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Some new properties of null curves on 3-null cone and unit semi-Euclidean 3-spheres

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Abstract

The null curves on 3-null cone have the applications in the studying of horizon types. Via the pseudo-scalar product and Frenet equations, the differential geometry of null curves on 3-null cone is obtained. In the local sense, the curvature describes the contact of submanifolds with pseudo-spheres. We introduce the geometric properties of the null curves on 3-null cone and unit semi-Euclidean 3-spheres, respectively. On the other hand, we give the existence conditions of null Bertrand curves on 3-null cone and unit semi-Euclidean 3-spheres.

Keywords: Null Bertrand curve, AW(k)-type curve, Frenet frame, null cone.
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1. Preliminaries

Einstein formulated general relativity as a theory of space, time and gravitation in semi-Euclidean space in 1915. However, this subject has remained dormant for much of its history because its understanding requires advanced mathematics knowledge. Since the end of the twentieth century, semi-Euclidean geometry has been an active area of mathematical research, and it has been applied to a variety of subjects related to differential geometry and general relativity [16]. In the view of physical point, null curves and null surfaces attracted many physicists' attentions [10, 5, 8, 17, 11]. Meanwhile, the study of the differential geometry of null curves on 3-null cone has special interest in Relativity Theory. Penrose R. indicated that null curves on null cone were null geodesics [22]. Meanwhile, C. Kozameh etc. have considered the global topology

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properties of the smooth mapping (light cone cut), which takes the sphere of null directions at a point into future null boundary, via the null geodesics [12]. Conformal geometries can be studied via ambient spaces with two timelike directions (semi-Euclidean space with index two) [4]. However, these applications require signatures $(d, 2)$ where d is the spacetime dimension. In this letter, we mainly consider when $d = 2$, the same method can be applied on null curves in $\mathbb{R}_2^{d+2}(d > 2)$.

Studies related to Bertrand curve date back to the nineteenth century. The classic work in this area is that of Bertrand, who studied curve pairs which have common principal normals in Euclidean space. More recently, besides in Euclidean space [21], there have been a number of studies in other ambient space: in Riemann-Otuski space [27], in Riemann space [20], in Galilean space [19], in Minkowski space [23, 15, 9, 26, 3, 2, 7, 13], in semi-Euclidean space with index two and in non flat space (sphere). P. Lucas and J.A. Ortega-Yagües firstly gave the study of Bertrand curves in Euclidean 3-sphere [15], however, until recently, there has no any consequence about Bertrand curves in semi-Euclidean non flat space. In this letter, we mainly consider null Bertrand curves on 3-null cone and unit semi-Euclidean 3-spheres.

To put it another way, many of the classical Bertrand results from semi-Euclidean geometry focus on spacelike curves and timelike curves [2, 7, 13, 24]. More generally, from the differential geometric point of view, the study of null curves has its own geometric interest, especially for null curves on null cone [3, 9, 23, 25, 20]. H. Balgetir, M. Bektaş etc. have studied the null Bertrand curve and their characterizations in Minkowski 3-space [3]; M. Göçmen and S. Keleş introduced 2-degenerate Bertrand curve in Minkowski space [9]; the current authors have considered null Cartan Bertrand curves in Minkowski space [23]. Comparing with Minkowski space, one of the characterizations of semi-Euclidean space with index two is that there can exist null curves on null cone [20]. To the best of authors' knowledge, there is little information available in literature about null curves on null cone until recently. The main goal of this letter is to obtain some interesting properties of null Bertrand curves on 3-null cone. In order to the enrichment of this article, we also consider the properties of null Bertrand curves in unit semi-Euclidean 3-spheres.

K. Arslan and C. Özgür defined AW(k)-type submanifolds and discussed AW(k)-type curves in Euclidean space [1, 18]; M. Külahcı et al. studied the spacelike curves of AW(k)-type on 3-null cone [14]; recently, the current authors have spread the AW(k)-type theory to null Cartan curves in Minkowski space [23]. On the other hand, it should be noted that most papers and books on AW(k)-type null curves in Minkowski space. The purpose of the present letter is to prove that, in several cases of interest, the AW(k)-type theory can be provided to null curve on 3-null cone, also in unit semi-Euclidean 3-spheres.

The remainder of this letter is organized as follows: Section 2 summarises the required formalism of the basic notions concerning null curves on 3-null cone and unit semi-Euclidean 3-spheres, and the definitions of null Bertrand curves on 3-null cone and unit semi-Euclidean 3-spheres are also introduced. In section 3, we present the main results about null Bertrand curves on 3-null cone and unit semi-Euclidean 3-spheres and obtain the necessary conditions of null Bertrand curves, meanwhile, we obtain that there is no any nonnull Bertrand mates of null curves. Section 4 is devoted to presenting some results relating AW(k)-type null curves on 3-null cone and unit semi-Euclidean 3-spheres. We give some examples to illustrate the null curves and their Bertrand mates in the last section.

We shall assume that all the maps and manifolds in this letter are C^∞, unless the contrary is explicitly stated.

2. Preliminaries

The semi-Euclidean space with index two $(\mathbb{R}_2^n, \langle , \rangle)$ is the vector space $\mathbb{R}^n$ endowed with the metric induced by the pseudo-scalar product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\sum_{i=1}^{2} x_i y_i + \sum_{j=3}^{n} x_j y_j,$$

for two vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$. The non-zero vector $\boldsymbol{x} \in \mathbb{R}_2^n$ is called *spacelike*, *null* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, respectively. The norm of a vector $\boldsymbol{x}$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x}\rangle|}$. A curve $\boldsymbol{\gamma}(t)$ is *spacelike*, *null* or *timelike* if $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle > 0, \langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle = 0$ or $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle < 0$, respectively.

The pseudo vector product of any $n-1$ vectors $\boldsymbol{x}_i = (x_i^1, x_i^2, \dots, x_i^n)$ $(i = 1, 2, \dots, n-1)$ in $\mathbb{R}^n_2$ is defined by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1} = \begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 & \cdots & \boldsymbol{e}_n \\ x_1^1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n-1}^1 & x_{n-1}^2 & x_{n-1}^3 & \cdots & x_{n-1}^n \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^n_2$ and $\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1}\rangle = \det(\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{n-1})$. Thus, we know that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1}$ is pseudo-orthogonal to any $\boldsymbol{x}_i$ $(i = 1, 2, \dots, n-1)$. One can write unit pseudo $(n-1)$-sphere by

$$\mathbb{S}^{n-1}_2 = \{\boldsymbol{x} \in \mathbb{R}^n_2 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\},$$

anti-de Sitter $(n-1)$-space by

$$\mathbb{H}^{n-1}_1 = \{\boldsymbol{x} \in \mathbb{R}^n_2 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1\},$$

open null cone with vertex $\mathbf{0}$ by

$$\Lambda^{n-1}_1 = \{\boldsymbol{x} \in \mathbb{R}^n_2 \setminus \{\mathbf{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0\}.$$

One calls $\mathbb{M}^n_2(c) = \{\boldsymbol{x} \in \mathbb{R}^n_2 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = c\}$ the semi-Euclidean spheres with radius c. $\mathbb{M}^n_2(c)$ is called unit pseudo $(n-1)$-sphere, $(n-1)$-null cone or anti-de Sitter $(n-1)$-space if $c = 1$, $c = 0$ or $c = -1$, respectively. The spheres $\mathbb{M}^{(n-1)}_2(\varepsilon_i)$ $(\varepsilon_i = (-1)^i, i = 1, 2)$ are called unit semi-Euclidean $(n-1)$-spheres.

We can proof that $\boldsymbol{\gamma}^{(k)}(t)(k \geq 2)$ is null vector when curve $\boldsymbol{\gamma}(t)$ and its tangent vector $\boldsymbol{\gamma}'(t)$ are null vector. Hence, we can noted that any order derivatives of the null curves on 3-null cone is null vector.

Let $\boldsymbol{\gamma}(t)$ be a null curve on 3-null cone, then, there exists a *Natural Frenet frame* $\{\boldsymbol{\gamma}(t), \boldsymbol{\xi}(t), \boldsymbol{N}(t), \boldsymbol{W}(t)\}$ satisfying the following equations [6]

$$\begin{cases} \nabla_\xi \boldsymbol{\gamma}(t) = \boldsymbol{\xi}(t) \\ \nabla_\xi \boldsymbol{\xi}(t) = h(t)\boldsymbol{\xi}(t) + k_1(t)\boldsymbol{\gamma}(t) \\ \nabla_\xi \boldsymbol{N}(t) = h(t)\boldsymbol{N}(t) + k_2(t)\boldsymbol{\gamma}(t) - \boldsymbol{W}(t) \\ \nabla_\xi \boldsymbol{W}(t) = -k_2(s)\boldsymbol{\xi}(t) - k_1(t)\boldsymbol{N}(t) \end{cases}, \tag{2.1}$$

where $\boldsymbol{\xi}(t)$ is the tangent vector, $\boldsymbol{N}(t)$ is the unique null transversal vector to $\boldsymbol{\xi}(t)$, $\boldsymbol{W}(t)$ (binormal vector) is the unique null transversal vector to $\boldsymbol{\gamma}(t)$. And a positively oriented 4-tuple of vectors satisfying

$$\langle \boldsymbol{\gamma}(t), \boldsymbol{\gamma}(t)\rangle = \langle \boldsymbol{\gamma}(t), \boldsymbol{\xi}(t)\rangle = \langle \boldsymbol{\gamma}(t), \boldsymbol{N}(t)\rangle = 0,$$

$$\langle \boldsymbol{\xi}(t), \boldsymbol{\xi}(t)\rangle = \langle \boldsymbol{\xi}(t), \boldsymbol{W}(t)\rangle = 0,$$

$$\langle \boldsymbol{N}(t), \boldsymbol{N}(t)\rangle = \langle \boldsymbol{W}(t), \boldsymbol{W}(t)\rangle = \langle \boldsymbol{N}(t), \boldsymbol{W}(t)\rangle = 0,$$

$$\langle \boldsymbol{\xi}(t), \boldsymbol{N}(t)\rangle = \langle \boldsymbol{\gamma}(t), \boldsymbol{W}(t)\rangle = 1,$$

we call Eqs.(2.1) the Nature Frenet equations of null curve $\boldsymbol{\gamma}(t)$ on 3-null cone, where

$$h(t) = \langle \nabla_\xi \boldsymbol{\xi}(t), \boldsymbol{N}(t)\rangle, k_1(t) = \langle \nabla_\xi \boldsymbol{\xi}(t), \boldsymbol{W}(t)\rangle,$$

$$k_2(t) = \langle \nabla_\xi \boldsymbol{N}(t), \boldsymbol{W}(t)\rangle,$$

the functions $h(t), k_1(t)$ and $k_2(t)$ are called the curvature functions of $\boldsymbol{\gamma}$.

By the third chapter in reference [6] (p.63–81), using the same methods, we can obtain the following proposition,

Proposition 2.1. *Let $\boldsymbol{\gamma}(t)$ be a curve on 3-null cone. Then, the type of Natural Frenet equations is invariant to the transformations of the coordinate neighborhood and the screen vector bundle of $\boldsymbol{\gamma}(t)$. And it is possible to find a new parameter on $\boldsymbol{\gamma}$ such that the curvature function $h = 0$ in Frent equation (2.1) of all possible types, using the same screen bundle.*

Proof. By Duggal's method [6], we consider the following differential equation

$$\frac{d^2t}{d\widetilde{t}^2} - \widetilde{h}\frac{dt}{d\widetilde{t}} = 0 \tag{2.2}$$

whose general solution comes from

$$t = a\int_{\widetilde{t}_0}^{\widetilde{t}} \exp(\int_{\widetilde{s}_0}^{\widetilde{s}} \widetilde{h}(\widetilde{t})d\widetilde{t})ds + b.\ a, b \in \mathbb{R} \tag{2.3}$$

It follows that any of these solutions, with $a \neq 0$, might be taken as special parameter on curve $\boldsymbol{\gamma}(t)$ such that $h(t) = 0$. Denote one such solution by $s = \frac{t-b}{a}$, where t is the parameter as defined in above equation. We called s a *distinguished parameter* of $\boldsymbol{\gamma}$, in terms for which $h = 0$. □

Hence, we can get the general Frenet equation of null curve on 3-null cone as following,

$$\begin{cases} \nabla_\xi \boldsymbol{\gamma}(s) = \boldsymbol{\xi}(s) \\ \nabla_\xi \boldsymbol{\xi}(s) = k_1(s)\boldsymbol{\gamma}(s) \\ \nabla_\xi \boldsymbol{N}(s) = k_2(s)\boldsymbol{\gamma}(s) - \boldsymbol{W}(s) \\ \nabla_\xi \boldsymbol{W}(s) = -k_2(s)\boldsymbol{\xi}(s) - k_1(s)\boldsymbol{N}(s) \end{cases}. \tag{2.4}$$

On the other hand, let $\boldsymbol{\gamma}(s)$ be a null curve in unit semi-Euclidean 3-spheres $\mathbb{M}_2^3(\varepsilon_i)$ ($\varepsilon_i = (-1)^i, i = 1, 2$), $\boldsymbol{\gamma}'(s)$ is null vector. Then, as the same as Proposition 2.1, there exists a distinguished parameter s and general Frenet frame $\{\boldsymbol{\gamma}(s), \boldsymbol{\xi}(s), \boldsymbol{N}(s), \boldsymbol{W}(s)\}$ satisfying the following equations

$$\begin{cases} \nabla_\xi \boldsymbol{\gamma}(s) = \boldsymbol{\xi}(s) \\ \nabla_\xi \boldsymbol{\xi}(s) = k_1(s)\boldsymbol{W}(s) \\ \nabla_\xi \boldsymbol{N}(s) = k_2(s)\boldsymbol{W}(s) - \varepsilon_i\boldsymbol{\gamma}(s) \\ \nabla_\xi \boldsymbol{W}(s) = \varepsilon_i k_2(s)\boldsymbol{\xi}(s) + \varepsilon_i k_1(s)\boldsymbol{N}(s) \end{cases}, \tag{2.5}$$

where $\boldsymbol{\xi}(s)$ is the tangent vector, $\boldsymbol{N}(s)$ is the unique null transversal vector to $\boldsymbol{\xi}(s)$, $\boldsymbol{W}(s)$ is unit binormal vector. And a positively oriented 4-tuple of vectors satisfying

$$\langle\boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle = \langle\boldsymbol{W}(s), \boldsymbol{W}(s)\rangle = -\varepsilon_i,$$

$$\langle\boldsymbol{\gamma}(s), \boldsymbol{\xi}(s)\rangle = \langle\boldsymbol{\gamma}(s), \boldsymbol{N}(s)\rangle = \langle\boldsymbol{\gamma}(s), \boldsymbol{W}(s)\rangle = 0,$$

$$\langle\boldsymbol{\xi}(s), \boldsymbol{\xi}(s)\rangle = \langle\boldsymbol{N}(s), \boldsymbol{N}(s)\rangle = \langle\boldsymbol{\xi}(s), \boldsymbol{W}(s)\rangle = \langle\boldsymbol{N}(s), \boldsymbol{W}(s)\rangle = 0,$$

$$\langle\boldsymbol{\xi}(s), \boldsymbol{N}(s)\rangle = 1.$$

We call Eqs.(2.5) the general Frenet equations of null curve $\boldsymbol{\gamma}(s)$ in unit semi-Euclidean 3-spheres, the functions $k_1(s)$ and $k_2(s)$ are called the first and second curvature function of $\boldsymbol{\gamma}$, respectively.

For a curve $\boldsymbol{\gamma}(s)$ on 3-null cone or unit semi-Euclidean 3-spheres with a natural Frenet frame $\{\boldsymbol{\gamma}(s), \boldsymbol{\xi}(s), \boldsymbol{N}(s), \boldsymbol{W}(s)\}$, we call the plane generated by tangent vector $\boldsymbol{\xi}(s)$ and binormal vector $\boldsymbol{W}(s)$ *osculating plane* of $\boldsymbol{\gamma}(s)$.

Definition 2.2. A null curve $\boldsymbol{\gamma}(s)$ on unit semi-Euclidean 3-spheres or 3-null cone is said to be a Bertrand curve if there exists another immersed curve $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ and a one-to-one correspondence between $\boldsymbol{\gamma}(s)$ and $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$, such that both curves have parallel osculating plane at corresponding points. We will say that $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ is a Bertrand mate (or Bertrand partner) of $\boldsymbol{\gamma}(s)$, the curves $\boldsymbol{\gamma}(s)$ and $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ are called a pair of Bertrand curves.

3. Null Bertrand curves on 3-null cone and unit semi-Euclidean 3-spheres

In this section, we mainly consider the differential geometry properties of null Bertrand curves on 3-null cone and unit semi-Euclidean 3-spheres, respectively.

Theorem 3.1. *Let $\boldsymbol{\gamma}(s)$ and $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ be null curves on 3-null cone. $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ is the Bertrand mate of $\boldsymbol{\gamma}(s)$. Then, we can obtain*
(1) $\boldsymbol{\xi}(s)$ is parallel to $\widetilde{\boldsymbol{\xi}}(\widetilde{s})$ and $\boldsymbol{W}(s)$ is parallel to $\widetilde{\boldsymbol{W}}(\widetilde{s})$.
(2) Supposing $k_1(s) \neq 0$, we have $\boldsymbol{\gamma}(s)$ and $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ are the same curve, so to speak, there do not exist any Bertrand mate of $\boldsymbol{\gamma}(s)$ on 3-null cone.
(3) Supposing $k_1(s) = 0$, there exist a continue function $\lambda(s) = \widetilde{s}(s) - s + C$ $(C \in \mathbb{R})$ such that $\widetilde{\boldsymbol{\gamma}}(\widetilde{s}) = \boldsymbol{\gamma}(s) + \lambda(s)\boldsymbol{\xi}(s)$ and $\widetilde{k}_1(\widetilde{s}) = 0$.

Proof. Let $\boldsymbol{\gamma}(s)$ and $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ be a set of Bertrand mates with s the distinguished parameter on $\boldsymbol{\gamma}(s)$. The osculating plane of the null curve $\boldsymbol{\gamma}(s)$ is parallel to $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$. The tangent vector $\boldsymbol{\xi}(s)$ is perpendicular to binormal vector $\boldsymbol{W}(s)$. This implies that there is a function $\theta(s)$ satisfying

$$\widetilde{\boldsymbol{\xi}}(\widetilde{s}) = cos\theta(s)\boldsymbol{\xi}(s) + sin\theta(s)\boldsymbol{W}(s). \tag{3.1}$$

Analytically, $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ can be expressed as

$$\widetilde{\boldsymbol{\gamma}}(\widetilde{s}) = \boldsymbol{\gamma}(s) + \lambda(s)\boldsymbol{\xi}(s) + \mu(s)\boldsymbol{W}(s) \tag{3.2}$$

for two functions $\lambda(s), \mu(s)$. $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ is a null curve on null cone, thus, we can obtain $\mu(s) = 0$. Differentiating Eq.(3.2) with respect to s and using the Frenet formulas Eqs.(2.1),

$$\widetilde{\boldsymbol{\xi}}(\widetilde{s})\frac{d\widetilde{s}}{ds} = (1 + \lambda'(s))\boldsymbol{\xi}(s) + \lambda(s)k_1(s)\boldsymbol{\gamma}(s), \tag{3.3}$$

substitution Eq.(3.1) into Eq.(3.3), it follows that

$$sin\theta(s)\frac{d\widetilde{s}}{ds} = \lambda(s)k_1(s) = 0 \tag{3.4}$$

and

$$cos\theta(s)\frac{d\widetilde{s}}{ds} = 1 + \lambda'(s). \tag{3.5}$$

Considering Eq.(3.4) and Eq.(3.5), since $\frac{d\widetilde{s}}{ds} \neq 0$, we can have $\theta(s) = 0$, the assertion (1) is complete.

Supposing $k_1(s) \neq 0$, $\lambda(s) = 0$ is given by Eq.(3.4). In other words, though Eq.(3.5), the distinguished parameter differential $d\widetilde{s}$ is equate to ds, which proves the assertion (2).

Supposing $k_1(s) = 0$, we can obtain $\lambda(s) = \widetilde{s}(s) - s + C$ $(C \in \mathbb{R})$ by Eq.(3.5). Meanwhile, differentiating Eq.(3.3) to s, we obtain $\widetilde{k}_1(\widetilde{s})(\frac{d\widetilde{s}}{ds})^2 = k_1(s) + 2k_1(s)\lambda(s) + k_1'(s)\lambda(s)$. Thence, our theorem is proved. □

By the Theorem 3.1 and Eqs.(2.1), we can conclude that there exist null Bertrand curve on 3-null cone if and only if the first curvature equate to zero. To put it another way, together with Eqs.(2.5), we have

Theorem 3.2. *Let $\boldsymbol{\gamma}(s)$ be a null curve in unit semi-Euclidean 3-spheres $\mathbb{M}_2^3(\varepsilon_i)$ and $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ be a null curve in unit semi-Euclidean 3-spheres $\mathbb{M}_2^3(\varepsilon_j)$ $(i, j = 1, 2)$. $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ is the Bertrand mate of $\boldsymbol{\gamma}(s)$. Then we can obtain the following conclusions,*
(1) Supposing $k_1(s) \neq 0$, we have that $\boldsymbol{\gamma}(s)$ is the same curve as $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$.
(2) Supposing $k_1(s) = 0$, there exist a continue function $\lambda(s) = C(\widetilde{s}(s) - s \pm \varepsilon_i\sqrt{1 - \varepsilon_i\varepsilon_j}\int k_2(s)ds)$ $(C \in \mathbb{R})$ such that $\widetilde{\boldsymbol{\gamma}}(\widetilde{s}) = \boldsymbol{\gamma}(s) + \lambda(s)\boldsymbol{\xi}(s) \pm \sqrt{1 - \varepsilon_i\varepsilon_j}\boldsymbol{W}(s)$ and $\widetilde{k}_1(\widetilde{s}) = 0$.

Proof. Since $\boldsymbol{\gamma}(s)$ and $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ have parallel osculating planes at corresponding points, we can find there exist two functions $\eta(s)$ and $\zeta(s)$, and then we obtain

$$\widetilde{\boldsymbol{\xi}}(\widetilde{s}) = \eta(s)\boldsymbol{\xi}(s) \tag{3.6}$$

and

$$\widetilde{\boldsymbol{W}}(\widetilde{s}) = \zeta(s)\boldsymbol{\xi}(s) + \boldsymbol{W}(s), \tag{3.7}$$

where $\{\boldsymbol{\xi}(s), \boldsymbol{W}(s)\}$ constitute the osculating plane along $\boldsymbol{\gamma}(s)$ and $\{\widetilde{\boldsymbol{\xi}}(\widetilde{s}), \widetilde{\boldsymbol{W}}(\widetilde{s})\}$ constitute the osculating plane along $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$. On the other hand, the expression of $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ is as

$$\widetilde{\boldsymbol{\gamma}}(\widetilde{s}) = \boldsymbol{\gamma}(s) + \lambda(s)\boldsymbol{\xi}(s) + \mu(s)\boldsymbol{W}(s), \tag{3.8}$$

where $\lambda(s), \mu(s)$ are two real functions. We obtain $\mu(s) = \pm\sqrt{1 - \varepsilon_i\varepsilon_j}$ by $\widetilde{\boldsymbol{\gamma}}(\widetilde{s}) \in \mathbb{M}_2^3(\varepsilon_j)$. Differentiation the last equation to s, we arrive at the following equation

$$\widetilde{\boldsymbol{\xi}}(\widetilde{s})\frac{d\widetilde{s}}{ds} = (1 + \lambda'(s) + \varepsilon_i\mu k_2(s))\boldsymbol{\xi}(s) + \lambda(s)k_1(s)\boldsymbol{W}(s) + \varepsilon_i\mu k_1(s)\boldsymbol{N}(s). \tag{3.9}$$

Substitution Eq.(3.6) into Eq.(3.9), we can obtain the following equations

$$\eta(s)\frac{d\widetilde{s}}{ds} = 1 + \lambda'(s) + \varepsilon_i\mu k_2(s), \tag{3.10}$$

$$\lambda(s)k_1(s) = 0, \tag{3.11}$$

$$\mu k_1(s) = 0. \tag{3.12}$$

As the same computation, differentiation the Eq.(3.9), which gives

$$\widetilde{k}_1(\widetilde{s})\zeta(s)(\frac{d\widetilde{s}}{ds})^2 + \eta(s)\frac{d^2\widetilde{s}}{ds^2} = \lambda''(s) + \varepsilon_i\mu(s)k_2'(s) + \varepsilon_i\lambda(s)k_1(s)k_2(s) \tag{3.13}$$

$$\lambda(s)k_1^2(s) + \mu k_1'(s) = 0, \tag{3.14}$$

$$\widetilde{k}_1(\widetilde{s})(\frac{d\widetilde{s}}{ds})^2 = k_1(s) + 2k_1(s)\lambda'(s) + 2\varepsilon_i\mu k_1(s)k_2(s) + \lambda(s)k_1'(s). \tag{3.15}$$

Supposing $k_1(s) \neq 0$, in the view of Eq.(3.11) and Eq.(3.12), it is easy to get that $\lambda(s) = \mu = 0$, that jointly with Eq.(3.10) and Eq.(3.15) yields $\boldsymbol{\gamma}(s)$ is the same curve as $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$.

Next, we assume $k_1(s) = 0$, a straight forward substitution shows that $\widetilde{k}_1(\widetilde{s}) = 0$ by Eq.(3.15). Substitution $k_1(s) = \widetilde{k}_1(\widetilde{s}) = 0$ into Eqs.(3.10) and (3.13), in terms of a simple computing, we obtain $\eta(s) = constant$ and $\lambda(s) = C(\widetilde{s}(s) - s \pm \varepsilon_i\sqrt{1 - \varepsilon_i\varepsilon_j}\int k_2(s)ds)$ $(C \in \mathbb{R})$, the proof is finished. □

For a null curve $\boldsymbol{\gamma}(s)$ on 3-null cone, the osculating plane is generated by two orthorhombic null vectors $\boldsymbol{\xi}(s)$ and $\boldsymbol{W}(s)$. Any way, we can put our attentions on the fact that all vectors in osculating plane are null vectors. As the same, for a null curve $\boldsymbol{\gamma}(s)$ in unit semi-Euclidean 3-spheres, the osculating plane of which contains null vector $\boldsymbol{\xi}(s)$ and unit vector $\boldsymbol{W}(s)$. If $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ is the Bertrand mate of $\boldsymbol{\gamma}(s)$, excepting null curve in unit semi-Euclidean 3-spheres, $\widetilde{\boldsymbol{\gamma}}(\widetilde{s})$ can be a nonnull curve on 3-null cone, However, we can find contradiction by computing. Here and hereafter, the following fact is correct.

Theorem 3.3. *There is no any nonnull Bertrand mate for null curve on 3-null cone and unit semi-Euclidean 3-spheres. The Bertrand mates of null curves on 3-null cone are also belonged to 3-null cone, however, the Bertrand mates of null curves in anti de-Sitter 3-space can be in unit pseudo 3-sphere, the converse is also true.*

4. AW(k)-type null curves on 3-null cone and unit semi-Euclidean 3-spheres

Let $\boldsymbol{\gamma}: I \to \mathbb{R}^n_2$ be a curve in $\mathbb{R}^n_2$. The curve $\boldsymbol{\gamma}(s)$ is a Frenet curve of osculating order d when its higher order derivatives $\boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \ldots, \boldsymbol{\gamma}^{(d)}(s)$ are linearly independent, and $\boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \ldots, \boldsymbol{\gamma}^{(d+1)}(s)$ are no longer independent for all $s \in I$. Each Frenet curve of osculating order d is associated with an orthonormal d-frame $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d$ along $\boldsymbol{\gamma}(s)$ (such that $\boldsymbol{\gamma}'(s) = \boldsymbol{v}_1$) known as the Frenet frame as well as the functions $k_1(s), k_2(s), \ldots, k_{d-1}(s): I \to \mathbb{R}$ known as Frenet curvatures [1].

Proposition 4.1. *Let $\boldsymbol{\gamma}(s)$ be a null Cartan curve in unit semi-Euclidean 3-spheres, thus, we have*

$$\boldsymbol{\gamma}'(s) = \boldsymbol{\xi}(s),$$

$$\boldsymbol{\gamma}''(s) = \nabla_\xi \boldsymbol{\xi}(s) = k_1(s)\boldsymbol{W}(s),$$

$$\boldsymbol{\gamma}'''(s) = \varepsilon_i k_1(s)k_2(s)\boldsymbol{\xi}(s) + \varepsilon_i k_1^2(s)\boldsymbol{N}(s) + k_1'(s)\boldsymbol{W}(s),$$

$$\boldsymbol{\gamma}''''(s) = -k_1^2(s)\boldsymbol{\gamma}(s) + \varepsilon_i(2k_1'(s)k_2(s) + k_1(s)k_2'(s))\boldsymbol{\xi}(s) + 3\varepsilon_i k_1(s)k_1'(s)\boldsymbol{N}(s) + (k_1''(s) + 2\varepsilon_i k_1^2(s)k_2(s))\boldsymbol{W}(s)$$

Notation. Let us write

$$\boldsymbol{N}_1(s) = k_1(s)\boldsymbol{W}(s), \tag{4.1}$$

$$\boldsymbol{N}_2(s) = \varepsilon_i k_1^2(s)\boldsymbol{N}(s) + k_1'(s)\boldsymbol{W}(s), \tag{4.2}$$

$$\boldsymbol{N}_3(s) = 3\varepsilon_i k_1(s)k_1'(s)\boldsymbol{N}(s) + (k_1''(s) + 2\varepsilon_i k_1^2(s)k_2(s))\boldsymbol{W}(s). \tag{4.3}$$

Remark 4.2. $\boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s)$ and $\boldsymbol{\gamma}''''(s)$are linearly dependent if and only if $\boldsymbol{N}_1(s), \boldsymbol{N}_2(s)$ and $\boldsymbol{N}_3(s)$ are linearly dependent.

As the definition of AW(k)-type curves in [1, 14, 23],

Definition 4.3. Null curves are
(¡) of type AW(1) if they satisfy $\boldsymbol{N}_3(s) = \boldsymbol{0}$.
(¡¡) of type AW(2) if they satisfy

$$\|\boldsymbol{N}_2(s)\|^2\boldsymbol{N}_3(s) = \langle \boldsymbol{N}_3(s), \boldsymbol{N}_2(s)\rangle \boldsymbol{N}_2(s). \tag{4.4}$$

(¡¡¡) of type AW(3) if they satisfy

$$\|\boldsymbol{N}_1(s)\|^2\boldsymbol{N}_3(s) = \langle \boldsymbol{N}_3(s), \boldsymbol{N}_1(s)\rangle \boldsymbol{N}_1(s). \tag{4.5}$$

By definition of $AW(k)$-type curve, the choose of the vectors $\boldsymbol{N}_1(s), \boldsymbol{N}_2(s), \boldsymbol{N}_3(s)$ is essential. Furthermore, to null curve on 3-null cone, thought Eqs.(2.1), there do not exist three vectors satisfying the Remark 4.2. Therefore, it states that

Theorem 4.4. *There is no any $AW(k)$-type null curve on 3-null cone* $(k = 1, 2, 3)$.

In the following text, we consider the $AW(k)$-type null curve on unit semi-Euclidean 3-spheres $\mathbb{M}^3_2(\varepsilon_i)$ and the specify conditions that must be true for the requirements to be satisfied.

Theorem 4.5. *Let $\boldsymbol{\gamma}(s)$ be a null curve on $\mathbb{M}^3_2(\varepsilon_i)$ $(i = 1, 2)$. Thus, $\boldsymbol{\gamma}(s)$ is $AW(1)$-type null curve if and only if $k_1(s) = 0$ or $k_1(s) = constant, k_2(s) = 0$.*

Proof. If $\boldsymbol{\gamma}(s)$ is an $AW(1)$-type null curve, from Eq.(4.3), we can directly come to a conclusion. The converse statement is trivial. The proof is complete. □

Theorem 4.6. *Let $\boldsymbol{\gamma}(s)$ be a null curve in $\mathbb{M}^3_2(\varepsilon_i)$ $(i = 1, 2)$. Thus, $\boldsymbol{\gamma}(s)$ is $AW(2)$-type null curve if and only if $k_1(s) = constant$ or*

$$k_1(s)k_1''(s) - 3(k_1'(s))^2 + 2\varepsilon_i k_1^3(s)k_2(s) = 0. \tag{4.6}$$

Proof. Since $\boldsymbol{\gamma}(s)$ is an $AW(2)$-type null curve, in the view of the fact that $\|\boldsymbol{N}_2(s)\|^2 = -\varepsilon_i(k_1'(s))^2$, it follows from Eq.(4.5) that

$$3k_1(s)(k_1'(s))^3 = k_1'(s)k_1^2(s)(k_1''(s) + 2\varepsilon_i k_1^2(s)k_2(s))$$

Apart from outside $k_1'(s) = 0$, we can have the Eq.(4.6) and our theorem is proved. □

Through simple computing as above theorem, we can have the following theorem.

Theorem 4.7. *Let $\boldsymbol{\gamma}(s)$ be a null curve in $\mathbb{M}_2^3(\varepsilon_i)$ $(i = 1, 2)$. Thus, $\boldsymbol{\gamma}(s)$ is $AW(3)$-type null curve if and only if $k_1(s) = constant$.*

From the above three theorems, we can obtain following corollary,

Corollary 4.8. *Let $\boldsymbol{\gamma}(s)$ be an $AW(1)$-type null curve in $\mathbb{M}_2^3(\varepsilon_i)$ $(i = 1, 2)$, and that, $\boldsymbol{\gamma}(s)$ is an $AW(j)$-type null curve $(j = 2, 3)$. However, the reverse is not set up.*

In conclusion, the application in together with Bertrand curve theory and $AW(k)$ theory to null curves on 3-null cone and null curve in unit semi-Euclidean 3-spheres, we can obtain the significant conclusion,

Theorem 4.9. *Let $\boldsymbol{\gamma}(s)$ be a null curve on 3-null cone or unit semi-Euclidean 3-spheres. Thus,*
(1) there do not exist any null Bertrand AW(k)-type curves on 3-null cone $(k = 1, 2, 3)$.
(2) null Bertrand AW(k)-type curves in unit semi-Euclidean 3-spheres should satisfy $k_1(s) = 0$.

5. Examples

In this section, we give two examples about the null curves on 3-null cone and unit semi-Euclidean 3-spheres, respectively. The graphics of those null curves and the Bertrand mates are given in the following text.

Example 5.1. Let $\boldsymbol{\gamma}(s)$ be a null curve on 3-null cone with the general Frenet frame $\{\boldsymbol{\gamma}, \boldsymbol{\xi}, \boldsymbol{N}, \boldsymbol{W}\}$, where

$$\boldsymbol{\gamma}(s) = \{\gamma_1(s), \gamma_2(s), \gamma_3(s), \gamma_4(s)\} = \{\frac{1}{3}s^3 - s, \frac{1}{3}s^3 + s, \frac{\sqrt{2}}{3}s^3, \sqrt{2}s\},$$
$$\boldsymbol{\xi}(s) = \{s^2 - 1, s^2 + 1, \sqrt{2}s^2, \sqrt{2}\},$$
$$\boldsymbol{N}(s) = -\frac{1}{4}\{s^2 - 1, s^2 + 1, \sqrt{2}s^2, -\sqrt{2}\},$$
$$\boldsymbol{W}(s) = \{\frac{(6 + 2s^2)b + \sqrt{2}s}{6 - 2s^2 + 2s}, b, \frac{\sqrt{2}(3 + s^2)b + s^2 - 3}{6 - 2s^2 + 2s}, \frac{\sqrt{2}}{2s}\},$$

where

$$b = \frac{\sqrt{2}s^5 - 2\sqrt{2}s^4 - 6\sqrt{2}s^2 - 9\sqrt{2}s + 2\sqrt{(2t^6 + t^4 - 4t^3 + 14t^2 + 12t + 27)(t^2 - t - 3)^2}}{2(3s^4 - 4s^3 - 4s^2 + 12s + 27)s},$$

and we can obtain the Bertrand curve of $\boldsymbol{\gamma}(s)$ as $\widetilde{\boldsymbol{\gamma}} = \{\frac{4}{3}s^3 - 2s, \frac{4}{3}s^3 + 2s, \frac{4\sqrt{2}}{3}s^3, 2\sqrt{2}s\}$. The graphics of the null curve $\boldsymbol{\gamma}(s)$ and its Bertrand curve are following (Fig.1-Fig.3).

Example 5.2. Let $\boldsymbol{\gamma}(s)$ be a null curve on unit semi-Euclidean 3-sphere $\mathbb{M}_2^3(-1)$ (Anti de Sitter 3-space) with the general Frenet frame $\{\boldsymbol{\gamma}, \boldsymbol{\xi}, \boldsymbol{N}, \boldsymbol{W}\}$, where

$$\boldsymbol{\gamma}(s) = \{s^2, 2, s^2, \sqrt{3}\},$$
$$\boldsymbol{\xi}(s) = \{2s, 0, 2s, 0\}$$
$$\boldsymbol{N}(s) = \{\frac{s^3}{4} - \frac{1}{4s} - \frac{1}{4s^5}, s - \frac{\sqrt{3}}{2s^3}, \frac{s^3}{4} + \frac{1}{4s} - \frac{1}{4s^5}, \frac{\sqrt{3}}{2}s - \frac{1}{s^3}\}$$
$$\boldsymbol{W}(s) = \{\frac{1}{s^2}, \sqrt{3}, \frac{1}{s^2}, 2\}$$

we can obtain the Bertrand curve of $\boldsymbol{\gamma}(s)$ as

$$\widetilde{\boldsymbol{\gamma}} = \{-s^2 - 4s^{-4} + \sqrt{2}s^{-2}, 2 + \sqrt{6}, -s^2 - 4s^{-4} + \sqrt{2}s^{-2}, \sqrt{3} + 2\sqrt{2}\}.$$

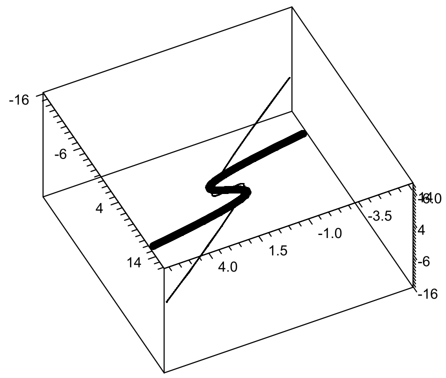

Fig. 1: the planform of the project graphics along the direction of $\gamma_2(s)$,the bold curve is $\widetilde{\boldsymbol{\gamma}}$

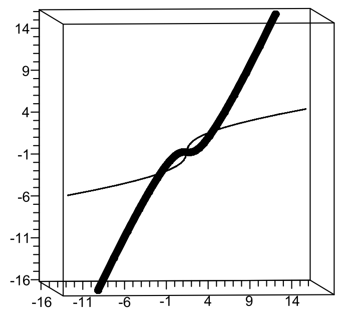

Fig. 2: the front view of the project graphics along the direction of $\gamma_2(s)$, the bold curve is $\widetilde{\boldsymbol{\gamma}}$

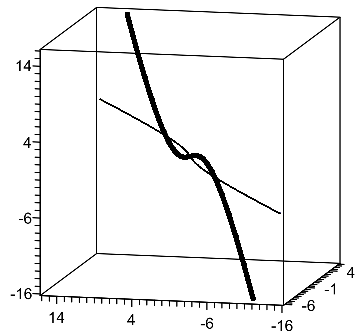

Fig. 3: the side view of the project graphics along the direction of $\gamma_2(s)$, the bold curve is $\widetilde{\boldsymbol{\gamma}}$

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