hadamard type inequalities for n–times differentiable (ρ, m)–geometrically convex function are established. The new inequalities recapture and give new estimates of the previous inequalities for first differentiable functions as special cases. The estimates for trapezoid, mid-point, averaged mid-point trapezoid and Simpson's inequalities can also be obtained for higher differentiable generalized geometrically convex functions.

Keywords: Hermite–Hadamard inequality, (ρ, m)–geometrically convex functions, n–times differentiable function.
2010 MSC: 26D15, 26A15.

1. Introduction

Since the establishment of theory of convex functions in the last century by a Danish mathematician, Jensen (1859-1925), the research on convex functions has gained much attention. However, the geometrically convex functions only appeared in [10, 11, 12] but has now become an active domain of definition. Convex and geometrically convex functions are used in parallel as tools to prove inequalities. The notion of geometric convexity was introduced by Montel [6], analogous to the notion of a convex function in n variables.

Now, we restate some basic convexity domains and related results.

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Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex, if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \tag{1.1}$$

holds for all $x, y \in I$, and $t \in [0,1]$.

The following is well known Hermite–Hadamard inequality which holds for a convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ where $a, b \in I$ with $a < b$,

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

As soon as an inequality appears, an attempt is made to generalize it [5]. The most earlier attempts of refining Hermite–Hadamard inequality can be found in [2, 3, 4].

In 2004, Zhang [9] presented the following concept of geometrically convex functions.

Definition 1.2. Let $f(x)$ be a positive function on $[0, b]$. If

$$f(x^t y^{(1-t)}) \leq [f(x)]^t [f(y)]^{(1-t)}, \tag{1.3}$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that the function $f(x)$ is geometrically convex on $[0, b]$.

In 2013, Ozdamir and Yildiz [7], presented some Ostrowski type inequalities for geometrically convex functions involving Logarithmic mean.

Xi et al. [8] in 2012, introduced the concept of m–geometrically convex functions and presented Hermite–Hadamard type inequalities for the generalized m–geometrically convex functions.

Definition 1.3. Let $f(x)$ be a positive function on $[0, b]$ and $m \in (0, 1]$. If

$$f(x^t y^{m(1-t)}) \leq [f(x)]^t [f(y)]^{m(1-t)}, \tag{1.4}$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that the function $f(x)$ is m–geometrically convex on $[0, b]$.

Theorem 1.4. ([8]) *Let $I \supset [0, \infty)$ be an open interval and $f : I \to (0, \infty)$ is differentiable. If $f' \in L([a, b])$ and $|f'(x)|$ is decreasing and m-geometrically convex on $[min\{1, a\}, b]$ for $a, b \in [0, \infty)$, with $a < b$ and $b \geq 1$, and $m \in (0, 1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left|f'(b)\right|^m [G_1(\alpha, m, q)]^{\frac{1}{q}}, \tag{1.5}$$

is valid for $q \geq 1$, where

$$G_1(1, m, q) = \int_0^1 |1-2t| \left(\frac{\left|f'(a)\right|}{|f'(b)^m|}\right)^{qt} dt.$$

Theorem 1.5. ([8]) *Let $I \supset [0, \infty)$ be an open interval and $f : I \to (0, \infty)$ is differentiable. If $f' \in L([a, b])$, $|f'(x)|$ is decreasing and m–geometrically convex on $[min\{1, a\}, b]$ for $a \in [0, \infty)$, $b \geq 1$, and $m \in (0, 1]$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{3}{q}} \left|f^{(n)}(b)\right|^m [G_2(1, m, q)]^{\frac{1}{q}},$$

is valid for $q \geq 1$*, where*

$$G_2(1,m,q) = \left[\int_0^{\frac{1}{2}} t\left(\frac{\left|f'(a)\right|}{\left|f'(b)^m\right|}\right)^{qt} dt\right]^{\frac{1}{q}} + \left[\int_{\frac{1}{2}}^{1} (1-t)\left(\frac{\left|f'(a)\right|}{\left|f'(b)^m\right|}\right)^{qt} dt\right]^{\frac{1}{q}}.$$

Recently, some Ostrowski type inequalities for m-geometrically convex functions are also established by Ozdamir and Yildiz [7].

Xi et al. [8] in 2012, introduced the concept of (α, m)–geometrically convex functions and established the generalized inequalities of this domain.

Definition 1.6. Let $f(x)$ be a positive function on $[0,b]$ and $\alpha, m \in (0,1] \times (0,1]$. If

$$f(x^t y^{m(1-t)}) \leq [f(x)]^{t^\alpha} [f(y)]^{m(1-t^\alpha)}, \tag{1.6}$$

holds for all $x, y \in [0,b]$, and $t \in [0,1]$, then we say that the function $f(x)$ is (α, m)–geometrically convex on $[0,b]$.

Remark 1.7. If $\alpha = m = 1$ in (1.6), then (α, m)–geometrically convex functions become geometrically convex functions.

Lemma 1.8. *If* $f(x)$ *is geometrically convex, and*

$$g(x) = ln\ f(e^x), \tag{1.7}$$

then g *is a convex function.*

Theorem 1.9. ([8]) *Let* $I \supset [0,\infty)$ *be an open interval and* $f : I \to (0,\infty)$ *is differentiable. If* $f' \in L([a,b])$ *and* $|f'(x)|$ *is decreasing and* (α,m)*-geometrically convex on* $[min\{1,a\},b]$ *for* $a \in [0,\infty)$*,* $b \geq 1$*, and* $(\alpha,m) \in (0,1] \times (0,1]$*, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left|f^{(n)}(b)\right|^m \left[G_1(\alpha,m,q)\right]^{\frac{1}{q}}, \tag{1.8}$$

is valid for $q \geq 1$*, where*

$$G_1(\alpha,m,q) = \int_0^1 |1-2t| \left(\frac{\left|f'(a)\right|}{\left|f'(b)^m\right|}\right)^{qt^\alpha} dt.$$

Theorem 1.10. ([8]) *Let* $I \supset [0,\infty)$ *be an open interval and* $f : I \to (0,\infty)$ *is differentiable. If* $f' \in L([a,b])$ *and* $|f'(x)|$ *is decreasing and* (α,m)*–geometrically convex on* $[min\{1,a\},b]$ *for* $a \in [0,\infty)$*,* $b \geq 1$ *and* $(\alpha,m) \in (0,1] \times (0,1]$*, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{3}{q}} \left|f^{(n)}(b)\right|^m \left[G_2(\alpha,m,q)\right]^{\frac{1}{q}}, \tag{1.9}$$

is valid for $q \geq 1$*, where*

$$\begin{aligned} G_2(\alpha,m,q) &= \left[\int_0^{\frac{1}{2}} t\left(\frac{\left|f'(a)\right|}{\left|f'(b)^m\right|}\right)^{qt^\alpha} dt\right]^{\frac{1}{q}} \\ &+ \left[\int_{\frac{1}{2}}^{1} (1-t)\left(\frac{\left|f'(a)\right|}{\left|f'(b)^m\right|}\right)^{qt^\alpha} dt\right]^{\frac{1}{q}}. \end{aligned}$$

Ostrowski type inequality for (α, m)–geometrically convex functions obtained by Ozdamir and Yildiz in [7] is stated in the form of the following theorem.

Theorem 1.11. *Let $I \subset \mathbb{R}_0$ be an open interval and $f : I \to (0, \infty)$ is differentiable. If $f \in L([a, b])$ and $|f^{'}(x)|^q$ is decreasing and (α, m)-geometrically convex on $[a, , b]$ with $a < b$ and $(\alpha, m) \in (0, 1] \times (0, 1]$, then we have*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \\ &\leq \frac{1}{(p+1)^{\frac{1}{p}}} \left\{ \frac{(x-a)^2}{b-a} \left| f^{'}(a) \right|^m M(t, q, \alpha) + \frac{(b-x)^2}{b-a} \left| f^{'}(b) \right|^m N(t, q, \alpha) \right\}, \end{aligned} \tag{1.10}$$

where

$$M(t, q, \alpha) = \left(\int_0^1 t \left(\frac{\left| f^{'}(x) \right|^q}{\left| f^{''}(a) \right|^q} \right)^{t^\alpha} dt \right)^{\frac{1}{q}}$$

and

$$N(t, q, \alpha) = \left(\int_0^1 t \left(\frac{\left| f^{'}(x) \right|^q}{\left| f^{''}(b) \right|^q} \right)^{t^\alpha} dt \right)^{\frac{1}{q}}.$$

Lemma 1.12. ([8]) *For $x, y \in [0, \infty)$ and $m, t \in (0, 1]$, if $x < y$ and $y \geq 1$, then*

$$x^t y^{m(1-t)} \leq tx + (1-t)y.$$

In this paper, we give some generalized inequalities for (ρ, m)–geometrically convex n-times differentiable function. The special case for first differentiable (ρ, m)–geometrically convex functions is a three point inequality of Hermite–Hadamard type which is capable of recapturing the previous results of this domain as well as some new inequalities can be obtained as a natural consequence.

2. Main Results

Lemma 2.1. *Let f be a real valued n-times differentiable mapping defined on $[a, b]$ such that $f^{(n)}(x)$ be absolutely continuous on $[a, b]$ with $\alpha : [a, b] \to [a, b]$ and $\beta : [a, b] \to [a, b]$, $\alpha(x) \leq x \leq \beta(x)$, then for all $x \in [a, b]$, the following identity holds*

$$(b-a)^{n+1} \int_0^1 k_n(x, t) f^{(n)}(ta + (1-t)b)dt = \int_a^b f(u)du - \sum_{k=1}^n \frac{1}{k!} \left[R_k(x) f^{(k-1)}(x) + S_k(x) \right] \tag{2.1}$$

where the kernel $k_n : [a, b] \times [0, 1] \to \mathbb{R}$ is given by

$$k_n(x, t) = \begin{cases} \frac{(t - \frac{b-\beta}{b-a})^n}{n!}, t \in [0, \frac{b-x}{b-a}] \\ \frac{(t - \frac{b-\alpha}{b-a})^n}{n!}, t \in (\frac{b-x}{b-a}, 1] \end{cases} \tag{2.2}$$

$$\begin{aligned} R_k(x) &= (\beta(x) - x)^k + (-1)^{k-1} (x - \alpha(x))^k, \\ S_k(x) &= (\alpha(x) - a)^k f^{(k-1)}(a) + (-1)^{k-1} (b - \beta(x))^k f^{(k-1)}(b). \end{aligned} \tag{2.3}$$

Proof. Under the given conditions, the following identity was proved by Cerone and Dragomir [1]

$$(-1)^n \int_a^b K_n(x,u) f^{(n)}(u)\,du = \int_a^b f(u)\,du - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x) f^{(k-1)}(x) + S_k(x)\right], \tag{2.4}$$

where the kernel $K_n : [a,b]^2 \to \mathbb{R}$ is given by

$$K_n(x,u) = \begin{cases} \frac{(u-\alpha(x))^n}{n!}, & u \in [a,x] \\ \frac{(u-\beta(x))^n}{n!}, & u \in (x,b] \end{cases}$$

Moreover, $R_k(x)$ and $S_k(x)$ are defined by (2.3).

Now, by considering the L. H. S. of the identity (2.4),

$$\begin{aligned} (-1)^n \int_a^b K_n(x,u) f^{(n)}(u)\,du &= \int_a^x \frac{(\alpha(x)-u)^n}{n!} f^{(n)}(u)\,du + \int_x^b \frac{(\beta(x)-u)^n}{n!} f^{(n)}(u)\,du \\ &= I_a + I_b \end{aligned} \tag{2.5}$$

Let $u = ta + (1-t)b$ in (2.5), then

$$I_a = (b-a)^{n+1} \int_{\frac{b-x}{b-a}}^{1} \frac{(t - \frac{b-\alpha(x)}{b-a})^n}{n!} f^{(n)}(ta + (1-t)b)\,dt,$$

and

$$I_b = (b-a)^{n+1} \int_0^{\frac{b-x}{b-a}} \frac{(t - \frac{b-\beta(x)}{b-a})^n}{n!} f^{(n)}(ta + (1-t)b)\,dt.$$

Thus,

$$(b-a)^{n+1} \int_0^1 k_n(x,t) f^{(n)}(ta + (1-t)b)\,dt = \int_a^b f(u)\,du - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x) f^{(k-1)}(x) + S_k(x)\right],$$

where

$$k_n(x,t) = \begin{cases} \frac{(t-\frac{b-\beta}{b-a})^n}{n!}, t \in [0, \frac{b-x}{b-a}] \\ \frac{(t-\frac{b-\alpha}{b-a})^n}{n!}, t \in (\frac{b-x}{b-a}, 1] \end{cases}$$

Hence, proved. □

The generalized Hermite–Hadamard type inequality for (ρ, m)–geometrically convex n–differentiable function is stated as follows.

Theorem 2.2. *Let $I \supset [0,\infty)$ be an open interval and $f : I \to (0,\infty)$ is n-differentiable. Let $|f^{(n)}(x)| \in L([a,b])$ is decreasing and (ρ,m)-geometrically convex on $[min\{1,a\}, b]$ for $a \in [0,\infty), b \geq 1$, $x \in [a,b]$ and $\rho, m \in (0,1] \times (0,1]$, then*

$$\begin{aligned} &\left| \int_a^b f(u)du - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x) f^{(k-1)}(x) + S_k(x)\right] \right| \\ \leq\ & \frac{(b-a)^{\frac{n+1}{q}} (n+1)^{\frac{1}{q}}}{(n+1)!} \left| f^{(n)}(b) \right|^m E_n(\rho, m, q), \end{aligned} \tag{2.6}$$

for $q \geqslant 1$,

$$E_n(\rho,m,q)=\begin{cases}\frac{1}{((n+1)(b-a)^{n+1})^{\frac{1}{q}}}\left[(b-\beta)^{n+1}+(\beta-x)^{n+1}+(x-\alpha)^{n+1}+(\alpha-a)^{n+1}\right], \\ \qquad \text{for } \mu=1 \\ \left((b-\beta)^{n+1}+(\beta-x)^{n+1}+(x-\alpha)^{n+1}+(\alpha-a)^{n+1}\right)^{1-\frac{1}{q}} \\ \left(\frac{\mu^{\frac{\rho q(b-\beta)}{b-a}}}{(\rho q\ln u)^{n+1}}\left(n!(-1)^{n+1}+\gamma(n+1,\ln\mu^{\frac{\rho q(b-\beta)}{b-a}})\right)\right. \\ +\mu^{\frac{\rho q(b-x)}{b-a}}\sum_{k=1}^{n+1}\left(\frac{(-1)^{k-1}(\frac{\beta-x}{b-a})^{n-k+1}+(-1)^{2k+1}(\frac{x-\alpha}{b-a})^{n-k+1}}{(n-k+1)!(\rho q\ln\mu)^k}\right) \\ \left.+\frac{\mu^{\frac{\rho q(b-\alpha)}{b-a}}}{(\rho q\ln\mu)^{n+1}}\left(n!+(-1)^{n+1}\gamma(n+1,\ln\mu^{-\frac{\rho q(\alpha-a)}{b-a}})\right)\right)^{\frac{1}{q}}, \\ \qquad \text{for } 0<\mu<1 \\ \left((b-\beta)^{n+1}+(\beta-x)^{n+1}+(x-\alpha)^{n+1}+(\alpha-a)^{n+1}\right)^{1-\frac{1}{q}} \\ \left(\frac{\rho^{n+1}\mu^{\frac{q(b-\beta)}{\rho(b-a)}}}{(q\ln\mu)^{n+1}}\left((-1)^{n+1}n!+\gamma(n+1,\ln\mu^{\frac{q(b-\beta)}{\rho(b-a)}})\right)\right. \\ +\mu^{\frac{q(b-x)}{\rho(b-a)}}\sum_{k=1}^{n+1}\left(\frac{(-1)^{k-1}\rho^k(\frac{\beta-x}{b-a})^{n-k+1}+(-1)^{2k+1}\rho^k(\frac{x-\alpha}{b-a})^{n-k+1}}{(n-k+1)!(q\ln\mu)^k}\right) \\ \left.+\frac{\rho^{n+1}\mu^{\frac{q(b-\alpha)}{\rho(b-a)}}}{(q\ln\mu)^{n+1}}\left(n!+(-1)^{n+1}\gamma(n+1,\ln\mu^{-\frac{q(\alpha-a)}{\rho(b-a)}})\right)\right)^{\frac{1}{q}}, \\ \qquad \text{for } \mu>1\end{cases} \tag{2.7}$$

for $\mu,\rho>0$, *where*

$$\mu=\frac{f^{(n)}(a)}{\left(f^{(n)}(b)\right)^m},$$

and $\gamma(a,x)$ *is the lower incomplete gamma function defined as*

$$\gamma(a,x)=\int_0^x t^{a-1}e^{-t}dt,$$

$$R_k(x)=(\beta(x)-x)^k+(-1)^{k-1}(x-\alpha(x))^k,$$
$$S_k(x)=(\alpha(x)-a)^k f^{(k-1)}(a)+(-1)^{k-1}(b-\beta(x)^k f^{(k-1)}(b). \tag{2.8}$$

Proof. Applying the definition of kernel, properties of modulus and Hölder's inequality on (2.1), we get

$$\left|\int_a^b f(u)\,du-\sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)f^{(k-1)}(x)+S_k(x)\right]\right|$$

$$\leq (b-a)^{n+1}\left(\int_0^1|k_n(x,t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|k_n(x,t)|\left|f^{(n)}(ta+(1-t)b)\right|^q\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^{n+1}}{n!}\left(\int_0^{\frac{b-\beta}{b-a}}(\frac{b-\beta}{b-a}-t)^n dt+\int_{\frac{b-\beta}{b-a}}^{\frac{b-x}{b-a}}(t-\frac{b-\beta}{b-a})^n dt\right.$$

$$
\left.+\int_{\frac{b-x}{b-a}}^{\frac{b-\alpha}{b-a}}(\frac{b-\alpha}{b-a}-t)^n dt+\int_{\frac{b-\alpha}{b-a}}^{1}(t-\frac{b-\alpha}{b-a})^n dt\right)^{1-\frac{1}{q}}
$$

$$
\times\left(\int_{0}^{\frac{b-\beta}{b-a}}\frac{(\frac{b-\beta}{b-a}-t)^n}{n!}\left|f^{(n)}(ta+(1-t)b)\right|^q dt+\int_{\frac{b-\beta}{b-a}}^{\frac{b-x}{b-a}}\frac{(t-\frac{b-\beta}{b-a})^n}{n!}\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right.
$$

$$
\left.+\int_{\frac{b-x}{b-a}}^{\frac{b-\alpha}{b-a}}\frac{(\frac{b-\alpha}{b-a}-t)^n}{n!}\left|f^{(n)}(ta+(1-t)b)\right|^q dt+\int_{\frac{b-\alpha}{b-a}}^{1}\frac{(t-\frac{b-\alpha}{b-a})^n}{n!}\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \tag{2.9}
$$

Applying the definition of (α, m)–geometric convexity on (2.9), we obtain

$$
\left|\int_a^b f(u)\,du-\sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)\,f^{(k-1)}(x)+S_k(x)\right]\right|
$$

$$
\leq \frac{(b-a)^{n+1}}{n!}\left(\int_{0}^{\frac{b-\beta}{b-a}}(\frac{b-\beta}{b-a}-t)^n dt+\int_{\frac{b-\beta}{b-a}}^{\frac{b-x}{b-a}}(t-\frac{b-\beta}{b-a})^n dt+\int_{\frac{b-x}{b-a}}^{\frac{b-\alpha}{b-a}}(\frac{b-\alpha}{b-a}-t)^n dt\right.
$$

$$
\left.+\int_{\frac{b-\alpha}{b-a}}^{1}(t-\frac{b-\alpha}{b-a})^n dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{b-\beta}{b-a}}(\frac{b-\beta}{b-a}-t)^n\left|f^{(n)}\left(a^t b^{m(1-t)}\right)\right|^q dt\right.
$$

$$
+\int_{\frac{b-\beta}{b-a}}^{\frac{b-x}{b-a}}(t-\frac{b-\beta}{b-a})^n\left|f^{(n)}\left(a^t b^{m(1-t)}\right)\right|^q dt+\int_{\frac{b-x}{b-a}}^{\frac{b-\alpha}{b-a}}(\frac{b-\alpha}{b-a}-t)^n\left|f^{(n)}\left(a^t b^{m(1-t)}\right)\right|^q dt
$$

$$
\left.+\int_{\frac{b-\alpha}{b-a}}^{1}(t-\frac{b-\alpha}{b-a})^n\left|f^{(n)}\left(a^t b^{m(1-t)}\right)\right|^q dt\right)^{\frac{1}{q}}.
$$

Therefore, upon simplification

$$
\left|\int_a^b f(u)\,du-\sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)\,f^{(k-1)}(x)+S_k(x)\right]\right|
$$

$$
\leq \frac{(b-a)^{n+1}}{n!}\left|f^{(n)}(b)\right|^m\left(\int_{0}^{\frac{b-\beta}{b-a}}(\frac{b-\beta}{b-a}-t)^n dt+\int_{\frac{b-\beta}{b-a}}^{\frac{b-x}{b-a}}(t-\frac{b-\beta}{b-a})^n dt+\int_{\frac{b-x}{b-a}}^{\frac{b-\alpha}{b-a}}(\frac{b-\alpha}{b-a}-t)^n dt\right.
$$

$$+\int_{\frac{b-\alpha}{b-a}}^{1}(t-\frac{b-\alpha}{b-a})^n dt\Bigg)^{1-\frac{1}{q}}\Bigg(\int_0^{\frac{b-\beta}{b-a}}(\frac{b-\beta}{b-a}-t)^n\left|\frac{f^{(n)}(a)}{f^{(n)}(b)^m}\right|^{qt^\rho}dt$$

$$+\int_{\frac{b-\beta}{b-a}}^{\frac{b-x}{b-a}}(t-\frac{b-\beta}{b-a})^n\left|\frac{f^{(n)}(a)}{f^{(n)}(b)^m}\right|^{qt^\rho}dt+\int_{\frac{b-x}{b-a}}^{\frac{b-\alpha}{b-a}}(\frac{b-\alpha}{b-a}-t)^n\left|\frac{f^{(n)}(a)}{f^{(n)}(b)^m}\right|^{qt^\rho}dt$$

$$+\int_{\frac{b-\alpha}{b-a}}^{1}(t-\frac{b-\alpha}{b-a})^n\left|\frac{f^{(n)}(a)}{f^{(n)}(b)^m}\right|^{qt^\rho}dt\Bigg)^{\frac{1}{q}}. \tag{2.10}$$

Let $\mu=\frac{f^{(n)}(a)}{\left(f^{(n)}(b)\right)^m}$ in (2.10), we have three cases:

Case 1: For $\mu=1$, (2.10) takes the form

$$\left|\int_a^b f(u)\,du-\sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)\,f^{(k-1)}(x)+S_k(x)\right]\right|$$

$$\leq \frac{(b-a)^{n+1}}{n!}\left|f^{(n)}(b)\right|^m\Bigg(\int_0^{\frac{b-\beta}{b-a}}(\frac{b-\beta}{b-a}-t)^n dt+\int_{\frac{b-\beta}{b-a}}^{\frac{b-x}{b-a}}(t-\frac{b-\beta}{b-a})^n dt+$$

$$\int_{\frac{b-x}{b-a}}^{\frac{b-\alpha}{b-a}}(\frac{b-\alpha}{b-a}-t)^n dt+\int_{\frac{b-\alpha}{b-a}}^{1}(t-\frac{b-\alpha}{b-a})^n dt\Bigg).$$

Thus, we have

$$\left|\int_a^b f(u)\,du-\sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)\,f^{(k-1)}(x)+S_k(x)\right]\right|$$

$$\leq \frac{(b-a)^{n+1}}{n!}\left|f^{(n)}(b)\right|^m\left[\frac{(\frac{b-\beta}{b-a})^{n+1}}{(n+1)}+\frac{(\frac{\beta-x}{b-a})^{n+1}}{(n+1)}+\frac{(\frac{x-\alpha}{b-a})^{n+1}}{(n+1)}+\frac{(\frac{\alpha-a}{b-a})^{n+1}}{(n+1)}\right],$$

$$\leq \frac{\left|f^{(n)}(b)\right|^m}{(n+1)!}\left[(b-\beta)^{n+1}+(\beta-x)^{n+1}+(x-\alpha)^{n+1}+(\alpha-a)^{n+1}\right]. \tag{2.11}$$

Case 2: For $\mu<1,\ 0<t,\rho\leq 1$, we have $\mu^{qt^\rho}\leq\mu^{\rho qt}$. Thus, (2.10) becomes

$$\left|\int_a^b f(u)\,du-\sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)\,f^{(k-1)}(x)+S_k(x)\right]\right|$$

$$\leq \frac{(b-a)^{\frac{n+1}{q}}(n+1)^{\frac{1}{q}}}{(n+1)!}\left|f^{(n)}(b)\right|^{m}\left((b-\beta)^{n+1}+(\beta-x)^{n+1}+(x-\alpha)^{n+1}\right.$$
$$\left.+(\alpha-a)^{n+1}\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{b-\beta}{b-a}}(\frac{b-\beta}{b-a}-t)^{n}\mu^{\rho qt}dt+\int_{\frac{b-\beta}{b-a}}^{\frac{b-x}{b-a}}(t-\frac{b-\beta}{b-a})^{n}\mu^{\rho qt}dt\right.$$
$$\left.+\int_{\frac{b-x}{b-a}}^{\frac{b-\alpha}{b-a}}(\frac{b-\alpha}{b-a}-t)^{n}\mu^{\rho qt}dt+\int_{\frac{b-\alpha}{b-a}}^{1}(t-\frac{b-\alpha}{b-a})^{n}\mu^{\rho qt}dt\right)^{\frac{1}{q}}. \tag{2.12}$$

Let,

$$I_1=\int_{0}^{\frac{b-\beta}{b-a}}(\frac{b-\beta}{b-a}-t)^{n}\mu^{\rho qt}dt,$$

$$I_2=\int_{\frac{b-\beta}{b-a}}^{\frac{b-x}{b-a}}(t-\frac{b-\beta}{b-a})^{n}\mu^{\rho qt}dt,$$

$$I_3=\int_{\frac{b-x}{b-a}}^{\frac{b-\alpha}{b-a}}(\frac{b-\alpha}{b-a}-t)^{n}\mu^{\rho qt}dt,$$

and

$$I_4=\int_{\frac{b-\alpha}{b-a}}^{1}(t-\frac{b-\alpha}{b-a})^{n}\mu^{\rho qt}dt.$$

Then, upon simplification, we have

$$I_1=\frac{\mu^{\rho q(\frac{b-\beta}{b-a})}}{(\rho q\ln u)^{n+1}}\gamma(n+1,\ln u^{\rho q(\frac{b-\beta}{b-a})}),$$

$$I_2=\mu^{\rho q(\frac{b-x}{b-a})}\sum_{k=1}^{n+1}\frac{(-1)^{k-1}(\frac{\beta-x}{b-a})^{n-k+1}+}{(n-k+1)!\,(\rho q\ln\mu)^{k}}+\frac{(-1)^{n+1}n!\mu^{\rho q(\frac{b-\beta}{b-a})}}{(\rho q\ln\mu)^{n+1}},$$

$$I_3=\mu^{\rho q(\frac{b-x}{b-a})}\sum_{k=1}^{n+1}\frac{(-1)^{2k+1}(\frac{x-\alpha}{b-a})^{n-k+1}}{(n-k+1)!\,(\rho q\ln\mu)^{k}}+\frac{n!\mu^{\rho q(\frac{b-\alpha}{b-a})}}{(\rho q\ln\mu)^{n+1}},$$

and

$$I_4=\frac{(-1)^{n+1}\mu^{\rho q(\frac{b-\alpha}{b-a})}}{(\rho q\ln\mu)^{n+1}}\gamma(n+1,\ln\mu^{-\rho q(\frac{\alpha-a}{b-a})}).$$

Re–substituting the values of the integrals in (2.12), we have

$$\left|\int_{a}^{b}f(u)\,du-\sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)\,f^{(k-1)}(x)+S_k(x)\right]\right|$$

$$\leq \frac{(b-a)^{\frac{n+1}{q}}(n+1)^{\frac{1}{q}}}{(n+1)!}\left|f^{(n)}(b)\right|^{m}\Big((b-\beta)^{n+1}+(\beta-x)^{n+1}+(x-\alpha)^{n+1}$$

$$+(\alpha-a)^{n+1}\Big)^{1-\frac{1}{q}}\left(\frac{\mu^{\rho q(\frac{b-\beta}{b-a})}}{(\rho q\ln u)^{n+1}}\left((-1)^{n+1}n!+\gamma(n+1,\ln\mu^{\rho q(\frac{b-\beta}{b-a})})\right)\right.$$

$$+\mu^{\rho q(\frac{b-x}{b-a})}\sum_{k=1}^{n+1}\left(\frac{(-1)^{k-1}(\frac{\beta-x}{b-a})^{n-k+1}+(-1)^{2k+1}(\frac{x-\alpha}{b-a})^{n-k+1}}{(n-k+1)!\,(\rho q\ln\mu)^{k}}\right)$$

$$\left.+\frac{\mu^{\rho q(\frac{b-\alpha}{b-a})}}{(\rho q\ln\mu)^{n+1}}\left(n!+(-1)^{n+1}\gamma(n+1,\ln\mu^{-\rho q(\frac{\alpha-a}{b-a})})\right)\right)^{\frac{1}{q}}, \text{for } 0<\mu<1. \tag{2.13}$$

Case 3: For $\mu>1$, $0<t,\rho\leq 1$, we have $\mu^{qt^{\rho}}\leq\mu^{\frac{qt}{\rho}}$. Thus, (2.10) becomes

$$\left|\int_a^b f(u)\,du-\sum_{k=1}^{n}\frac{1}{k!}\left[R_k(x)\,f^{(k-1)}(x)+S_k(x)\right]\right|$$

$$\leq \frac{(b-a)^{\frac{n+1}{q}}(n+1)^{\frac{1}{q}}}{(n+1)!}\left|f^{(n)}(b)\right|^{m}\Big((b-\beta)^{n+1}+(\beta-x)^{n+1}+(x-\alpha)^{n+1}$$

$$+(\alpha-a)^{n+1}\Big)^{1-\frac{1}{q}}\left(\int_0^{\frac{b-\beta}{b-a}}(\frac{b-\beta}{b-a}-t)^{n}\mu^{\frac{qt}{\rho}}\,dt+\int_{\frac{b-\beta}{b-a}}^{\frac{b-x}{b-a}}(t-\frac{b-\beta}{b-a})^{n}\mu^{\frac{qt}{\rho}}\,dt\right.$$

$$\left.+\int_{\frac{b-x}{b-a}}^{\frac{b-\alpha}{b-a}}(\frac{b-\alpha}{b-a}-t)^{n}\mu^{\frac{qt}{\rho}}\,dt+\int_{\frac{b-\alpha}{b-a}}^{1}(t-\frac{b-\alpha}{b-a})^{n}\mu^{\frac{qt}{\rho}}\,dt\right)^{\frac{1}{q}}. \tag{2.14}$$

Let,

$$I_1^{'}=\int_0^{\frac{b-\beta}{b-a}}(\frac{b-\beta}{b-a}-t)^{n}\mu^{\frac{qt}{\rho}}\,dt,$$

$$I_2^{'}=\int_{\frac{b-\beta}{b-a}}^{\frac{b-x}{b-a}}(t-\frac{b-\beta}{b-a})^{n}\mu^{\frac{qt}{\rho}}\,dt,$$

$$I_3^{'}=\int_{\frac{b-x}{b-a}}^{\frac{b-\alpha}{b-a}}(\frac{b-\alpha}{b-a}-t)^{n}\mu^{\frac{qt}{\rho}}\,dt,$$

and

$$I_4' = \int_{\frac{b-\alpha}{b-a}}^{1} (t - \frac{b-\alpha}{b-a})^n \mu^{\frac{qt}{\rho}} dt.$$

Then, upon simplification, we have

$$I_1' = \frac{\rho^{n+1}\mu^{\frac{q(b-\beta)}{\rho(b-a)}}}{(q\ln u)^{n+1}} \gamma(n+1, \ln\mu^{\frac{q(b-\beta)}{\rho(b-a)}}),$$

$$I_2' = \mu^{\frac{q(b-x)}{\rho(b-a)}} \sum_{k=1}^{n+1} \frac{\rho^k(-1)^{k-1}(\frac{\beta-x}{b-a})^{n-k+1}}{(n-k+1)!\,(q\ln\mu)^k} + \frac{\rho^{n+1}(-1)^{n+1}n!\mu^{\frac{q(b-\beta)}{\rho(b-a)}}}{(\rho q\ln\mu)^{n+1}},$$

$$I_3' = \mu^{\frac{q(b-x)}{\rho(b-a)}} \sum_{k=1}^{n+1} \frac{\rho^k(-1)^{2k+1}(\frac{x-\alpha}{b-a})^{n-k+1}}{(n-k+1)!\,(q\ln\mu)^k} + \frac{\rho^{n+1}n!\mu^{\frac{q(b-\alpha)}{\rho(b-a)}}}{(\rho q\ln\mu)^{n+1}},$$

and

$$I_4' = \frac{\rho^{n+1}(-1)^{n+1}\mu^{\frac{q(b-\alpha)}{\rho(b-a)}}}{(q\ln\mu)^{n+1}} \gamma(n+1, \ln\mu^{-\frac{q}{\rho}(\frac{\alpha-a}{b-a})}).$$

Re–substituting the values of the integrals in (2.14), we have

$$\begin{aligned}
&\left| \int_a^b f(u)\,du - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x) f^{(k-1)}(x) + S_k(x)\right] \right| \\
&\leq \frac{(b-a)^{\frac{n+1}{q}}(n+1)^{\frac{1}{q}}}{(n+1)!} \left| f^{(n)}(\frac{b}{m}) \right|^m \Big((b-\beta)^{n+1} + (\beta-x)^{n+1} + (x-\alpha)^{n+1} \\
&+ (\alpha-a)^{n+1}\Big)^{1-\frac{1}{q}} \Bigg(\frac{\rho^{n+1}\mu^{\frac{q(b-\beta)}{\rho(b-a)}}}{(q\ln\mu)^{n+1}} \left((-1)^{n+1}n! + \gamma(n+1, \ln\mu^{\frac{q(b-\beta)}{\rho(b-a)}}) \right) \\
&+ \mu^{\frac{q(b-x)}{\rho(b-a)}} \sum_{k=1}^{n+1} \rho^k \left(\frac{(-1)^{k-1}(\frac{\beta-x}{b-a})^{n-k+1} + (-1)^{2k+1}(\frac{x-\alpha}{b-a})^{n-k+1}}{(n-k+1)!\,(q\ln\mu)^k} \right) \\
&+ \frac{\rho^{n+1}\mu^{\frac{q(b-\alpha)}{\rho(b-a)}}}{(q\ln\mu)^{n+1}} \left(n! + (-1)^{n+1}\gamma(n+1, \ln\mu^{-\frac{q(\alpha-a)}{\rho(b-a)}}) \right) \Bigg)^{\frac{1}{q}}, \mathit{for}\ \mu > 1.
\end{aligned} \tag{2.15}$$

Therefore, (2.11), (2.13) and (2.15) are required inequalities. □

Remark 2.3. If we take

$$\alpha(x) = a,$$

$$\beta(x) = b,$$

$x = \frac{a+b}{2}$ and $n = 1$ in (2.10), then the inequalities for the m and (α, m)–geometrically convex functions by Xi et al.[8] are recaptured.

The generalized inequality for m–geometrically convex n–differentiable function is stated as follows.

Corollary 2.4. *Let f, α, β, μ be defined as in Theorem 2.2. If $\left|f^{(n)}\right|$ is m–geometrically convex on $[min\{1, a\}, b]$, for $m \in (0,1)$, then for all $x \in [a,b]$,*

$$\left|\int_a^b f(u)du - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x) f^{(k-1)}(x) + S_k(x)\right]\right|$$

$$\leq \frac{(b-a)^{\frac{n+1}{q}}(n+1)^{\frac{1}{q}}}{(n+1)!}\left|f^{(n)}(b)\right|^m E_n(\rho, m, q), \tag{2.16}$$

for $q \geqslant 1$,

$$E_n(1,m,q) = \begin{cases} \frac{1}{((n+1)(b-a)^{n+1})^{\frac{1}{q}}}\left[(b-\beta)^{n+1} + (\beta-x)^{n+1} + (x-\alpha)^{n+1} + (\alpha-a)^{n+1}\right], \\ \qquad \textit{for } \mu = 1 \\ \left((b-\beta)^{n+1} + (\beta-x)^{n+1} + (x-\alpha)^{n+1} + (\alpha-a)^{n+1}\right)^{1-\frac{1}{q}} \\ \times\Bigg(\frac{\mu^{\frac{q(b-\beta)}{b-a}}}{(q\ln u)^{n+1}}\left((-1)^{n+1}n! + \gamma(n+1, \ln\mu^{\frac{q(b-\beta)}{b-a}})\right) \\ +\mu^{\frac{q(b-x)}{b-a}}\sum_{k=1}^{n+1}\left(\frac{(-1)^{k-1}(\frac{\beta-x}{b-a})^{n-k+1} + (-1)^{2k+1}(\frac{x-\alpha}{b-a})^{n-k+1}}{(n-k+1)!(q\ln\mu)^k}\right) \\ +\frac{\mu^{\frac{q(b-\alpha)}{b-a}}}{(q\ln\mu)^{n+1}}\left(n! + (-1)^{n+1}\gamma(n+1, \ln\mu^{-\frac{q(\alpha-a)}{b-a}})\right)\Bigg)^{\frac{1}{q}}, \\ \qquad \textit{for } 0 < \mu < 1 \\ \left((b-\beta)^{n+1} + (\beta-x)^{n+1} + (x-\alpha)^{n+1} + (\alpha-a)^{n+1}\right)^{1-\frac{1}{q}} \\ \times\Bigg(\frac{\mu^{\frac{q(b-\beta)}{b-a}}}{(q\ln\mu)^{n+1}}\left(n!(-1)^{n+1} + \gamma(n+1, \ln\mu^{\frac{q(b-\beta)}{b-a}})\right) \\ +\mu^{\frac{q(b-x)}{b-a}}\sum_{k=1}^{n+1}\left(\frac{(-1)^{k-1}(\frac{\beta-x}{b-a})^{n-k+1} + (-1)^{2k+1}(\frac{x-\alpha}{b-a})^{n-k+1}}{(n-k+1)!(q\ln\mu)^k}\right) \\ +\frac{\mu^{\frac{q(b-\alpha)}{b-a}}}{(q\ln\mu)^{n+1}}\left(n! + (-1)^{n+1}\gamma(n+1, \ln\mu^{-\frac{q(\alpha-a)}{b-a}})\right)\Bigg)^{\frac{1}{q}}, \\ \qquad \textit{for } \mu > 1. \end{cases} \tag{2.17}$$

where

$$\gamma(a,x)$$

is the lower incomplete gamma function. Moreover,

$$R_k(x)$$

and

$$S_k(x)$$

are defined by (2.8).

Proof. Substituting $\rho = 1$, in (2.6)–(2.8), we have the required inequality. □

The generalized three point Hermite–Hadamard type inequality for (ρ, m)–geometrically convex n-differentiable function is stated as,

Theorem 2.5. *Let f, ρ, m, μ be defined as in Theorem 2.2, then the following inequality holds for any $r \in [0,1]$ and for all $x \in [a,b]$,*

$$\begin{aligned}&\left|\int_a^b f(u)\,du - \sum_{k=1}^{n}\frac{1}{k!}\left[\left((1-r)^k(b-x)^k + (-1)^{k-1}(x-a)^k\right)f^{(k-1)}(x)\right.\right.\\&\left.\left.+r^k\left((x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right)\right]\right|\\ \leq\ & \frac{(b-a)^{\frac{n+1}{q}}(n+1)^{\frac{1}{q}}}{(n+1)!}\left|f^{(n)}(b)\right|^m E_n(\rho,m,q),\end{aligned} \tag{2.18}$$

for $q \geqslant 1$, and

$$E_n(\rho,m,q) = \frac{\left(r^{n+1} + \left(1-r^{n+1}\right)\right)\left((b-x)^{n+1} + (x-a)^{n+1}\right)}{\left((n+1)(b-a)^{n+1}\right)^{\frac{1}{q}}},\ \textit{for } \mu = 1 \tag{2.19}$$

$$E_n(\rho,m,q) = \begin{cases} \left(\left(r^{n+1} + \left(1-r^{n+1}\right)\right)\left((b-x)^{n+1} + (x-a)^{n+1}\right)\right)^{1-\frac{1}{q}} \\ \times\left(\frac{\mu^{\frac{r\rho q(b-x)}{b-a}}}{(\rho q\ln\mu)^{n+1}}\left((-1)^{n+1}n! + \gamma(n+1,\ \ln\mu^{\frac{rq\rho(b-x)}{b-a}})\right)\right. \\ +\mu^{\frac{q\rho(b-x)}{b-a}}\sum_{k=1}^{n+1}\left(\frac{(-1)^{k-1}(\frac{(1-r)(b-x)}{b-a})^{n-k+1} + (-1)^{2k+1}(\frac{(1-r)(x-a)}{b-a})^{n-k+1}}{(n-k+1)!(\rho q\ln\mu)^k}\right) \\ \left. + \frac{\mu^{q\rho\left(1-\frac{r(x-a)}{(b-a)}\right)}}{(\rho q\ln\mu)^{n+1}}\left(n! + (-1)^{n+1}\gamma(n+1,\ \ln\mu^{-\frac{rq\rho(x-a)}{b-a}})\right)\right)^{\frac{1}{q}}, \\ \textit{for } 0<\mu<1 \end{cases} \tag{2.20}$$

$$E_n(\rho,m,q) = \begin{cases} \left(\left(r^{n+1} + \left(1-r^{n+1}\right)\right)\left((b-x)^{n+1} + (x-a)^{n+1}\right)\right)^{1-\frac{1}{q}} \\ \times\left(\frac{\rho^{n+1}\mu^{\frac{rq(b-x)}{\rho(b-a)}}}{(q\ln\mu)^{n+1}}\left((-1)^{n+1}n! + \gamma(n+1,\ \ln\mu^{\frac{rq(b-x)}{\rho(b-a)}})\right)\right. \\ +\mu^{\frac{q(b-x)}{\rho(b-a)}}\sum_{k=1}^{n+1}\rho^k\left(\frac{(-1)^{k-1}(\frac{(1-r)(b-x)}{b-a})^{n-k+1} + (-1)^{2k+1}(\frac{(1-r)(x-a)}{b-a})^{n-k+1}}{(n-k+1)!(q\ln\mu)^k}\right) \\ \left. + \frac{\rho^{n+1}\mu^{\frac{q}{\rho}\left(1-\frac{r(x-a)}{(b-a)}\right)}}{(q\ln\mu)^{n+1}}\left(n! + (-1)^{n+1}\gamma(n+1,\ \ln\mu^{-\frac{rq(x-a)}{\rho(b-a)}})\right)\right)^{\frac{1}{q}}, \\ \textit{for } \mu>1 \end{cases} \tag{2.21}$$

for $\rho > 0$.

Proof. Let $\alpha(x)$ and $\beta(x)$ be defined as follows in Theorem 2.2,

$$\alpha(x) = rx + (1-r)a,$$

and

$$\beta(x) = rx + (1-r)b.$$

Then, we have the required result. □

The generalized three point inequality for (ρ, m)–geometrically convex first differentiable function is stated as follows.

Corollary 2.6. *Let f, ρ, m, μ be defined as in Theorem 2.5, then the following inequality holds for any $r \in [0,1]$ and $x \in [a,b]$,*

$$\left| \frac{1}{b-a} \int_a^b f(u)\, du - (1-r) f(x) + \frac{r\left((x-a)f(a) + (b-x)f(b)\right)}{b-a} \right| \leq \frac{(b-a)^{\frac{2}{q}-1}}{2^{1-\frac{1}{q}}} \left| f'(b) \right|^m E_1(\rho, m, q), \tag{2.22}$$

for $q \geqslant 1$, where

$$E_1(\rho, m, q) = \frac{\left(r^2 + (1-r)^2\right)\left((b-x)^2 + (x-a)^2\right)}{\left(2(b-a)^2\right)^{\frac{1}{q}}}, \text{ for } \mu = 1. \tag{2.23}$$

$$E_1(\rho, m, q) = \begin{cases} \left(\left(r^2 + (1-r)^2\right)\left((b-x)^2 + (x-a)^2\right)\right)^{1-\frac{1}{q}} \\ \times \left(\frac{\mu^{\frac{r\rho q(b-x)}{b-a}}}{(\rho q \ln \mu)^2} \left(1 + \mu^{-\frac{r\rho q(b-x)}{b-a}} \left(\ln \mu^{-\frac{r\rho q(b-x)}{b-a}} - 1 + \mu^{\frac{r\rho q(b-x)}{b-a}}\right)\right) \right. \\ \quad + \mu^{\frac{\rho q(b-x)}{b-a}} \left(\frac{(1-r)(b-x)}{(b-a)\rho q \ln \mu} - \frac{(1-r)(x-a)}{(b-a)\rho q \ln \mu} - \frac{2}{(\rho q \ln \mu)^2} \right) \\ \left. + \frac{\mu^{\rho q\left(1-\frac{r(x-a)}{(b-a)}\right)}}{(\rho q \ln \mu)^2} \left(1 + \mu^{\frac{r\rho q(b-x)}{b-a}} \left(\ln \mu^{\frac{r\rho q(x-a)}{b-a}} - 1 + \mu^{-\frac{r\rho q(x-a)}{b-a}}\right)\right)\right)^{\frac{1}{q}}, \\ \text{for } 0 < \mu < 1 \end{cases} \tag{2.24}$$

$$E_1(\rho, m, q) = \begin{cases} \left(\left(r^2 + (1-r)^2\right)\left((b-x)^2 + (x-a)^2\right)\right)^{1-\frac{1}{q}} \\ \times \left(\frac{\rho^2 \mu^{\frac{rq(b-x)}{\rho(b-a)}}}{(q \ln \mu)^2} \left(1 + \mu^{-\frac{rq(b-x)}{\rho(b-a)}} \left(\ln \mu^{-\frac{rq(b-x)}{\rho(b-a)}} - 1 + \mu^{\frac{rq(b-x)}{\rho(b-a)}}\right)\right) \right. \\ \quad + \mu^{\frac{q(b-x)}{\rho(b-a)}} \left(\frac{\rho(1-r)(b-x)}{(b-a)q \ln \mu} - \frac{\rho(1-r)(x-a)}{(b-a)q \ln \mu} - \frac{2\rho^2}{(q \ln \mu)^2} \right) \\ \left. + \frac{\rho^2 \mu^{\frac{q}{\rho}\left(1-\frac{r(x-a)}{(b-a)}\right)}}{(q \ln \mu)^2} \left(1 + \mu^{\frac{rq(b-x)}{\rho(b-a)}} \left(\ln \mu^{\frac{rq(x-a)}{\rho(b-a)}} - 1 + \mu^{-\frac{rq(x-a)}{\rho(b-a)}}\right)\right)\right)^{\frac{1}{q}}, \\ \text{for } \mu > 1. \end{cases} \tag{2.25}$$

Proof. Substituting $n = 1$, in (2.18)-(2.21), we have the required inequality. □

Remark 2.7. For $r = 0$ in (2.22)-(2.25), we can get a three-point Ostrowski type inequality for (ρ, m)–geometrically convex function.

Remark 2.8. The following are the Hermite–Hadamard type inequalities for (ρ, m)–geometrically convex first differentiable function for different choices of r in (2.22)–(2.25).

1. When $r = 0$ and $x = \frac{a+b}{2}$ in (2.22)–(2.25), then

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left|f'(b)\right|^m \left[G_1(\rho,m,q)\right]^{\frac{1}{q}}, \tag{2.26}$$

for $q \geqslant 1$ and μ is defined as

$$\mu = \frac{f'(a)}{(f'(b))^m},$$

where

$$G_1(\rho,m,q) = \begin{cases} \frac{1}{2}, \text{ for } \mu = 1 \\ \frac{2+2\mu^{\rho q}-4\mu^{\frac{\rho q}{2}}}{\rho^2 q^2 \ln^2 \mu}, \text{ for } 0 < \mu < 1 \\ \frac{\rho^2\left(2+2\mu^{\frac{q}{\rho}}-4\mu^{\frac{q}{2\rho}}\right)}{q^2 \ln^2 \mu}, \text{ for } \mu > 1 \end{cases} \tag{2.27}$$

for $\rho > 0$.

2. When $r = 1$ and $x = \frac{a+b}{2}$ in (2.22)-(2.25), then

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du + \frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left|f'(b)\right|^m \left[G_2(\rho,m,q)\right]^{\frac{1}{q}}, \tag{2.28}$$

for $q \geqslant 1$ where μ is defined as

$$\mu = \frac{f'(a)}{(f'(b))^m},$$

where

$$G_2(\rho,m,q) = \begin{cases} \frac{1}{2}, \text{ for } \mu = 1 \\ \frac{\left(\mu^{\rho q}-1\right)\ln u^{\rho q}-2\left(\mu^{\frac{\rho q}{2}}-1\right)^2}{\rho^2 q^2 \ln^2 \mu}, \text{ for } 0 < \mu < 1 \\ \frac{\rho^2\left(\left(\mu^{\frac{q}{\rho}}-1\right)\ln \mu^{\frac{q}{\rho}}-2\left(\mu^{\frac{q}{2\rho}}-1\right)^2\right)}{q^2 \ln^2 \mu}, \text{ for } \mu > 1 \end{cases} \tag{2.29}$$

for $\rho > 0$.

3. When $r = \frac{1}{2}$ and $x = \frac{a+b}{2}$ in (2.22)-(2.25), then

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - \frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left|f'(b)\right|^m \left[G_3(\rho,m,q)\right]^{\frac{1}{q}}, \tag{2.30}$$

μ is defined as

$$\mu = \frac{f'(a)}{f'(b)^m},$$

and

$$G_3(\rho,m,q)=\begin{cases}\frac{1}{2}, \text{ for } \mu=1\\ \frac{-4\mu^{\rho q}+8\mu^{\frac{3\rho q}{4}}-8\mu^{\frac{\rho q}{2}}+8\mu^{\frac{\rho q}{4}}+4(\mu^{\rho q}-1)\ln\mu^{\frac{\rho q}{4}}-4}{\rho^2q^2\ln^2\mu}, \text{ for } 0<\mu<1\\ \frac{\rho^2\left(-4\mu^{\frac{q}{\rho}}+8\mu^{\frac{3q}{4\rho}}-8\mu^{\frac{q}{2\rho}}+8\mu^{\frac{q}{4\rho}}+4\left(\mu^{\frac{q}{\rho}}-1\right)\ln\mu^{\frac{q}{4\rho}}-4\right)}{q^2\ln^2\mu}, \text{ for } \mu>1\end{cases} \tag{2.31}$$

for $\rho>0$.

4. When $r=\frac{1}{3}$ and $x=\frac{a+b}{2}$ in (2.22)-(2.25), then

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|$$
$$\leq \frac{5(b-a)}{18}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left|f'(b)\right|^m\left[G_4(\rho,m,q)\right]^{\frac{1}{q}}, \tag{2.32}$$

for $q\geqslant 1$ where μ is defined as

$$\mu=\frac{f'(a)}{f'(b)^m},$$

where

$$G_4(\rho,m,q)=\begin{cases}\frac{1}{2}, \text{ for } \mu=1\\ \frac{9(-2\mu^{\rho q}+4\mu^{\frac{5\rho q}{6}}-4\mu^{\frac{\rho q}{2}}+4\mu^{\frac{\rho q}{6}}+2(\mu^{\rho q}-1)\ln\mu^{\frac{\rho q}{6}}-2)}{5(\rho^2q^2\ln^2\mu)}, \text{ for } 0<\mu<1\\ \frac{9\rho^2\left(-2\mu^{\frac{q}{\rho}}+4\mu^{\frac{5q}{6\rho}}-4\mu^{\frac{q}{2\rho}}+4\mu^{\frac{q}{6\rho}}+2\left(\mu^{\frac{q}{\rho}}-1\right)\ln\mu^{\frac{q}{6\rho}}-2\right)}{5(q^2\ln^2\mu)}, \text{ for } \mu>1\end{cases} \tag{2.33}$$

for $\rho>0$.

3. Conclusions

Some generalized inequalities for (ρ,m)–geometrically convex and n-differentiable mappings are given which are capable of giving bounds of the one point, two point and three point Hadamard type inequalities for first and higher differentiable functions. The special cases recapture inequalities given by Xi et al. [8]. Some new estimates of the average mid-point trapezoid and Simpson's inequality are given for first differentiable generalized geometrically convex function as special cases. The estimates for higher differentiable function can also be obtained from (2.18)-(2.21) for all these cases.

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