Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 255–266
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

# Two different distributions of limit cycles in a quintic system

Hongwei Li*, Yinlai Jin

*School of Science, Linyi University, Linyi, 276005 China.*

Communicated by Yeol Je Cho

**Abstract**

In this paper, the conditions for bifurcations of limit cycles from a third-order nilpotent critical point in a class of quintic systems are investigated. Treaty the system coefficients as parameters, we obtain explicit expressions for the first fourteen quasi Lyapunov constants. As a result, fourteen or fifteen small amplitude limit cycles with different distributions could be created from the third-order nilpotent critical point by two different perturbations. 

*Keywords:* Third-order nilpotent critical point, center-focus problem, bifurcation of limit cycles, quasi-Lyapunov constant.
*2010 MSC:* 34C05, 34C07.

## 1. Introduction

Recently years, many works have been devoted to study the center–focus problem which is also related to the so–called *cyclicity* of the point, see [1, 3, 4]. As far as the maximum number of small-amplitude limit cycles are concerned, there have been many results. For an elementary center or an elementary focus, one of the best-known results is $M(2) = 3$, which was solved by Bautin [2]. For $n = 3$, Yu and Tian have proved that there could be twelve limit cycles around a center point in a planar cubic-degree polynomial system [12]. For $n = 4$, an example of a quartic system with eight limit cycles bifurcated from a fine focus [5] was given. As far as bifurcation of limit cycles from degenerate critical points were concerned, they also have been investigated intensively. Especially, for nilpotent critical point, there were also many results about limit cycles, see [7, 9]. So far, regarding the family of polynomial differential systems, a complete

*Corresponding author
*Email addresses:* hongweifx@163.com (Hongwei Li), jinyinlai@sina.com (Yinlai Jin)

*Received 2014-10-22*

classification of centers and isochronous centers has only been solved for quadratic polynomial systems, or simply quadratic systems. Recently, the conditions of center and isochronous center at the origin for a class of non-analytic quintic systems were studied in [8]. A class of nilpotent-Poincar system was discussed in [10]. Two kinds of bifurcation phenomena in a quartic system were investigated in [11].

In this paper, we consider a quintic systems

$$\begin{aligned} \frac{dx}{dt} &= y + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{14}xy^4 + a_{05}y^5 + a_{31}x^3y \\ &\quad - \frac{3}{2}b_{13}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4, \\ \frac{dy}{dt} &= -2x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{14}xy^4 + b_{05}y^5 \\ &\quad - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4. \end{aligned} \tag{1.1}$$

We will show that two different distributions of fourteen or fifteen cycles can be given by different perturbations.

The rest of this paper will be organized as follows. In Section 2, some preliminary results in [6] will be given. In Section 3, the linear recursive formulae in [6] are used to compute the first fourteen quasi–Lyapunov constants and then obtain the sufficient and necessary conditions for a center. In Section 4, one kind of different bifurcation are discussed to confirm that fourteen limit cycles can bifurcate from quintic systems. In Section 5, another kind of interesting bifurcation phenomenon was discussed to confirm that fifteen limit cycles can bifurcate from quintic systems.

To perform the computations in this paper, we have used the computer algebra system–MATHEMATICA 7.

## 2. Preliminary results

In this section, some important results taken from [6] for center-focus problem of third-order nilpotent critical points in the planar dynamical systems are presented for convenience in future, for more detail, see [6].

It is well known that the origin of a system with a third-order monodromic critical point can be written in the following form of real autonomous planar system:

$$\begin{aligned} \frac{dx}{dt} &= y + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij}x^iy^j = X(x,y), \\ \frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{i+2j=4}^{\infty} b_{ij}x^iy^j = Y(x,y). \end{aligned} \tag{2.1}$$

**Theorem 2.1.** *For any positive integer $s$ and a given real number sequence,*

$$\{c_{0\beta}\}, \beta \geq 3, \tag{2.2}$$

*one can construct successively the terms with the coefficients $c_{\alpha\beta}$ satisfying $\alpha \neq 0$ of the formal series,*

$$M(x,y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta}x^\alpha y^\beta = \sum_{k=2}^{\infty} M_k(x,y), \tag{2.3}$$

*such that*

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - (s+1)\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right) = \sum_{m=3}^{\infty} \omega_m(s,\mu)x^m, \tag{2.4}$$

*where$M_k(x,y)$ is a $kth-$ degree homogeneous polynomial of $x,y$ for all $k$ and $s\mu = 0$.*

**Theorem 2.2.** *For $\alpha \geq 1, \alpha + \beta \geq 3$ in (2.3) and (2.4), $c_{\alpha\beta}$ can be uniquely determined by the recursive formula,*

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha}(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}); \tag{2.5}$$

*and for $m \geq 1$, $\omega_m(s,\mu)$ can be uniquely determined by the recursive formula,*

$$\omega_m(s,\mu) = A_{m,0} + B_{m,0}, \tag{2.6}$$

*where*

$$\begin{aligned} A_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [k - (s+1)(\alpha - k + 1)] a_{kj} c_{\alpha-k+1,\beta-j}, \\ B_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [j - (s+1)(\beta - j + 1)] b_{kj} c_{\alpha-k,\beta-j+1}. \end{aligned} \tag{2.7}$$

have been set. The $mth-$order quasi-Lyapunov constant is defined as

$$\lambda_m = \frac{\omega_{2m+4}(s,\mu)}{2m - 4s - 1}. \tag{2.8}$$

Clearly, the recursive formulae in Theorem 2.2 are linear with respect to all $c_{\alpha\beta}$. Therefore, it is convenient to develop programs for computing quasi–Lyapunov constants by using computer algebraic system such as MATHEMATICA.

## 3. Quasi–Lyapunov constants and center conditions

According to Theorem 2.1, for system (1.1), we can find a positive integer $s$ and a formal series $M(x, y) = x^4 + y^2 + o(r^4)$, such that (2.4) holds. Applying the recursive formulae in Theorem 2.2 to carry out calculations, we have

$$\begin{aligned} &\omega_3 = \omega_4 = \omega_5 = 0, \\ &\omega_6 = -\frac{1}{3} b_{21}(-1 + 4s), \\ &\omega_7 \sim 3(s+1)c_{03}, \\ &\omega_8 \sim -\frac{2}{5}(a_{12} + 3b_{03})(-3 + 4s), \\ &\omega_9 \sim 0, \\ &\omega_{10} \sim -\frac{4}{7} b_{03}(a_{21} + b_{12})(-5 + 4s), \\ &\omega_{11} \sim -\frac{3}{8}(4a_{04} - 3a_{21}b_{13} - 3b_{12}b_{13} + 4a_{04}s + 2a_{21}b_{13}s + 2b_{12}b_{13}s - 10c_{05} - 10sc_{05}), \\ &\omega_{12} \sim -\frac{4}{15}(a_{14} + 5b_{05})(-7 + 4s), \\ &\omega_{13} \sim -\frac{1}{5}(a_{21} + b_{12})(4a_{04} + 2a_{21}b_{13} - 3b_{12}b_{13})(-2 + s), \end{aligned} \tag{3.1}$$

(2.8) and (3.1) yield that

$$\begin{aligned} c_{03} &= 0, \\ c_{05} &= \frac{4a_{04} - 3a_{21}b_{13} - 3b_{12}b_{13} + 4a_{04}s + 2a_{21}b_{13}s + 2b_{12}b_{13}s}{10(1+s)}. \end{aligned}$$

Furthermore, the quasi-Lyapunov constants can be computed in two cases and we obtain the following results.

**Proposition 3.1.** *For system* (1.1), *one can determine successively the terms of the formal series* $M(x,y) = x^4 + y^2 + o(r^4)$, *such that*

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - 2\left(\frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y\right) = \sum_{m=1}^{14} \mu_m[(2m-5)x^{2m+4} + o(r^{30})], \tag{3.2}$$

*where* $\mu_m$ *is the mth-order quasi-Lyapunov constant at the origin of system* (1.1), $m = 1, 2, \cdots, 14$.

**Theorem 3.2.** *For system* (1.1), *the first 14 quasi-Lyapunov constants at the origin are given by*

$$\begin{aligned}
\mu_1 &= -\frac{1}{3} b_{21}, \\
\mu_2 &= \frac{2}{5}(a_{12} + 3b_{03}), \\
\mu_3 &= -\frac{4}{7} b_{03}(a_{21} + b_{12}), \\
\mu_4 &= -\frac{4}{15}(a_{14} + 5b_{05}),
\end{aligned} \tag{3.3}$$

**Case 1** $s = 2$

$$\mu_5 = \frac{5}{77}(a_{21} + b_{12})(8b_{05} - 3a_{31}b_{13}),$$

**Subcase 1.1** *If* $a_{31} \neq 0$

$$\begin{aligned}
\mu_6 &= \frac{7}{195}(a_{21} + b_{12})(4a_{04}a_{31} + 2a_{21}a_{31}b_{13} + 20b_{04}b_{13} - 3a_{31}b_{12}b_{13}), \\
\mu_7 &= \frac{4}{3456} b_{13}(a_{21} + b_{12}) b_{04}(338a_{21}a_{31} + 3780b_{04} - 607a_{31}b_{12}), \\
\mu_8 &= -\frac{2}{136769457825} a_{31}b_{13}(a_{21} + b_{12})(80445352a_{21}^3 - 156314652a_{21}^2 b_{12} \\
&\quad - 71017440a_{21}b_{12}^2 + 165742564b_{12}^3 + 5569796925b_{14}), \\
\mu_9 &= \frac{a_{31}b_{13}(a_{21} + b_{12})}{9009513854294703750}(-342782234947152 00a_{03}a_{21}^2 + 1220801069532512a_{21}^4 \\
&\quad + 2544039835214377 50a_{21}a_{31}^2 + 2728059798839760 0a_{03}a_{21}b_{12} + 1577422505111120a_{21}^3 b_{12} \\
&\quad - 456873426028144125a_{31}^2 b_{12} + 615588214831128 00a_{03}b_{12}^2 - 9680067969729072a_{21}^2 b_{12}^2 \\
&\quad - 233037428820256a_{21}b_{12}^3 + 9803651976487424b_{12}^4 + 7112774095495818 75b_{13}^2),
\end{aligned}$$

*If* $188200a_{21} - 109097b_{12} \neq 0$

$$\begin{aligned}
\mu_{10} &= -\frac{a_{31}b_{13}(a_{21} + b_{12})}{6238962685090061920743750}(338a_{21} - 607b_{12})(-1013559170267508480 0a_{03}a_{21}^2 \\
&\quad + 728855954800765888a_{21}^4 + 4507151800580178750 00a_{21}a_{31}^2 - 1507964520262329360 0a_{03}a_{21}b_{12} \\
&\quad + 4622881761161239120a_{21}^3 b_{12} - 2612735069011135818 75a_{31}^2 b_{12} - 4944053499948208800a_{03}b_{12}^2 \\
&\quad - 2612963902077875568a_{21}^2 b_{12}^2 - 5712510416295551744a_{21}b_{12}^3 + 7944792921427970 56b_{12}^4), \\
\mu_{11} &= -\frac{8a_{31}b_{13}(a_{21} + b_{12})^2(338a_{21} - 607b_{12})}{6376771232489338117222358228906 25(188200a_{21} - 109097b_{12})} \\
&\quad \times (-1556420284845858977651781675000 00a_{03}^2 a_{21} + 3367439481410463961615563450000 00a_{05}a_{21} \\
&\quad - 4035564815191783825880187372960 0a_{03}a_{21}^3 + 1148400493414922637218817901376a_{21}^5 \\
&\quad + 9022358332403224064021064048750 0a_{03}^2 b_{12} - 1952059219465660928907402368250 00a_{05}b_{12} \\
&\quad + 1003228827801904976809211242722 00a_{03}a_{21}^2 b_{12} - 1326293171407512954279596130584a_{21}^4 b_{12}
\end{aligned}$$

$$- 7651089845283807635860720818630 0a_{03}a_{21}b_{12}^2 - 617062440859539475583800010623 6a_{21}^3b_{12}^2$$
$$+ 1372707520839637795811614825440 0a_{03}b_{12}^3 + 1230115253998720017325771896739 1a_{21}^2b_{12}^3$$
$$- 5863030103422599488045010426776a_{21}b_{12}^4 - 1321258049837728241312651405 68b_{12}^5),$$

$$\mu_{12} = \frac{4a_{31}b_{13}(a_{21}+b_{12})^2(338a_{21}-607b_{12})f_1}{10679729857381660502598890741386436337890625(188200a_{21}-109097b_{12})^2}$$
$$\mu_{13} = \frac{4a_{31}b_{13}(a_{21}+b_{12})^2(338a_{21}-607b_{12})f_2}{3722803564484153491242701746054580400229397753906 25(188200a_{21}-109097b_{12})^2}$$
$$\mu_{14} = \frac{a_{31}b_{13}(a_{21}+b_{12})^2(338a_{21}-607b_{12})f_3}{825256202960589209324717152258395164801251976494921875000 0(188200\ a_{21}-109097b_{12})^2}.$$

*If* $188200a_{21} - 109097b_{12} = 0$

$$\mu_{10} = -\frac{a_{31}b_{13}(a_{21}+b_{12})b_{12}^3}{50840139392092053281250000000 00}(14266438601064554520000 00a_{03}$$
$$+ 201350610870420881440183b_{12}^2),$$
$$\mu_{11} = -\frac{52169a_{31}b_{13}(a_{21}+b_{12})^2b_{12}^2}{41433860909626881541380479149261901535017250000000000000000}$$
$$\times (-150500519428614395475074557359365049484512000000000000a_{05}$$
$$+ 56268386998393642923015480077596363920739272000000000a_{31}^2b_{12}$$
$$+ 5627637341410830712071105513808163188812874249432 57b_{12}^4),$$
$$\mu_{12} = \frac{4013a_{31}b_{13}(a_{21}+b_{12})b_{12}f_4}{26589376090671664215061872566004321936380747731500000000000000000000}$$
$$\mu_{13} = \frac{4013a_{31}b_{13}(a_{21}+b_{12})^2b_{12}^2f_5}{118016191431192450780373323438877262672731845160211853958507144782000000000000000000000}.$$

**Subcase 1.2** *If* $a_{31} = 0$

$$\mu_6 = \frac{28}{13}(a_{21}+b_{12})b_{04}b_{13},$$
$$\mu_7 = -\frac{48}{11}b_{04}(a_{21}+b_{12})a_{04}.$$

**Case 2** $a_{04} = -\frac{1}{4}(2a_{21} - 3b_{12})b_{13}$.

$$\mu_6 = \frac{28}{39}(a_{21}+b_{12})b_{04}b_{13},$$
$$\mu_7 = 0,$$
$$\mu_8 = -\frac{3}{3094}b_{04}(a_{21}+b_{12})a_{31}b_{13}b_{14}.$$

*Where* $f_i, i = 1, \cdots, 6$ *are given in Appendix. In the above expressions of* $\mu_k$*, for each* $k = 2, \cdots, 14$*,* $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0$ *have been set.*

Theorem 3.2 directly gives the following assertion.

**Proposition 3.3.** *The first fourteen quasi-Lyapunov constants at the origin of system* (1.1) *are zero if and only if one of the following conditions is satisfied:*

$$b_{21} = 0, a_{12} = -3b_{03}, a_{21} = -b_{12}, a_{14} = -5b_{05}; \tag{3.4}$$

$$b_{21} = a_{12} = b_{03} = a_{14} = b_{05} = a_{04} = b_{13} = 0; \tag{3.5}$$

$$b_{21} = a_{12} = b_{03} = a_{14} = b_{05} = a_{31} = b_{04} = 0. \tag{3.6}$$

From Propositions 3.3 we have the following theorem.

**Theorem 3.4.** *The origin of system* (1.1) *is a center if and only if the first fourteen quasi–Lyapunov constants are zero, that is, one of the condition in Proposition 3.3 is satisfied.*

*Proof.* When condition (3.4) is satisfied, system (1.1) can be brought to

$$\begin{aligned}\frac{dx}{dt} &= y - b_{12}x^2y - 3b_{03}xy^2 + a_{03}y^3 - 5b_{05}xy^4 + a_{05}y^5 + a_{31}x^3y\\ &\quad - \frac{3}{2}b_{13}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4,\\ \frac{dy}{dt} &= -2x^3 + b_{12}xy^2 + b_{03}y^3 + b_{14}xy^4 + b_{05}y^5 - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4,\end{aligned} \tag{3.7}$$

which has an analytic first integral

$$\begin{aligned}H(x,y) &= \frac{1}{2}y^2 + \frac{1}{2}x^4 - \frac{1}{2}b_{12}x^2y^2 - b_{03}xy^3 + \frac{1}{4}a_{03}y^4 - b_{05}xy^5 + \frac{1}{6}a_{05}y^6\\ &\quad + \frac{1}{2}a_{31}x^3y - \frac{1}{2}b_{13}x^2y^3 - b_{04}xy^3 + \frac{1}{5}a_{04}y^5.\end{aligned} \tag{3.8}$$

When condition (3.5) holds, system (1.1) can be rewritten as

$$\begin{aligned}\frac{dx}{dt} &= y + a_{03}y^3 + a_{05}y^5 + a_{31}x^3y - 4b_{04}xy^3,\\ \frac{dy}{dt} &= -2x^3 + b_{12}xy^2 + b_{14}xy^4 - \frac{3}{2}a_{31}x^2y^2 + b_{04}y^4,\end{aligned} \tag{3.9}$$

whose vector field is symmetric with respect to the $x$–axis.

When condition (3.6) holds, system (1.1) becomes

$$\begin{aligned}\frac{dx}{dt} &= y + a_{21}x^2y + a_{03}y^3 + a_{05}y^5 + a_{04}y^4,\\ \frac{dy}{dt} &= -2x^3 + b_{12}xy^2 + b_{14}xy^4 + b_{13}xy^3,\end{aligned} \tag{3.10}$$

whose vector field is symmetric with respect to the $y$–axis. □

## 4. Existence of fourteen limit cycles

Now, we will prove that fourteen limit cycles enclosing an elementary node at the origin of unperturbed system (1.1) can be bifurcated from the perturbed system of (1.1) the third–order nilpotent critical point $O(0,0)$ is a 14th-order weak focus.

$\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_8 = \mu_9 = \mu_{10} = \mu_{11} = \mu_{12} = \mu_{13} = 0, \mu_{14} \neq 0$, means that

**Theorem 4.1.** *The origin of system* (1.1) *is a 14th-order weak focus if and only if* $188200a_{21} - 109097b_{12} \neq 0$ *and*

$$\begin{aligned}&b_{21} = b_{03} = a_{12} = 0,\\ &a_{14} = -\frac{15}{8}a_{31}b_{13}, b_{05} = \frac{3}{8}a_{31}b_{13},\\ &a_{04} = -\frac{10}{189}(a_{21} + b_{12})b_{13},\\ &b_{04} = -\frac{1}{3780}a_{31}(338a_{21} - 607b_{12}),\\ &b_{14} = -\frac{52}{5569796925}(4577a_{21} - 5251b_{12})(338a_{21} - 607b_{12})(a_{21} + b_{12}),\end{aligned} \tag{4.1}$$

$$a_{31}^2 = -\frac{16(a_{21}+b_{12})}{2394873432826875(188200a_{21}-109097b_{12})}(-633474481417192800a_{03}a_{21} + 45553497175047868a_{21}^3 - 309003343746763050a_{03}b_{12} + 243376612897529577a_{21}^2 b_{12} - 406686856777396800a_{21}b_{12}^2 + 49654955758924816b_{12}^3),$$

$$b_{13}^2 = \frac{338a_{21}-607b_{12}}{711277409549581875}(101414862410400a_{03}a_{21} - 3611837483824a_{21}^3 - 752674507459875a_{31}^2 + 101414862410400a_{03}b_{12} - 11153277685776a_{21}^2 b_{12} + 8609551522080a_{21}b_{12}^2 + 16150991724032b_{12}^3).$$

*Proof.* Solving $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_8 = \mu_9 = \mu_{10} = \mu_{11} = 0$, we obtain above relations. Furthermore, we denote

$$F_1 = Factor[Resultant[f_1, f_2, b_{12}]],$$

$$F_2 = Factor[Resultant[f1, f_3, b_{12}]],$$

then

$$G = Resultant[F_1, F_2, a_{03}] = -9.21710139189961 \times 10^{18640} a_{21}^{966} \neq 0$$

so the origin of system (1.1) is a 14th-order weak focus. □

Now, we study the perturbed system of (1.1), given by:

$$\begin{aligned} \frac{dx}{dt} &= \delta x + y + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{14}xy^4 + a_{05}y^5 + a_{31}x^3y - \frac{3}{2}b_{13}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4, \\ \frac{dy}{dt} &= \delta y - 2x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{14}xy^4 + b_{05}y^5 - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4. \end{aligned} \tag{4.2}$$

When conditions in (4.1) hold,

$$\frac{\partial(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\mu_8,\mu_9,\mu_{10},\mu_{11},\mu_{12},\mu_{13})}{\partial(b_{21},\ a_{12},\ b_{03},\ a_{14},\ b_{05},\ a_{04},\ b_{04},\ b_{14},\ b_{13},\ a_{31},\ b_{12},a_{03})} \neq 0. \tag{4.3}$$

Further, it follows from Theorem 2.1 in [6] that

**Theorem 4.2.** *If the origin of system* (1.1) *is a 14th–order weak focus, for* $0 < \delta \ll 1$*, with a small perturbation to the coefficients of system* (1.1)*, then, for system* (4.2)*, in a small neighborhood of the origin, there exist exactly fourteen small amplitude limit cycles enclosing the origin* $O(0,0)$*, which is an elementary node, see figure 1.*

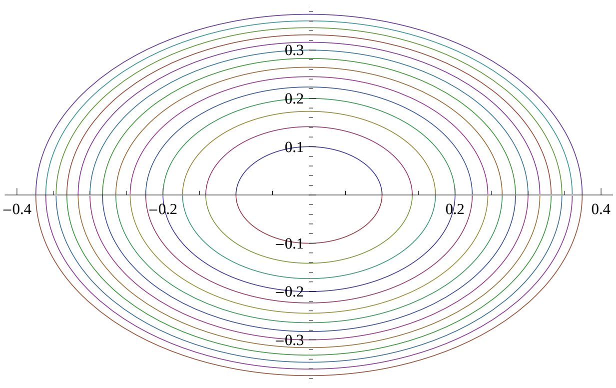


Fig1:Phase portrait of system (4.2)

## 5. Existence of fifteen limit cycles

An interesting bifurcation of limit cycles which is different from the first kind of bifurcation will be considered in this section. It is first time to consider this kind of bifurcation phenomena in a quintic system. The following perturbed system of (1.1)

$$\begin{aligned}\frac{dx}{dt} &= y + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{14}xy^4 + a_{05}y^5 + a_{31}x^3y \\ &\quad - \frac{3}{2}b_{13}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4, \\ \frac{dy}{dt} &= 4\delta\varepsilon y - (x^2 - \varepsilon^2)(2x - b_{21}y) + b_{12}xy^2 \\ &\quad + b_{03}y^3 + b_{14}xy^4 + b_{05}y^5 - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4.\end{aligned} \tag{5.1}$$

which is called double perturbed system of system (1.1) will be considered in this section. Obviously, when $0 <|\ \varepsilon\ |\ll 1$, system (5.1) has three real singular points in the neighborhood of the origin, namely $O(0, 0)$ and $P_{1,2}(\pm\varepsilon, 0)$.

The following transformation

$$\begin{gathered} x = \varepsilon(u \pm 1), \\ y = 2\varepsilon^2 \frac{\delta u - \rho v}{1 \pm \varepsilon(\pm a_{21}\varepsilon + a_{31}\varepsilon^2)}, t = \frac{\tau}{2\rho\varepsilon}, \\ \rho = \sqrt{(1 \pm \varepsilon(\pm a_{21}\varepsilon + a_{31}\varepsilon^2)) - \delta^2}, \end{gathered}$$

can shift $P_{1,2}(\pm\varepsilon, 0)$ of system (5.1) to origin, and obtain a new system in the form of

$$\begin{aligned}\frac{d\xi}{d\tau} &= \Phi(\xi, \eta, \varepsilon, \delta) = \frac{\delta\xi}{\rho} - \eta + \sum_{k+j=2}^{\infty} A_{kj}(\varepsilon, \delta)\xi^k\eta^j, \\ \frac{d\eta}{d\tau} &= \Psi(\xi, \eta, \varepsilon, \delta) = \xi + \frac{\delta\eta}{\rho} + \sum_{k+j=2}^{\infty} B_{kj}(\varepsilon, \delta)\xi^k\eta^j,\end{aligned} \tag{5.2}$$

where $\Phi(\xi, \eta, \varepsilon, \delta)$ and $\Psi(\xi, \eta, \varepsilon, \delta)$ are power series in $(u, v, \varepsilon, \delta)$ with nonzero convergence radius. So $P_{1,2}(\pm\varepsilon, 0)$ of (5.1) are fine foci when $\delta \neq 0$, and weak foci or centers when $\delta = 0$. Especially for $\delta = 0$, let $A = 1 + a_{21}\varepsilon^2 - a_{31}\varepsilon^3$, corresponding to $P_{1,2}(\pm\varepsilon, 0)$, system (5.1) are changed into the same systems

$$\begin{aligned}\frac{du}{dt} &= -v + \frac{1}{8\varepsilon^2}(3b_{13}u^2v^2 - 2(a_{21} - 3a_{31}\varepsilon)u^2v) - \frac{\sqrt{A}}{8\varepsilon^3}a_{31}u^3v \\ &\quad - \frac{1}{2A}(2a_{05}v^5 + 2v^4(a_{04} - a_{14}\varepsilon) + 2v^3(a_{03} + 4b_{04}\varepsilon) + v^2\varepsilon(2a_{12} + 3b_{13}\varepsilon)) \\ &\quad + \frac{1}{2\varepsilon\sqrt{A}}(4b_{04}uv^3 - a_{14}uv^4 - uv\varepsilon(-2a_{21} + 3a_{31}\varepsilon) - uv^2(a_{12} + 3b_{13}\varepsilon)), \\ \frac{dv}{dt} &= u + \frac{1}{2\varepsilon}b_{21}uv - \frac{1}{4\varepsilon^2}(b_{13}uv^3 + b_{14}uv^4 - uv^2(b_{12} + 3a_{31}\varepsilon)) - \frac{A}{8\varepsilon^4}u^3 \\ &\quad + \frac{1}{4\varepsilon\sqrt{A}}(v^2\varepsilon(2b_{12} + 3a_{31}\varepsilon) - 2b_{05}v^5 + 2v^3(-b_{03} + b_{13}\varepsilon) + v^4(-b_{04} + b_{14}\varepsilon)) \\ &\quad + \frac{\sqrt{A}}{16\varepsilon^3}(3a_{31}u^2v^2 - 2b_{21}u^2v - 12\varepsilon u^2).\end{aligned} \tag{5.3}$$

The first Lyapunov constant at origin for system (5.3) is given by

$$\begin{aligned}V_1 = &-i(-4(a_{12} + 3b_{03})\varepsilon^2 + b_{21}(2 + \varepsilon^2(2a_{21} - 2b_{12} - 5a_{31}\varepsilon))(1 + \varepsilon^2(a_{21} - a_{31}\varepsilon)) \\ &+ 4\varepsilon^4(-3a_{21}b_{03} + 3(a_{31}b_{03} + (a_{21} + b_{12})b_{13})\varepsilon + a_{12}(a_{21} + 2b_{12} + a_{31}\varepsilon)))\end{aligned}$$

When the the origin of system (1.1) is a 14th–order weak focus, the first Lyapunov constant of system (5.3) at origin is

$$V_1 = 3i(a_{21} + b_{12})b_{13}\varepsilon\sqrt{\frac{\varepsilon^2}{1 + \varepsilon^2(a_{21} - a_{31}\varepsilon)}} \neq 0.$$

Summarizing the above results yields the following theorem.

**Theorem 5.1.** *If the origin of system* (1.1) *is a 14–order weak focus, choosing proper coefficients in system* (1.1)*, when* $0 <| \varepsilon |\ll 1$*, there exist fifteen limit cycles with the distribution of one limit cycle enclosing each of* $P_{1,2}(\pm\varepsilon, 0)$*, and thirteen limit cycles enclosing both* $(\varepsilon, 0)$ *and* $(-\varepsilon, 0)$ *in the neighborhood of origin, see figure 2.*

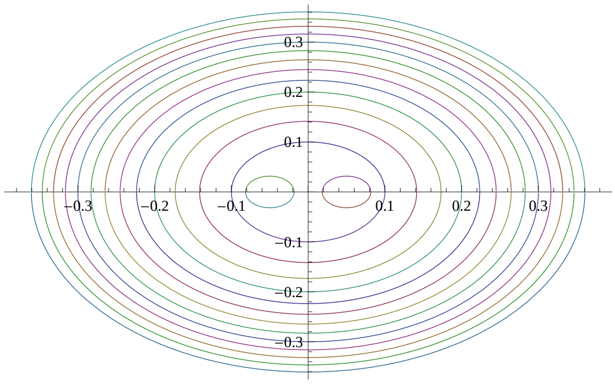


Fig2: Phase portrait of system (5.1)

We have studied an interesting bifurcation which, different from the first kind of bifurcation, can generate 15 limit cycles by perturb the quintic system with a nilpotent critical point. We set a new record of limit cycles bifurcated from an isolated critical point in a quintic systems.

## 6. Appendix

$$\begin{aligned}
f_1 = {} & 398966693980473495044364802094971820834902560000a_{03}^2a_{21}^3 \\
& - 484896365942328518493743174310189235231894272000a_{03}a_{21}^5 \\
& + 79536911994553425825150874536982021652927641 6a_{21}^7 \\
& - 71840001429360414896183163598760294040351972 0000a_{03}^2a_{21}^2b_{12} \\
& + 152921183082778076676566272028978847544528804800a_{03}a_{21}^4b_{12} \\
& - 25108335503574414079983136466991363880421456 96a_{21}^6b_{12} \\
& + 356317132340031901544189053015226441369740192500a_{03}^2a_{21}b_{12}^2 \\
& - 142549870063093476910061839908358922606626741800a_{03}a_{21}^3b_{12}^2 \\
& - 34554289519653052297959890935504674414726140 16a_{21}^5b_{12}^2 \\
& - 340191955277036405798081836500383882213537775 00a_{03}^2b_{12}^3 \\
& + 420461932527141608920450298524989926959798713 00a_{03}a_{21}^2b_{12}^3 \\
& + 210519757703303423688595819626728634428835949 40a_{21}^4b_{12}^3 \\
& - 124984247624001349096389542626817020393429893 00a_{03}a_{21}b_{12}^4 \\
& - 287610999273526122299778970962478886398602500 35a_{21}^3b_{12}^4 \\
& + 928605373761878938510607429829188502343891560 0a_{03}b_{12}^5 \\
& + 172326372969109819749527584074272328274273560 21a_{21}^2b_{12}^5
\end{aligned}$$

$$
\begin{aligned}
&- 53938915322236121238194535406782907235714689 76a_{21}b_{12}^{6}\\
&+ 1041396652581455625890548077863148594041425624b_{12}^{7}.\\
f_2 = &-247145634066961423295736841874117765228828889375000000000a_{03}^{3}a_{21}^{2}\\
&+ 1017727874198815914002109649602054965255954430301285920000a_{03}^{2}a_{21}^{4}\\
&- 148422427930752803914126116795233952782387191533318950400a_{03}a_{21}^{6}\\
&+ 268210923298189766113326828505325409340358407574 6227712a_{21}^{8}\\
&+ 28653397704360563652810841910670165603793353181875 0000000a_{03}^{3}a_{21}b_{12}\\
&- 12996744200314865965978362239672669650761739236299629 20000a_{03}^{2}a_{21}^{3}b_{12}\\
&+ 4600708871032595732039358712777924642708586866325676672 00a_{03}a_{21}^{5}b_{12}\\
&- 827680165274214066810766290996809130064571006020959 6320a_{21}^{7}b_{12}\\
&- 83049939674618076855225941018288604061558008822609375000a_{03}^{3}b_{12}^{2}\\
&- 69440448669777049376409085900289684617949062201082467500a_{03}^{2}a_{21}^{2}b_{12}^{2}\\
&- 44362126578124089646080076319316519203820397417387266 9000a_{03}a_{21}^{4}b_{12}^{2}\\
&- 105208254003438650250066840447893043162236978167091035 84a_{21}^{6}b_{12}^{2}\\
&+ 41568875740331272182675253844801835067454096664124413 0000a_{03}^{2}a_{21}b_{12}^{3}\\
&+ 21776478458394405396204561252547578666563499551942212 6500a_{03}a_{21}^{3}b_{12}^{3}\\
&+ 630182552287918670931272799404197292294028977494662653 08a_{21}^{5}b_{12}^{3}\\
&- 55511671671230969621513993548815913178060524469785680000a_{03}^{2}b_{12}^{4}\\
&- 150405163297367214664803827521331260798692606577521099750a_{03}a_{21}^{2}b_{12}^{4}\\
&- 81212786659593733006465951076881491007503048628520436715a_{21}^{4}b_{12}^{4}\\
&+ 62638746628191947529935527186175452863981811173582531600a_{03}a_{21}b_{12}^{5}\\
&+ 44237642447032098451551492019908714431134104741251139502a_{21}^{3}b_{12}^{5}\\
&+ 4160463031238415844056885463309129897149201232764013200a_{03}b_{12}^{6}\\
&- 11935302354245668551207290055170662557283042763600481520a_{21}^{2}b_{12}^{6}\\
&+ 1287096054394367010552995175059704614495457345335390736a_{21}b_{12}^{7}\\
&+ 715421429145973100067384633366635978935926698260090208b_{12}^{8}.\\
f_3 = &-2424251117860679766112121822250621432226067050352406057702000000000a_{03}^{3}a_{21}^{3}\\
&+ 28617679570951781161330079249551963439587606622988353132698848 0000a_{03}^{2}a_{21}^{5}\\
&- 4412576284174883577076504394125870165503481851007801395339701760 0a_{03}a_{21}^{7}\\
&+ 83747187811182270866044631472899472272436831467448977260410572 8a_{21}^{9}\\
&+ 312138302472308913359697042864978793128990046500043773448830000000a_{03}^{3}a_{21}^{2}b_{12}\\
&- 305263154519473514379906570282196551888125765520185141662879200000a_{03}^{2}a_{21}^{4}b_{12}\\
&+ 13135459599615082878572253637155177866360951281892772691946956320 0a_{03}a_{21}^{6}b_{12}\\
&- 2447136222999108280771835384233498744422637393908750630069481152a_{21}^{8}b_{12}\\
&- 930071332168687063111686251036609983006171390602565046690493750 00a_{03}^{3}a_{21}b_{12}^{2}\\
&- 674916468472858358014892524639376746756461809756070635360400750 00a_{03}^{2}a_{21}^{3}b_{12}^{2}\\
&- 1141470670679377162529390706272553818389995063686238549302670492 00a_{03}a_{21}^{5}b_{12}^{2}
\end{aligned}
$$

$$
\begin{aligned}
&- 34367892146132281100603043358266766303631424871222047345330596 16a_{21}^{7}b_{12}^{2}\\
&- 66794079457238662732097311971117822679361764472533820103918750 00a_{03}^{3}b_{12}^{3}\\
&+ 49675063664528688899519027071566553193432156094515703994477462500a_{03}^{2}a_{21}^{2}b_{12}^{3}\\
&+ 40244690648760616897412434507383958963317715594253349285594897000a_{03}a_{21}^{4}b_{12}^{3}\\
&+ 1815573174772776035900703815061885190106948285440207248501431469 6a_{21}^{6}b_{12}^{3}\\
&+ 51667687988519077921098737236038449592418283405559297256423762500a_{03}^{2}a_{21}b_{12}^{4}\\
&- 13725916410357744991150569902830560754549755488285599769406776250a_{03}a_{21}^{3}b_{12}^{4}\\
&- 2066437496551796939982907081107422997217096662626296550429014711 8a_{21}^{5}b_{12}^{4}\\
&- 1261945892368079132078750023791202113969547649480668740703234500 0a_{03}^{2}b_{12}^{5}\\
&- 11928164637411067449940797361536394448675173021913953233870482850a_{03}a_{21}^{2}b_{12}^{5}\\
&+ 8085098291409163429839177180184469247412418128808767492480312503a_{21}^{4}b_{12}^{5}\\
&+ 13650448892781387770450511811599955198758287787657241068621418200a_{03}a_{21}b_{12}^{6}\\
&+ 1198185205090676852769025750482985544605141288793846590161853 83a_{21}^{3}b_{12}^{6}\\
&- 6915125814161225604829506802549490274600461632622031561227772 00a_{03}b_{12}^{7}\\
&- 1277626368239126953717300563535651351675419666165946771422284586a_{21}^{2}b_{12}^{7}\\
&+ 590394321779533618268138537203223052339144766435359212612924376a_{21}b_{12}^{8}\\
&+ 339503300252926114857929254770134966666300658424188473654597 92b_{12}^{9},\\
f_4 &= 4196091763511947009497011770511840390597811942750000000000000000a_{31}^{4}\\
&+ 1007024473472318477336913314809162472385973480089193230000000000a_{31}^{2}b_{12}^{3}\\
&- 1202931658308219887576021440917870023330474424551091103542581b_{12}^{6},\\
f_5 &= 9617913044128961826508623766390251295931974383225365765171228894291700000000000000000a_{31}^{4}\\
&+ 240490664152978819978234052280101093876830358741540951676460504724830703604000000000a_{31}^{2}b_{12}^{3}\\
&- 226915475884484993649083006160099247951764715240651441934006791003655409045130 7671b_{12}^{6}.
\end{aligned}
$$

**Acknowledgements:**

Hongwei Li and Yinlai Jin were supported by the National Natural Science Foundation of China(Grant No.11201211) and AMEP of Linyi University.

## References


[1] M.J. Álvarez, A. Gasull, *Momodromy and stability for nilpotent critical points*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **15** (2005), 1253–1265. 1

[2] N. N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, American Math. Soc. Translation, **1954** (1954). no. 100, 19 pp. 1

[3] J. Chavarriga, I. García, J. Giné, *Integrability of centers perturbed by quasi–homogeneous polynomials*, J. Math. Anal. Appl., **211** (1997), 268–278. 1

[4] A. Gasull, J. Torregrosa, *A new algorithm for the computation of the Lyapunov constans for some degenerated critical points*, Nonlin. Anal.: Proc. $\text{III}_{rd}$ World Congress on Nonlinear Analysis, **47** (2001), 4479–4490. 1

[5] W. Huang, A. Chen, *Bifurcation of limit cycles and isochronous centers for a quartic system*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **23** (2013), 10 pages. 1

[6] Y. Liu, J. Li, *On third-order nilpotent critical points: integral factor method*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **21** (2011), 1293–1309. 1, 2, 4

[7] F. Li, Y. Liu, H. Li, *Center conditions and bifurcation of limit cycles at three-order nilpotent critical point in a septic Lyapunov system*, Math. Comput. Simulation, **81** (2011), 2595–2607. 1

[8] F. Li, J. Qiu, J. Li, *Bifurcation of limit cycles, classification of centers and isochronicity for a class of non-analytic quintic systems*, Nonlinear Dynamics, **76** (2014), 183–197. 1

[9] F. Li, M. Wang, *Bifurcation of limit cycles in a quintic system with ten parameters*, Nonlinear Dynamics, **71** (2013), 213–222. 1

[10] F. Li, Y. Wu, *Center conditions and limit cycles for a class of nilpotent-Poincar systems*, Appl. Math. Comput., **243:2** (2014), 114–120. 1

[11] J. Qiu, F. Li, *Two kinds of bifurcation phenomena in a quartic system*, Adv. Difference Equ., **2015** (2015), DOI 10.1186/s13662-015-0365-2. 1

[12] P. Yu, Y. Tian, *Twelve limit cycles around a singular point in a planar cubic-degree polynomial*, Commun. Nonlinear Sci. Numer. Simul., **19** (2014), 2690–2705.