Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 467–477
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Singularity properties of one parameter lightlike hypersurfaces in Minkowski 4-space

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Communicated by C. Park

Abstract

In this paper, we give one parameter families of extrinsic differential geometries on spacelike curves in Minkowski 4-space. We investigate the nonlinear properties of one parameter lightlike hypersurfaces. Meanwhile, the classification of singularities to one parameter lightlike hypersurfaces is considered by singularity theory.

Keywords: Minkowski space, singularity, one parameter lightlike hypersurfaces, height function.
2010 MSC: 58K27, 53B50, 53A35.

1. Introduction

Minkowski space is a real vector space with a symmetric bilinear form. And Minkowski space form with the positive curvature is called de Sitter space. We know that de Sitter 3-space is a vacuum solution of the Einstein equation and an important cosmological model for physical universe [1, 4, 6, 10, 11]. Authors have given the geometrical properties of spacelike or timelike curves in Minkowski space [6, 8, 12]; B. Mustafa obtained the geometrical properties of involutes of spacelike curves in Minkowski 3-space [9]. However, most of papers and books studying the geometrical properties of general surfaces generated by spacelike curves in Minkowski 4-space nor their hypersurfaces. Authors had obtained the horizon of the black hole is a lightlike hypersurface or a part [3, 7]. In this paper, we consider, however, the one parameter lightlike hypersurfaces, which are generated by spacelike curves in de Sitter 3-space, as the most elementary case for the study of the lowest codimensional submanifolds in non-flat Lorentzian space forms.

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Received 2015-2-26

On the other hand, singularity theory, which is a direct descendant of differential calculus, is certain to have a great deal of interest to say about geometry, equation, physic, astronomy and other disciplines [2, 4, 14]. In general, the current theory always does not allow for singularities, however, it is unavoidable in some real life circumstances. Thus, we apparently need to understand the ontology of singularities if we want to research the nature of space and time in the actual universe. By now, the studying of singularities has been concentrated in general surfaces [2, 6, 12, 13].

Meanwhile, the one parameter lightlike hypersurfaces are a bundle along a spacelike curve whose fibres are lightlike lines or spacelike curves. The most interesting case is the contact of spacelike curves and lightcone. Moreover, from the point of view of physics, lightlike hypersurfaces are of importance because they are models of different types of horizons studied in relativity theory [5, 7, 16, 17]. The authors considered a classification of the singularities of lightlike surfaces with codimensional two for generic spacelike curves in de Sitter 3-space and a geometric characterization of the singularities [6]. Except for the difference that we consider the one parameter lightlike hypersurfaces of spacelike curves and the geometric characterizations of their singularities.

The remainder of this paper is organized as follows: Section 2 reviews some basic notions about the Minkowski space and gives the main result about the classifications of singularities (Theorem 2.2). Section 3 considers the one parameter spacelike height function on spacelike curves. Also, the versal property of one parameter height function is used to prove Theorem 2.2 in Section 4. Section 5 gives the generic properties of spacelike curves to introduce the stability of singularity. In the last section of this paper, we supply an example to explain the singular locus of one parameter lightlike hypersurfaces.

We shall assume that all the maps and manifolds in this paper are C^∞, unless the contrary is explicitly stated.

2. Preliminaries and the main result

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) | x_i \in \mathbb{R}\ (i = 1, 2, 3, 4)\ \}$ be a 4-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ and $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the *symmetric bilinear form* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$. $(\mathbb{R}^4, \langle , \rangle)$ is called *four dimensional Minkowski space* and written by $\mathbb{R}^4_1$.

A vector $\boldsymbol{x}$ in $\mathbb{R}^4_1 \setminus \{\boldsymbol{0}\}$ is called a *spacelike vector*, a *lightlike vector* or a *timelike vector* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ is positive, zero or negative, respectively. The *norm* of $\boldsymbol{x} \in \mathbb{R}^4_1$ is defined by $\|\boldsymbol{x}\| = (\mathrm{sign}(\boldsymbol{x})\langle \boldsymbol{x}, \boldsymbol{x} \rangle)^{1/2}$, where $\mathrm{sign}(\boldsymbol{x})$ denotes the signature of $\boldsymbol{x}$ which is given by $\mathrm{sign}(\boldsymbol{x})$=1, 0 or -1 when $\boldsymbol{x}$ is a spacelike, lightlike or timelike vector [15]. For any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^4_1$, we say that $\boldsymbol{x}$ is *pseudo-perpendicular* to $\boldsymbol{y}$ if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$. For vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ and $\boldsymbol{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{R}^4_1$, we define a vector $\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z}$ by

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & -\mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is the canonical base of $\mathbb{R}^4_1$. One can easily show that

$$\langle \boldsymbol{a}, \boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} \rangle = \det(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}).$$

In $\mathbb{R}^4_1$, we introduce some typical manifolds,

$$\textit{de Sitter space} \quad \mathbb{S}^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\},$$

$$\textit{hyperbolic space} \quad \mathbb{H}^3 = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

$$\textit{lightcone} \quad \mathbb{LC}^* = \{\boldsymbol{x} \in \mathbb{R}^4_1 \setminus \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\},$$

$$\textit{one paramater de Sitter space} \quad \mathbb{S}^3_1(\sin^2\varphi) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \sin^2\varphi\}, \varphi \in [0, \pi/2].$$

For a vector $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in \mathbb{LC}^*$, if $v_4 = 1$, vector $\boldsymbol{v}$ is denoted by $\widetilde{\boldsymbol{v}}$.

Let $\boldsymbol{\gamma} : I \to \mathbb{R}^4_1$ by $\boldsymbol{\gamma}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be a regular curve in $\mathbb{R}^4_1$ (*i.e.*, $\dot{\boldsymbol{\gamma}}(t) \neq 0$ for any $t \in I$), where I is an open interval. For any $t \in I$, the curve $\boldsymbol{\gamma}$ is called a spacelike curve, a lightlike curve or a timelike curve if $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle > 0, \langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle = 0$ or $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle < 0$, respectively. The arc-length of a nonlightlike curve $\boldsymbol{\gamma}(t)$ measured from $\boldsymbol{\gamma}(t_0)$ $(t_0 \in I)$ is

$$s(t) = \int_{t_0}^{t} \|\dot{\boldsymbol{\gamma}}(t)\| dt.$$

The parameter s is determined as $\|\boldsymbol{\gamma}'(s)\| = 1$ for a nonlightlike curve, where $\boldsymbol{\gamma}'(s) = (d\boldsymbol{\gamma}/ds)(s)$. For a spacelike curve $\boldsymbol{\gamma}(s)$ in de Sitter 3-space, there are a spacelike tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ and a normal vector $\boldsymbol{n}_1(s)$. A new unit normal vector as following:

$$\boldsymbol{n}_2(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}_1(s)/\|\boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}_1(s)\|,$$

where $\mathrm{sign}(\boldsymbol{n}_1(s))\mathrm{sign}(\boldsymbol{n}_2(s)) = -1$. In the following, we only consider normal vector $\boldsymbol{n}_1(s)$ is spacelike and normal vector $\boldsymbol{n}_2(s)$ is timelike. The other case is the same. Then we have a pseudo-orthonormal frame $\{\boldsymbol{\gamma}(s),\ \boldsymbol{t}(s),\ \boldsymbol{n}_1(s),\ \boldsymbol{n}_2(s)\}$ satisfying

$$\langle \boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle = \langle \boldsymbol{t}(s), \boldsymbol{t}(s)\rangle = \langle \boldsymbol{n}_1(s), \boldsymbol{n}_1(s)\rangle = -\langle \boldsymbol{n}_2(s), \boldsymbol{n}_2(s)\rangle = 1,$$

$$\langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s)\rangle = \langle \boldsymbol{\gamma}(s), \boldsymbol{n}_1(s)\rangle = \langle \boldsymbol{\gamma}(s), \boldsymbol{n}_2(s)\rangle = 0,$$
$$\langle \boldsymbol{t}(s), \boldsymbol{n}_1(s)\rangle = \langle \boldsymbol{t}(s), \boldsymbol{n}_2(s)\rangle = \langle \boldsymbol{n}_1(s), \boldsymbol{n}_2(s)\rangle = 0$$

and the Frenet type formulas as following:

$$\begin{cases} \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) = -\boldsymbol{\gamma}(s) - \kappa_g(s)\boldsymbol{n}_1(s) \\ \boldsymbol{n}_1'(s) = k_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_2(s) \\ \boldsymbol{n}_2'(s) = -\tau_g(s)\boldsymbol{n}_1(s) \end{cases}. \tag{2.1}$$

If $k_g(s) = 0$, one can obtain $\boldsymbol{t}'(s) = -\boldsymbol{\gamma}(s)$ and $\boldsymbol{\gamma}'(s) = \boldsymbol{t}(s)$, so $\boldsymbol{\gamma}(s) = \cos s + \sin s$ is a plane fixed curve. Therefore, we only consider $k_g(s)$ does not equal to zero in the following sections. For $\langle \boldsymbol{t}(s), \boldsymbol{\gamma}(s)\rangle = 0$, we have $\langle \boldsymbol{t}'(s) + \boldsymbol{\gamma}(s), \boldsymbol{t}(s)\rangle = 0$, $\langle \boldsymbol{t}'(s) + \boldsymbol{\gamma}(s), \boldsymbol{t}'(s) + \boldsymbol{\gamma}(s)\rangle = \langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle - 1 \neq 0$ and $\langle \boldsymbol{t}'(s) + \boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle = 0$. One denotes the unit normal vector as

$$\boldsymbol{n}_1(s) = (\boldsymbol{t}'(s) + \boldsymbol{\gamma}(s))/\|\boldsymbol{t}'(s) + \boldsymbol{\gamma}(s)\|$$

and

$$\boldsymbol{n}_2(s) = (\boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}_1(s))/\|\boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}_1(s)\|,$$

where $k_g(s) = \|\boldsymbol{t}'(s) + \boldsymbol{\gamma}(s)\|$ and $\tau_g(s) = (1/k_g^2(s))det(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s))$.

Let $\boldsymbol{\gamma} : I \to \mathbb{S}^3_1$ be a unit speed spacelike curve, we define a map $\mathbb{L}^{\pm}_{\varphi} : I \times \mathbb{R} \times [0, \pi/2] \to \mathbb{S}^3_1(\sin^2\varphi)$ by

$$\mathbb{L}^{\pm}_{\varphi}(s, \mu) = \sin\varphi\boldsymbol{\gamma}(s) + \mu\cos\varphi(\widetilde{\boldsymbol{n}_1(s) \pm \boldsymbol{n}_2}(s)).$$

We call the image of $\mathbb{L}^{\pm}_{\varphi}$ the *one parameter lightlike hypersurfaces* associated to the spacelike curve $\boldsymbol{\gamma}(s)$.

Remark 2.1. When $\varphi = 0$, the hypersurface $\mathbb{L}^{\pm}_{\varphi}$ is a lightlike vector in lightcone. Hence, we only consider $\varphi \in (0, \pi/2]$ in the next section.

Let $F : \mathbb{S}^3_1 \longrightarrow \mathbb{R}$ be a submersion and $f : I \longrightarrow \mathbb{S}^3_1$ be a spacelike curve. We say that f and $F^{-1}(0)$ have *k-point* contact at $t = t_0$ if the function $g(t) = F \circ f(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$ and $g^{(k)}(t_0) \neq 0$. We also say that f and $F^{-1}(0)$ have at least *k-point* contact at $t = t_0$ if the function $g(t) = F \circ f(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$. The main result in this paper as following:

Theorem 2.2. *Let $\boldsymbol{\gamma}(s)$ be a unit speed spacelike curve in de Sitter 3-space, $\boldsymbol{v}_0 = \mathbb{L}^{\pm}_{\varphi}(s_0, \mu_0)$ and $LC(\varphi)(\boldsymbol{v}_0) = \{\boldsymbol{u} \in \mathbb{S}^3_1 \mid \langle \boldsymbol{u}, \boldsymbol{v}_0\rangle = \sin\varphi\}$, we have the following:*

1. *$\boldsymbol{\gamma}(s)$ and $LC(\varphi)(\boldsymbol{v}_0)$ have at least 2-point contact for s_0.*
2. *$\boldsymbol{\gamma}(s)$ and $LC(\varphi)(\boldsymbol{v}_0)$ have at least 3-point contact for s_0 if and only if*

$$\boldsymbol{v}_0 = \sin\varphi\boldsymbol{\gamma}(s_0) - (\tan\varphi/k_g(s_0))\cos\varphi(\widetilde{\boldsymbol{n}_1(s_0)\pm\boldsymbol{n}_2(s_0)})$$

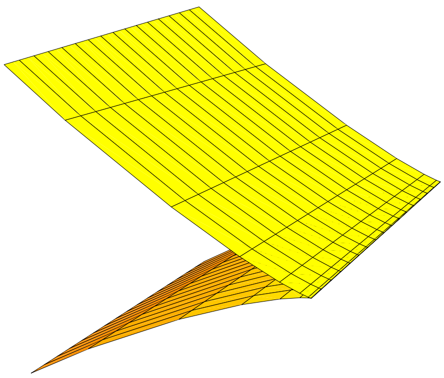
and $k'_g(s) \pm k_g(s)\tau_g(s) \neq 0$. Under this condition, the germ of image $\mathbb{L}^{\pm}_{\varphi}$ at $\mathbb{L}^{\pm}_{\varphi}(s_0, \lambda_0)$ is diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ (Fig. 1).

3. *$\boldsymbol{\gamma}(s)$ and $LC(\varphi)(\boldsymbol{v}_0)$ have at least 4-point contact for s_0 if and only if*

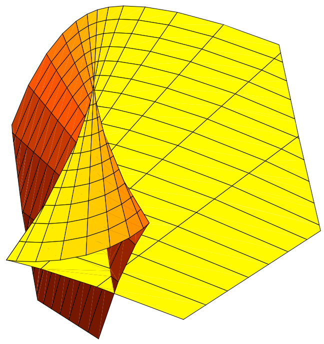
$$\boldsymbol{v}_0 = \sin\varphi\boldsymbol{\gamma}(s_0) - (\tan\varphi/k_g(s_0))\cos\varphi(\widetilde{\boldsymbol{n}_1(s_0)\pm\boldsymbol{n}_2(s_0)})$$

and $k'_g(s)\pm k_g(s)\tau_g(s) = 0, (k'_g(s)\pm k_g(s)\tau_g(s))' \neq 0$. Under this condition, the germ of image $\mathbb{L}^{\pm}_{\varphi}$ at $\mathbb{L}^{\pm}_{\varphi}(s_0, \lambda_0)$ is diffeomorphic to the swallowtail SW (Fig. 2).

Here $C\times\mathbb{R} = \{(x_1, x_2, x_3) \mid x_1 = u, x_2 = \pm v^{1/2}, x_3 = v^{1/3}\}$ is the cuspidal edge and $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallowtail.



cuspidaledge
Fig. 1



swallowtail
Fig. 2

3. One parameter spacelike height function

Let $\boldsymbol{\gamma}(s)$ be a unit speed spacelike curve in de Sitter 3-space, for any parameter $\varphi \in (0, \pi/2]$, we define a function $H : I \times \mathbb{S}^3_1(\sin^2\varphi) \times (0, \pi/2] \to \mathbb{R}$ by

$$H(s, \boldsymbol{v}, \varphi) = \langle\boldsymbol{\gamma}(s), \boldsymbol{v}\rangle - \sin\varphi,$$

which is called *one parameter spacelike height function* of $\boldsymbol{\gamma}(s)$. Denoted $h_{v,\varphi}(s) = H(s, \boldsymbol{v}, \varphi)$ for any fixed vector $(\boldsymbol{v}, \varphi) \in \mathbb{S}^3_1(\sin^2\varphi) \times (0, \pi/2]$.

Proposition 3.1. *Suppose $\boldsymbol{\gamma}(s)$ is a unit speed spacelike curve in de Sitter 3-space with $k_g(s) \neq 0$ and $(\boldsymbol{v}, \varphi) \in \mathbb{S}^3_1(\sin^2\varphi) \times (0, \pi/2]$. Then,*

1. *$h_{v,\varphi}(s) = h'_{v,\varphi}(s) = 0$ if and only if there exists a real number λ such that $\boldsymbol{v} = \sin\varphi\boldsymbol{\gamma}(s)+\lambda(\widetilde{\boldsymbol{n}_1(s)\pm\boldsymbol{n}_2(s)})$.*

2. $h_{v,\varphi}(s) = h'_{v,\varphi}(s) = h''_{v,\varphi}(s) = 0$ *if and only if*

$$\boldsymbol{v} = \sin\varphi\boldsymbol{\gamma}(s) + (\tan\varphi/k_g(s))\cos\varphi(\widetilde{\boldsymbol{n}_1(s) \pm \boldsymbol{n}_2(s)})$$

and $\mu = \sin\varphi/k_g(s)$.

3. $h_{v,\varphi}(s) = h'_{v,\varphi}(s) = h''_{v,\varphi}(s) = h^{(3)}_{v,\varphi}(s) = 0$ *if and only if*

$$\boldsymbol{v} = \sin\varphi\boldsymbol{\gamma}(s) + (\tan\varphi/k_g(s))\cos\varphi(\widetilde{\boldsymbol{n}_1(s) \pm \boldsymbol{n}_2(s)})$$

and $k'_g(s) \pm k_g(s)\tau_g(s) = 0$, $(k'_g(s) \pm k_g(s)\tau_g(s))' \neq 0$.

4. $h_{v,\varphi}(s) = h'_{v,\varphi}(s) = h''_{v,\varphi}(s) = h^{(3)}_{v,\varphi}(s) = h^{(4)}_{v,\varphi}(s) = 0$ *if and only if*

$$\boldsymbol{v} = \sin\varphi\boldsymbol{\gamma}(s) + (\tan\varphi/k_g(s))\cos\varphi(\widetilde{\boldsymbol{n}_1(s) \pm \boldsymbol{n}_2(s)})$$

and $k'_g(s) \pm k_g(s)\tau_g(s) = 0$, $(k'_g(s) \pm k_g(s)\tau_g(s))' = 0$.

Proof.

1. Supposing there are three real numbers η, ω, λ satisfying $\boldsymbol{v} = \eta\boldsymbol{\gamma}(s) + \omega\boldsymbol{t}(s) + \lambda(\widetilde{\boldsymbol{n}_1(s) + \boldsymbol{n}_2(s)}) \in \mathbb{S}^3_1(\sin^2\varphi)$, we obtain $\omega = 0$ and $\eta = \sin\varphi$ by $h_{v,\varphi}(s) = h'_{v,\varphi}(s) = 0$. Therefore, the assertion 1 follows.

2. The easy computation that

$$h''_{v,\varphi}(s) = \langle \boldsymbol{\gamma}''(s), \boldsymbol{v}\rangle = \langle -\gamma(s) + k_g(s)\boldsymbol{n}_1(s), \boldsymbol{v}\rangle = 0. \tag{3.1}$$

Substituting the condition $\boldsymbol{v} = \sin\varphi\boldsymbol{\gamma}(s) + \lambda(\widetilde{\boldsymbol{n}_1(s) \pm \boldsymbol{n}_2(s)})$ and using Equations (2.1), we obtain $\lambda = \sin\varphi/k_g(s)$ and $\mu = \tan\varphi/k_g(s)$. Hence, the assertion 2 holds.

3. Basing on the above assumption and using Equations (2.1), we have

$$\begin{aligned} h^{(3)}_{v,\varphi}(s) &= \langle -\boldsymbol{\gamma}'(s) + k'_g(s)\boldsymbol{n}_1(s) + k_g(s)\boldsymbol{n}'_1(s), \boldsymbol{v}\rangle \\ &= \langle -\boldsymbol{t}(s) + k'_g(s)\boldsymbol{n}_1(s) + k_g(s)(k_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_2(s)), \boldsymbol{v}\rangle \\ &= (\sin\varphi/k_g(s))(k'_g(s) \mp k_g(s)\tau_g(s)) \\ &= 0. \end{aligned} \tag{3.2}$$

As $k_g(s) \neq 0$ and $\sin\varphi$ does not always equate to zero for $\varphi \in (0, \pi/2]$, therefore, $k'_g(s) \mp k_g(s)\tau_g(s) = 0$ and $(k'_g(s) \mp k_g(s)\tau_g(s))' \neq 0$.

4. By Equations (2.1) and (3.2), we have

$$\begin{aligned} h^{(4)}_{v,\varphi}(s) &= \langle ((k^2_g(s) - 1)\boldsymbol{t}(s) + k'_g(s)\boldsymbol{n}_1(s) + k_g(s)\tau_g(s)n_2(s))', \boldsymbol{v}\rangle \\ &= (\sin\varphi/k_g(s))(k''_g(s) + k_g(s)\tau^2_g(s) \mp (k'_g(s)\tau_g(s) + (k_g(s)\tau_g(s))')) \\ &= (\sin\varphi/k_g(s))(k''_g(s) \mp (k_g(s)\tau_g(s))' + (k_g(s)\tau_g(s) \mp k'_g(s))\tau_g(s)) \\ &= 0. \end{aligned} \tag{3.3}$$

By assertion 3, we know $k'_g(s) \mp k_g(s)\tau_g(s) = 0$ and $k''_g(s) \mp (k_g(s)\tau_g(s))' = (k'_g(s) \mp k_g(s)\tau_g(s))' = 0$. □

4. Singularities of one parameter lightlike hypersurfaces

In this section, we study the geometric properties of one parameter lightlike hypersurfaces of spacelike curves in de Sitter 3-space. Meanwhile, we use some general results on the singularity theory for families of function germs [2]. These properties will be stated in the following.

Proposition 4.1. *Suppose $\boldsymbol{\gamma}(s)$ is a unit speed spacelike curve in de Sitter 3-space with $k_g(s) \neq 0$, the following assertions are established:*

1. The singularities of $\mathbb{L}^{\pm}_{\varphi}(s,\mu)$ are the set $\{(s,\mu) \mid \mu = -\tan\varphi/k_g(s),\ s \in I\}$.

2. If $\boldsymbol{v}_0 = \mathbb{L}^{\pm}_{\varphi}(s, -\tan\varphi/k_g(s))$ is a constant vector, then $\boldsymbol{\gamma}(s) \in LC(\varphi)(\boldsymbol{v}_0)$ and $k'_g(s) \pm k_g(s)\tau_g(s) = 0$.

Proof.

1. Since $\mathbb{L}^{\pm}_{\varphi}(s,\mu) = \sin\varphi\boldsymbol{\gamma}(s) + \mu\cos\varphi(\boldsymbol{n}_1\widetilde{(s) \pm \boldsymbol{n}}_2(s))$, we have

$$\begin{aligned}\partial\mathbb{L}^{\pm}_{\varphi}(s,\mu)/\partial s &= \sin\varphi\boldsymbol{\gamma}'(s) + \mu\cos\varphi(\boldsymbol{n}'_1\widetilde{(s) \pm \boldsymbol{n}'_2}(s)) \\ &= (\sin\varphi + \mu\cos\varphi k_g(s))\boldsymbol{t}(s) + \tau_g(s)\mu\cos\varphi(\boldsymbol{n}_1\widetilde{(s) \pm \boldsymbol{n}}_2(s)),\end{aligned} \tag{4.1}$$

$$\partial\mathbb{L}^{\pm}_{\varphi}(s,\mu)/\partial\mu = \cos\varphi(\boldsymbol{n}_1\widetilde{(s) \pm \boldsymbol{n}}_2(s)),$$
$$\partial\mathbb{L}^{\pm}_{\varphi}(s,\mu)/\partial\varphi = -\cos\varphi,$$

the above three vectors are linearly dependent if and only if $\sin\varphi + \lambda\cos\varphi k_g(s) = 0$ and $\lambda = -\tan\varphi/k_g(s)$. Hence, the assertion (1) is complete.

2. For a smooth function $\nu : I \to \mathbb{R}$, we define a mapping $f_\nu : I \to \mathbb{R}^4_1$ by

$$f_\nu(s) = \sin\varphi\boldsymbol{\gamma}(s) + \nu(s)(\boldsymbol{n}_1\widetilde{(s) \pm \boldsymbol{n}}_2(s)).$$

Supposing $\boldsymbol{v}_0 = \mathbb{L}^{\pm}_{\varphi}(s, -\tan\varphi/k_g(s))$ is constant, we have

$$\begin{aligned}df_\nu(s)/ds &= \sin\varphi\boldsymbol{\gamma}'(s) + \nu(s)(\boldsymbol{n}'_1\widetilde{(s) \pm \boldsymbol{n}'_2}(s)) + \nu'(s)(\boldsymbol{n}_1\widetilde{(s) \pm \boldsymbol{n}}_2(s)) \\ &= (\sin\varphi + \nu(s)k_g(s))\boldsymbol{t} + (\nu'(s) \pm \nu(s)\tau_g(s))\boldsymbol{n}_1 + (\nu(s)\tau_g \pm \nu'(s))\boldsymbol{n}_2 \\ &= 0.\end{aligned} \tag{4.2}$$

One obtains $\sin\varphi + \nu(s)k_g(s) = 0$ and $\nu(s)\tau_g(s) \pm \nu'(s) = 0$. Moreover,

$$\begin{aligned}\langle\boldsymbol{\gamma}(s), \boldsymbol{v}_0\rangle &= \langle\boldsymbol{\gamma}(s), \sin\varphi\boldsymbol{\gamma}(s) - (\tan\varphi/k_g(s))\cos\varphi(\boldsymbol{n}_1\widetilde{(s) \pm \boldsymbol{n}}_2(s))\rangle \\ &= \sin\varphi.\end{aligned} \tag{4.3}$$

Hence, $\boldsymbol{\gamma}(s)$ is belonged to $LC(\varphi)(\boldsymbol{v}_0)$ and $k'_g(s) \pm k_g(s)\tau_g(s) = 0$. □

Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be a function germ. We call F an r-parameter unfolding of f, where $f(s) = F_{x_0}(s, \boldsymbol{x}_0)$. We call $f(s)$ has A_k-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$. We say that $f(s)$ has $A_{\geq k}$-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let F be an unfolding of f and $f(s)$ has A_k-singularity $(k \geq 1)$ at s_0. Denote the $(k-1)$-jet of the $\partial F/\partial x_i$ at s_0 by $j^{(k-1)}(\partial F/\partial x_i)(s, \boldsymbol{x}_0)(s_0) = \Sigma_{j=1}^{k-1}\alpha_{ji}(s-s_0)^j$ for $i = 1, 2, \ldots, r$. Then F is called a (p) versal unfolding if the $(k-1)\times r$ matrix of coefficients α_{ji} has rank $(k-1)(k-1 \leq r)$. Under the same as the above, F is called a versal unfolding if the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank $k(k \leq r)$, where $\alpha_{0i} = (\partial F/\partial x_i)(s_0, x_0)$. Let a function germ $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an unfolding of f. Then the discriminant set of F is given by

$$\mathcal{D}_F = \{x \in \mathbb{R}^r \mid F(s,x) = (\partial F/\partial s)(s,x) = 0\}.$$

And the main theorem is the following [2].

Theorem 4.2. *Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has A_k-singularity at s_0. Suppose that F is a versal unfolding of f.*

1. If $k = 1$ then $\mathcal{D}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.

2. If $k = 2$ then $\mathcal{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.

3. If $k = 3$ then $\mathcal{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$, a point $\boldsymbol{x}_0 \in \mathbb{R}^r$ is called a fold point of a map germ $f : (\mathbb{R}^r, \boldsymbol{x}_0) \to (\mathbb{R}^r, f(\boldsymbol{x}_0))$ if there exist diffeomorphism germs $\phi : (\mathbb{R}^r, \boldsymbol{x}_0) \to (\mathbb{R}^r, 0)$ and $\psi : (\mathbb{R}^r, f(\boldsymbol{x}_0)) \to (\mathbb{R}^r, 0)$ such that $\psi \circ \phi(x_1, \ldots, x_r) = (x_1, \ldots, x_{r-1}, x^2)$.

By Proposition 3.1, the discriminant set of the height function $H(s, \boldsymbol{v}, \varphi)$ is given by

$$\mathcal{D}_H = \{\boldsymbol{v} = \sin\varphi\boldsymbol{\gamma}(s) + \mu\cos\varphi(\boldsymbol{n}_1\widetilde{(s) \pm \boldsymbol{n}}_2(s)) \mid s, \mu \in \mathbb{R}, \varphi \in (0, \pi/2]\}$$

Theorem 4.3. *Let $H(s, \boldsymbol{v}, \varphi)$ be a spacelike height function of spacelike curve $\boldsymbol{\gamma}(s)$ and $\boldsymbol{v} \in \mathcal{D}_H$. If h_v has A_k-singularity at s $(k = 1, 2, 3)$, then H is a versal unfolding of $h_{\boldsymbol{v}}$.*

Proof. We denote

$$\boldsymbol{\gamma}(s) = (x_1(s), x_2(s), x_3(s), x_4(s)) \text{ and } \boldsymbol{v} = (v_1, v_2, v_3, v_4) \in \mathbb{S}^3_1(\sin^2\varphi),$$

for any parameter $\varphi \in (0, \pi/2]$, then

$$H(s, \boldsymbol{v}, \varphi) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle - \sin\varphi = (x_1v_1 + x_2v_2 + x_3v_3 \mp x_4v_4) - \sin\varphi. \tag{4.4}$$

Thus,

$$\begin{gathered}(\partial H/\partial v_i)(s, v) = x_i \mp (v_i/v_4)x_4,\\ (\partial H/\partial\varphi)(s, v) = -\cos\varphi,\\ \partial(\partial H/\partial v_i)/\partial s(s, v) = x_i' \mp (v_i/v_4)x_4',\\ \partial(\partial H/\partial\varphi)/\partial s(s, v) = \partial^2(\partial H/\partial\varphi)/\partial s^2(s, v) = 0,\\ \partial^2(\partial H/\partial v_i)/\partial^2 s(s, v) = x_i'' \mp (v_i/v_4)x_4'',\end{gathered}$$

where $v_4 = \pm\sqrt{v_1^2 + v_2^2 + v_3^2 - \sin^2\varphi}$ $(i = 1, 2, 3)$, so the 2-jet of $\partial H/\partial v_i$ at s_0 is

$$(x_i's + (1/2)x_i''s^2) \mp (v_i/v_4)(x_4''s + (1/2)x_4''s^2).$$

The condition for (p) versal can be checked as following:

(1): By Proposition 3.1, h has A_1-singularity at s if and only if $\boldsymbol{v} = \sin\varphi\boldsymbol{\gamma}(s) + \lambda(\boldsymbol{n}_1\widetilde{(s) \pm \boldsymbol{n}}_2(s))$ and $k_g(s) \neq 0$, when h has A_1-singularity at s, we require the 1×4 matrix $(x_1 \mp (v_1/v_4)x_4, x_2 \mp (v_2/v_4)x_4, x_3 \mp (v_3/v_4)x_4, -\cos\varphi)$ to have rank 1, which it always does since $\boldsymbol{\gamma}(s)$ is regular.

(2): It also follows from Proposition 3.1 that h has $A_{\geq 2}$-singularity at s if and only if $\boldsymbol{v} = \sin\varphi\boldsymbol{\gamma}(s) + (\tan\varphi/k_g(s))\cos\varphi(\boldsymbol{n}_1\widetilde{(s) \pm \boldsymbol{n}}_2(s))$ and $k_g'(s) \pm k_g(s)\tau_g(s) \neq 0$, when h has $A_{\geq 2}$-singularity at s, we require the 2×4 matrix

$$\begin{pmatrix} x_1 \mp (v_1/v_4)x_4 & x_1' \mp (v_1/v_4)x_4' \\ x_2 \mp (v_2/v_4)x_4 & x_2' \mp (v_2/v_4)x_4' \\ x_3 \mp (v_3/v_4)x_4 & x_3' \mp (v_3/v_4)x_4' \\ -\cos\varphi & 0 \end{pmatrix}$$

to have rank 2, which follows from the proof of the case (3).

(3): By Proposition 3.1, h has the A_3-singularity at s if and only if

$$\boldsymbol{v} = \sin\varphi\boldsymbol{\gamma}(s) + (\tan\varphi/k_g(s))\cos\varphi(\boldsymbol{n}_1\widetilde{(s) \pm \boldsymbol{n}}_2(s))$$

and $k_g'(s) \pm k_g(s)\tau_g(s) = 0$, $(k_g'(s) \pm k_g(s)\tau_g(s))' \neq 0$, when h has A_3-singularity at s, we require the 3×4 matrix

$$A = \begin{pmatrix} \alpha_{0,1} & \alpha_{0,2} & \alpha_{0,3} \\ \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ -\cos\varphi & 0 & 0 \end{pmatrix}$$

to be nonsingular, where

$$\begin{aligned} & j^2(\partial H/\partial v_i)(s,v_0)(s_0) \\ = \;& \frac{\partial H}{\partial v_i}(s_0,v_0)+\frac{\partial}{\partial s}(\frac{\partial H}{\partial v_i})(s_0,v_0)(s-s_0)+(1/2)\frac{\partial^2}{\partial^2 s}(\frac{\partial H}{\partial v_i})(s_0,v_0)(s-s_0)^2 \\ = \;& \alpha_{0,i}+\alpha_{1,i}(s-s_0)+(1/2)\alpha_{2,i}(s-s_0)^2. \end{aligned} \tag{4.5}$$

One denotes that

$$A(i,j,k)=det\begin{pmatrix} x_i(s) & x_j(s) & x_k(s) \\ x_i'(s) & x_j'(s) & x_k'(s) \\ x_i''(s) & x_j''(s) & x_k''(s) \end{pmatrix}.$$

We have

$$\begin{aligned} detA &= -(A(1,2,3)\mp(v_1/v_4)A(4,2,3)\mp(v_2/v_4)A(1,4,3)\mp(v_3/v_4)A(1,2,4)) \\ &= \pm(1/v_4)\langle \boldsymbol{v},\boldsymbol{\gamma}(s)\wedge\boldsymbol{\gamma}'(s)\wedge\boldsymbol{\gamma}''(s)\rangle. \end{aligned} \tag{4.6}$$

Since $\boldsymbol{v}\in\mathcal{D}_H$ is a singular point, $\boldsymbol{v}=\sin\varphi\boldsymbol{\gamma}(s)-(\tan\varphi/k_g(s))\cos\varphi(\boldsymbol{n}_1\widetilde{(s)\pm\boldsymbol{n}}_2(s))$ and

$$\boldsymbol{\gamma}(s)\wedge\boldsymbol{\gamma}'(s)\wedge\boldsymbol{\gamma}''(s) \quad =\boldsymbol{\gamma}(s)\wedge\boldsymbol{\gamma}'(s)\wedge(-\boldsymbol{\gamma}(s)+k_g(s)\boldsymbol{n}_1(s))=k_g(s)\boldsymbol{n}_2(s). \tag{4.7}$$

Therefore,

$$\begin{aligned} detA &= \pm(1/v_4)\langle\sin\varphi\boldsymbol{\gamma}(s)-(\tan\varphi/k_g(s))\cos\varphi(\boldsymbol{n}_1\widetilde{(s)\pm\boldsymbol{n}}_2(s)),k_g(s)\boldsymbol{n}_2(s)\rangle \\ &= \pm(\sin\varphi/v_4)\neq 0. \end{aligned} \tag{4.8}$$

This completes the proof. □

Proof of Theorem 2.2. Let $\boldsymbol{\gamma}(s)$ be a spacelike curve in de Sitter 3-space. We define a function $\mathrm{G}:\mathbb{S}^3_1\to\mathbb{R}$ by $\mathrm{G}(\boldsymbol{u})=\langle\boldsymbol{u},\boldsymbol{v}\rangle-\sin\varphi$. Then we have $g_{v_0,\varphi_0}(s)=H(\boldsymbol{\gamma}(s),\boldsymbol{v}_0,\varphi_0)$, since $LC(\varphi)(\boldsymbol{v}_0)=\mathrm{G}^{-1}(0)$ and 0 is a regular value of G. g_{v_0,φ_0} has the A_k-singularity at s_0 if and only if $\boldsymbol{\gamma}(s)$ and $LC(\varphi)(\boldsymbol{v}_0)$ have $(k+1)$-point contact at s_0. By Proposition 3.1 and Theorems 4.2, 4.3, we get the results of Theorem 2.2.

5. Generic properties

In this section, we consider generic properties of spacelike curves in $\mathbb{S}^3_1$. The main tool is the transversality theorem. Let $Emb_s(I,\mathbb{S}^3_1)$ be a space of spacelike embeddings $\boldsymbol{\gamma}:I\to\mathbb{S}^3_1$ with $k_g(s)\neq 0$ equipped with Whitney C^∞-topology. The function $\mathfrak{H}:\mathbb{S}^3_1\times\mathbb{S}^3_1(\sin^2\varphi)\times(0,\pi/2]\to\mathbb{R}$ defined by $\mathfrak{H}(\boldsymbol{u},\boldsymbol{v},\varphi)=\langle\boldsymbol{u},\boldsymbol{v}\rangle-\sin\varphi$, we claim that $\mathfrak{H}_{v,\varphi}$ is a submersion for any fixed $(\boldsymbol{v},\varphi)\in\mathbb{S}^3_1(\sin^2\varphi)\times(0,\pi/2]$. For any $\boldsymbol{\gamma}(s)\in Emb_s(I,\mathbb{S}^3_1)$, we have $H=\mathfrak{H}\circ(\boldsymbol{\gamma}(s)\times id_{\mathbb{S}^3_1(\sin^2\varphi)}\times(0,\pi/2])$ and the l-jet extension $j^l_1H:I\times\mathbb{S}^3_1(\sin^2\varphi)\times(0,\pi/2]\to J^l(I,\mathbb{R})$ defined by

$$j^l_1H(s,\boldsymbol{v},\varphi)=j^lh_{v,\varphi}(s,\boldsymbol{v},\varphi)$$

and the trivialization

$$J^l(I,\mathbb{R})=U\times\mathbb{R}\times J^l(1,1).$$

For any submanifold $\mathcal{O}\subset J^l(1,1)$, we denote that $\widetilde{\mathcal{O}}=I\times\{0\}\times\mathcal{O}$. Then we have the following proposition as a corollary of Lemma 6 in [2].

Proposition 5.1. *Let $\mathcal{O}$ be a submanifold of $J^l(1,1)$. Then the set $T_{\mathcal{O}}=\{\boldsymbol{\gamma}\in Emb_s(I,\mathbb{S}^3_1)\mid j^l_1(H)$ is transversal to $\mathcal{O}\}$ is a residual subset of $Emb_s(I,\mathbb{S}^3_1)$. If $\mathcal{O}$ is a closed subset, then $T_{\mathcal{O}}$ is open.*

Let $f:(\mathbb{R},0)\to(\mathbb{R},0)$ be a function germ which has A_k-singularity at 0. It is well known that there exists a diffeomorphism germ $\phi:(\mathbb{R},0)\to(\mathbb{R},0)$ such that $f\circ\phi(s)=\pm s^{k+1}$. This is the classification of A_k-singularity. For any $z=j^lf(0)$ in $J^l(1,1)$, we have the orbit $L^l(z)$ given by the action of the Lie group of l-jet diffeomorphism germ. If f has an A_k-singularity, then the codimension of the orbit is k. There is another characterization of versal unfolds.

Proposition 5.2. [3] *Let $F : (\mathbb{R}\times\mathbb{R}^r, \mathbf{0}) \to (\mathbb{R}, 0)$ be an r-parameter unfolding of $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ which has A_k-singularity at 0. Then F is a versal unfolding if and only if $j_1^l F$ is transversal to the orbit $L^l(j^l f(0))$ for $l \geq k+1$, where $j_1^l F : (\mathbb{R}\times\mathbb{R}^r, \mathbf{0}) \to J^l(\mathbb{R}, \mathbb{R})$ is the l-jet extension of F given by $j_1^l F(s, x) = j^l F_x(s)$.*

The generic classification theorem is given as following:

Proposition 5.3. *There exists an open and dense subset $T_{L_k^l} \subset Emb_s(I, \mathbb{S}_1^3)$ such that for any $\boldsymbol{\gamma}(s) \in T_{L_k^l}$. The one parameter lightlike hypersurfaces of $\boldsymbol{\gamma}(s)$ is locally diffeomorphic to the cuspidal edge or the swallowtail at a singular point.*

Proof. For $l \geq 4$, we consider the decomposition of the jet space $J^l(1,1)$ into L_1^l orbits. We now define a semi-algebraic set by

$$\Sigma^l = \{z = j^l f(0) \in J^l(1,1) \mid f \text{ has an } A_{\geq 4}\text{-singularity}\},$$

the codimension of Σ^l is 4. Therefore, the codimension of $\widetilde{\Sigma_0^l} = I \times \{0\} \times \Sigma^l$ is 5 and the orbit decomposition of $j^l(1,1) - \Sigma^l$ is

$$j^l(1,1) - \Sigma^l = L_0^l \cup L_1^l \cup L_2^l \cup L_3^l,$$

where L_k^l is the orbit through an A_k-singularity. Thus the codimension of L_k^l is $(k+1)$. We consider the *l-jet* extension $j_1^l(H)$ of the spacelike height function H. By Proposition 5.1, there exists an open and dense subset $T_{L_k^l} \subset Emb_s(I, \mathbb{S}_1^3)$ such that $j_1^l(H)$ is transversal to L_k^l $(k = 0, 1, 2, 3)$ and the orbit decomposition of Σ^l. This means that $j_1^l(H)(I\times\mathbb{S}_1^3\times(0,\pi/2])\cap\Sigma^l = \emptyset$ and H is a versal unfolding of h at any point $(s_0, \boldsymbol{v}_0, \varphi_0)$. The discriminant set of H is locally diffeomorphic to cuspidal edge or the swallowtail at a singular point as in [[6], Theorem 4.1]. □

6. Example

In this section, an example is given in order to verify the idea of Theorem 2.2.

Example 6.1. Let $\boldsymbol{\gamma}(s)$ be a spacelike curve in $\mathbb{S}_1^3$ defined by

$$\boldsymbol{\gamma}(s) = \{\frac{\sqrt{3}}{3}s, \frac{1}{18}s^2 - 1, 2, \frac{1}{18}s^2 + 2\}$$

with respect to a distinguished parameter s (Fig. 3), the Frenet frames as following

$$\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s) = \sqrt{3}\{\frac{\sqrt{3}}{3}, \frac{1}{9}s, 0, \frac{1}{9}s\},$$

$$\boldsymbol{n}_1(s) = \sqrt{3}\{\frac{\sqrt{3}}{3}s, \frac{1}{18}s^2 - 1 + \frac{\sqrt{3}}{9}, 2, \frac{1}{18}s^2 + 2 + \frac{\sqrt{3}}{9}\},$$

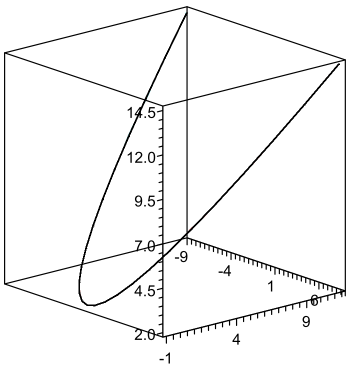
we can obtain

$$\begin{aligned}\boldsymbol{n}_2(s) = \wp\{-\tfrac{\sqrt{3}}{54}s^3 - \tfrac{2}{9}s^2 + \tfrac{7\sqrt{3}}{9}s, \quad & -\tfrac{1}{324}s^4 + \tfrac{\sqrt{3}}{81}s^3 + \tfrac{14}{81}s^2 - \tfrac{55}{9}, -\tfrac{1}{9}s^2 - \tfrac{\sqrt{3}}{27}s - \tfrac{16}{9},\\ & \tfrac{1}{324}s^4 - \tfrac{\sqrt{3}}{81}s^3 - \tfrac{1}{162}s^2 - \tfrac{2\sqrt{3}}{9}s + \tfrac{8}{9}\},\end{aligned}$$

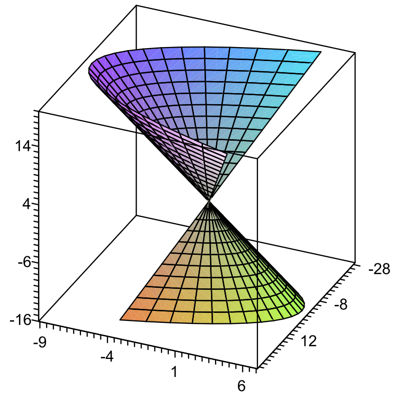
where $\wp = \frac{162}{2\sqrt{3}s^7 - 48s^6 + 84\sqrt{3}s^5 + 8163s^4 - 7128s^3 - 756s^2 + 6712\sqrt{3}s + 521154}$.

At the moment, the curvatures $k_g = \sqrt{3}$ and $\tau_g = -\frac{s}{6} + \frac{7\sqrt{3}}{3s+1} - 7\sqrt{3}$. Thus, the one parameter lightlike hypersurfaces when $\varphi = 0, \frac{\pi}{4}, \frac{\pi}{2}$ are following,

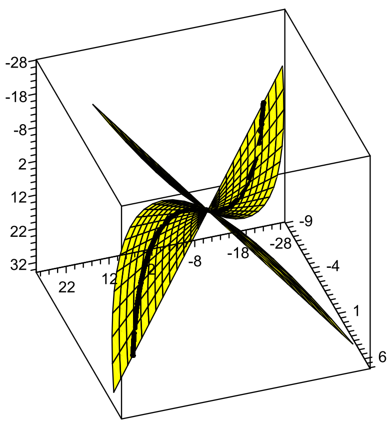
$$\begin{aligned}\mathbb{L}_0^+(s,\mu) = \quad & \{\tfrac{\sqrt{3}}{3}s + \wp(-\tfrac{\sqrt{3}}{54}s^3 - \tfrac{2}{9}s^2 + \tfrac{7\sqrt{3}}{9}s),\\ & \tfrac{1}{18}s^2 - 1 + \tfrac{\sqrt{3}}{9}\wp(-\tfrac{1}{324}s^4 + \tfrac{\sqrt{3}}{81}s^3 + \tfrac{14}{81}s^2 - \tfrac{55}{9}),\\ & 2 + \wp(-\tfrac{1}{9}s^2 - \tfrac{\sqrt{3}}{27}s - \tfrac{16}{9}),\\ & \tfrac{1}{18}s^2 + 2 + \tfrac{\sqrt{3}}{9} + \wp(\tfrac{1}{324}s^4 - \tfrac{\sqrt{3}}{81}s^3 - \tfrac{1}{162}s^2 - \tfrac{2\sqrt{3}}{9}s + \tfrac{8}{9})\}, (Fig.4)\end{aligned}$$



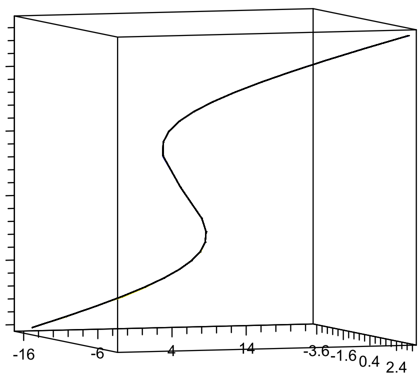
spacelike curve $\boldsymbol{\gamma}(s)$
Fig. 3



one parameter lightlike hypersurface with $\varphi = 0$
Fig. 4



one parameter lightlike hypersurface with $\varphi = \frac{\pi}{4}$ and its singularities
Fig. 5



Singular locus of one parameter lightlike hypersurface with $\varphi = \frac{\pi}{4}$
Fig. 6

$$\begin{aligned}\mathbb{L}^{+}_{\frac{\pi}{4}}(s,\mu) \;&= \tfrac{\sqrt{2}}{2}\{\tfrac{\sqrt{3}}{3}s+\mu(\tfrac{\sqrt{3}}{3}s+\wp(-\tfrac{\sqrt{3}}{54}s^3-\tfrac{2}{9}s^2+\tfrac{7\sqrt{3}}{9}s)),\\
&\tfrac{1}{18}s^2-1+\mu(\tfrac{1}{18}s^2-1+\tfrac{\sqrt{3}}{9}\wp(-\tfrac{1}{324}s^4+\tfrac{\sqrt{3}}{81}s^3+\tfrac{14}{81}s^2-\tfrac{55}{9})),\\
&2+\mu(2+\wp(-\tfrac{1}{9}s^2-\tfrac{\sqrt{3}}{27}s-\tfrac{16}{9})),\\
&\tfrac{1}{18}s^2+2+\mu(\tfrac{1}{18}s^2+2+\tfrac{\sqrt{3}}{9}+\wp(\tfrac{1}{324}s^4-\tfrac{\sqrt{3}}{81}s^3-\tfrac{1}{162}s^2-\tfrac{2\sqrt{3}}{9}s+\tfrac{8}{9}))\}, (Fig.5)\end{aligned}$$

$$\mathbb{L}^{+}_{\frac{\pi}{2}}(s,\mu)=\{\frac{\sqrt{3}}{3}s,\frac{1}{18}s^2-1,2,\frac{1}{18}s^2+2\}.$$

On the other hand, we get the singularities of $\frac{\pi}{4}$-lightlike hypersurface and the singularities satisfying $\mu=\frac{1}{\sqrt{3}}$ (Fig. 6). We can calculate the geometric invariant

$$\sigma(s)=k'_g(s)\mp k_g(s)\tau_g(s)$$

and

$$\sigma'(s)=k''_g(s)\mp k'_g(s)\tau_g(s)\mp k_g(s)\tau'_g(s).$$

We see $\sigma(s)=0$ gives two real roots $s=0,-(126\sqrt{3}+1)/3$ and $\sigma'(s)=0$ gives two complex roots $s=-1/3\pm(252\sqrt{3})^{\frac{1}{2}}i$. Hence, we have $\mathbb{L}^{\pm}_{\varphi}(s,\mu)$ is locally diffeomorphic to the cuspidal edge at $s=0,-(126\sqrt{3}+1)/3$ and $\mu=tan\varphi/\sqrt{3}$. Moreover, $\mathbb{L}^{\pm}_{\varphi}(s,\mu)$ is locally diffeomorphic to the swallowtail at $s=-1/3\pm(252\sqrt{3})^{\frac{1}{2}}i$ and $\mu=tan\varphi/\sqrt{3}$.

Acknowledgements

The authors express thank to the referee for his/her valuable suggestions. The first author was supported by NSF of Shandong ZR2014AQ016 and BRFCU 15CX02068A, 14CX02147A; the second author was partially supported by NSF of China No.11271063.

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