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Strong convergence of a Halpern-type iteration algorithm for fixed point problems in Banach spaces

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Abstract

In this paper, we studied a Halpern-type iteration algorithm involving pseudo-contractive mappings for solving some variational inequality in a q-uniformly smooth Banach space. We show the studied algorithm has strong convergence under some mild conditions. Our result extends and improves many results in the literature.

Keywords: Halpern iterative algorithm, pseudocontractive mapping, fixed point, variational inequality.
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1. Introduction

Variational inequality problems were initially studied by Stampacchia [13] in 1964. Variational inequalities have applications in diverse disciplines such as partial differential equations, physical, optimal control, optimization, mathematical programming, mechanics and finance, see [6, 7, 8, 9, 10, 12, 13, 17] and the references therein. Variational inequalities have been extended and generalized in several directions using novel and innovative techniques. It is common practice to study these variational inequalities in the setting of convexity. It has been observed that the optiminality conditions of the differentiable convex functions can be characterized by the variational inequalities. In recent years, it has been shown that the minimum of the differentiable nonconvex functions can also be characterized by the variational inequalities. Motivated

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and inspired by these developments, Noor [8] has introduced a new type of variational inequality involving two nonlinear operators, which is called the general variational inequality. It is worth mentioning that this general variational inequality is remarkable different from the so-called general variational inequality which was introduced by Noor [6] in 1988. Noor [8] proved that the general variational inequalities are equivalent to nonlinear projection equations and the Wiener-Hopf equations by using the projection technique. Using this equivalent formulation, Noor [8] suggested and analyzed some iterative algorithms for solving the special general variational inequalities and further proved these algorithms have strong convergence. Related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is our purpose in this paper, we studied a Halpern-type iteration algorithm involving pseudo-contractive mappings for solving some variational inequality in a q-uniformly smooth Banach space. We show the studied algorithm has strong convergence under some mild conditions. Our result extends and improves many results in the literature.

To be more precise, let C be a nonempty closed convex subset of a real Banach space E. Let $A : C \to E$ be a nonlinear operator. The variational inequality problem is formulated as finding a point $x^* \in C$ such that, for some $j(x - x^*) \in J(x - y)$,

$$\langle Ax^*, j(x - x^*)\rangle \geq 0,$$

for all $x \in C$.

Recall that a mapping $T : C \to C$ is said to be strictly pseudo-contractive (in the terminology of Browder-Petryshyn) if there exists a constant $\lambda > 0$ such that, for all $x, y \in C$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle (I - T)x - (I - T)y, j_q(x - y)\rangle \geq \lambda\|(I - T)x - (I - T)y\|^q, \tag{1.1}$$

where I denotes the identity operator on C. We denote by $F(T)$ the set of fixed points of a mapping $T : C \to C$, that is $F(T) = \{x \in C : Tx = x\}$. This class of mappings was introduced actually in a Hilbert space by Browder and Petryshyn [1].

Recall also that a mapping $f : C \to C$ is said to be contractive if there exists a constant $\rho \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho\|x - y\|,$$

for all $x, y \in C$.

Numerous papers have been written on the approximation of fixed points of strictly pseudo-contractive mappings (see [3, 4, 5, 11, 18, 19, 20, 21, 22] and the references contained therein). In particular, recently, Chidume and Souza [2] introduced a Halpern-type iterative algorithm for a strictly pseudo-contractive mapping and proved the following strong convergence theorem:

Theorem 1.1. *Let E be a real reflexive Banach space with uniformly Gâteaux differentiable norm. Let C be a nonempty bounded closed and convex subset of E. Let $T : C \to C$ be a strictly pseudo-contractive mapping. Assume $F(T) \neq \emptyset$ and let $z \in F(T)$. Fix $\delta \in (0, 1)$ and let δ^* be such that $\delta^* := \delta L \in (0, 1)$. Define $S_n x := (1 - \delta_n)x + \delta_n Tx$ for all $x \in C$, where $\delta_n \in (0, 1)$ and $\lim \delta_n = 0$. Let $\{\alpha_n\}$ be a real sequence in $(0, 1)$ which satisfies the following conditions:*

$(C1)$ $\lim \alpha_n = 0$;

$(C2)$ $\sum_{n=1}^{\infty} \alpha_n = \infty$.

For arbitrary $x_0, u \in C$, define a sequence $\{x_n\}$ in K by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)S_n x_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

In the proof lines of Theorem 1.1, we point out some problems as follows:

Remark 1.2. First, we note that there exists a big gap in the proof of Theorem 1.1. In Theorem 1.1, they asserted that the sequence $\{z_{t_n}\}$ generated by $z_{t_n} = t_n u + (1-t_n)S_n z_{t_n}$ converges to a fixed point of T. Unfortunately, this conclusion is false. Indeed, noting that

$$z_{t_n} = t_n u + (1-t_n)(1-\delta_n)z_{t_n} + (1-t_n)\delta_n T z_{t_n},$$

it follows that

$$\begin{aligned} z_{t_n} &= \frac{t_n}{\delta_n + t_n - t_n\delta_n}u + \frac{(1-t_n)\delta_n}{\delta_n + t_n - t_n\delta_n}Tz_{t_n} \\ &= \frac{1}{\frac{\delta_n}{t_n} + 1 - \delta_n}u + \frac{(1-t_n)\delta_n}{\delta_n + t_n - t_n\delta_n}Tz_{t_n}. \end{aligned}$$

Thus, from the conditions $\delta_n \to 0$ and $\delta_n = o(t_n)$, we have $\frac{1}{\frac{\delta_n}{t_n}+1-\delta_n} \to 1$ and so the application of Lemma MJ fails. This indicates that $\{z_{t_n}\}$ does not converge to a fixed point of T. Therefore, Theorem 1.1 is dubious.

In this paper, we studied a Halpern-type viscosity iteration algorithm involving pseudo-contractive mapping T in a q-uniformly smooth Banach space. for solving some variational inequality We show the studied algorithm strongly converges to a fixed point of T which solves some variational inequality in Banach spaces under some mild conditions. Our result modifies the main result in Chidume and Souza [2] and extends and improves many other results in the literature.

2. Preliminaries

Let E be a real Banach space. The modulus of smoothness of E is defined as the function $\rho_E : [0,\infty) \to [0,\infty)$:

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \le 1, \|y\| \le \tau\}.$$

E is said to be uniformly smooth if and only if $\lim_{\tau\to 0+}(\rho_E(\tau)/\tau) = 0$. Let $q > 1$. The space E is said to be q-uniformly smooth (or to have a modulus of smoothness of power type $q > 1$), if there exists a constant $c_q > 0$ such that $\rho_E(\tau) \le c_q\tau^q$. It is well known that Hilbert spaces, L_p and l_p spaces, $1 < p < \infty$, as well as the Sobolev spaces, W_m^p, $1 < p < \infty$, are q-uniformly smooth.

Now, we give some lemmas which will be used in the proof of the main result in the next section.

Lemma 2.1. ([15]) *Let $q > 1$ and E be a real smooth Banach space. Then the following are equivalent:*

(1) *E is q-uniformly smooth;*

(2) *There exists a constant $c_q > 0$ such that, for all $x, y \in E$,*

$$\|x+y\|^q \le \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q.$$

Lemma 2.2. ([16]) *Let C be a nonempty closed convex subset of a uniformly smooth Banach space E. Let $f : C \to C$ be a ρ-contraction. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. For $t \in (0,1)$, defined a net $\{x_t\}$ in C by $x_t = tf(x_t) + (1-t)Tx_t$. Then as $t \to 0$, the net $\{x_t\}$ converges strongly to $p \in F()T$ which solves the following variational inequality*

$$\langle (I-f)p, j(x-p)\rangle \ge 0, \text{ for all } \quad x \in F(T).$$

Lemma 2.3. ([14]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E such that*

$$x_{n+1} = \sigma_n x_n + (1-\sigma_n)y_n$$

where $\{\sigma_n\}$ is a sequence in $[0,1]$ such that

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1.$$

Assume

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Then $\lim_{n\to\infty}\|y_n-x_n\|=0$.

Lemma 2.4. ([16]) *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that* $a_{n+1}\le(1-\gamma_n)a_n+\delta_n$ *where* $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty}\gamma_n=\infty$;

(2) $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}\le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n|<\infty$.

Then $\lim_{n\to\infty}a_n=0$.

3. Main Results

Now, we give the main results in this paper.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a q-uniformly smooth Banach space* E. *Let* $f:C\to C$ *be a* ρ*-contraction. Let* $T:C\to C$ *be a strictly pseudo-contractive mapping such that* $F(T)\ne\emptyset$. *For* $t\in(0,1)$, *defined a net* $\{x_t\}$ *by* $x_t=tf(x_t)+(1-t)Tx_t$. *Then, as* $t\to 0$, *the net* $\{x_t\}$ *converges strongly to* $p\in F(T)$ *which solves the following variational inequality*

$$\langle (I-f)p, j(x-p)\rangle\ge 0,\ \textit{for all}\quad x\in F(T).$$

Proof. First, we note that $(1-\delta)I+\delta T$ is nonexpansive mapping for all $\delta\in(0,\min\{1,(\frac{q\lambda}{c_q})^{\frac{1}{q-1}}\})$. Indeed, from (1.1) and Lemma 2.1, we have

$$\begin{aligned}
&\|(1-\delta)(x-y)+\delta(Tx-Ty)\|^q\\
&=\|(x-y)-\delta[x-Tx-(y-Ty)]\|^q\\
&\le\|x-y\|^q-q\delta\langle x-Tx-(y-Ty),j_q(x-y)\rangle\\
&\quad+c_q\delta^q\|x-Tx-(y-Ty)\|^q\\
&\le\|x-y\|^q-q\delta\lambda\|x-Tx-(y-Ty)\|^q\\
&\quad+c_q\eta^q\|x-Tx-(y-Ty)\|^q\\
&=\|x-y\|^q+(c_q\delta^q-q\delta\lambda)\|x-Tx-(y-Ty)\|^q\\
&\le\|x-y\|^q
\end{aligned}$$

and so

$$\|(1-\delta)(x-y)+\delta(Tx-Ty)\|\le\|x-y\|.$$

Hence $(1-\delta)I+\delta T$ is nonexpansive.

For $s\in(0,1)$, we consider the mapping $S:C\to C$ defined by

$$Sx=sf(x)+(1-s)[(1-\delta)x+\delta Tx],\quad\forall x\in C.$$

It is clear that S is a contraction on C. Therefore, there exists a unique fixed point x_s of S in C. That is, x_s solves the equation

$$x_s=sf(x_s)+(1-s)[(1-\delta)x_s+\delta Tx_s],x\in C.$$

It follows that

$$x_s=\frac{sf(x_s)}{\delta+(1-\delta)s}+\frac{\delta(1-s)}{\delta+(1-\delta)s}Tx_s. \tag{3.1}$$

Taking $s=\frac{\delta t}{1-(1-\delta)t}$ in (3.1), we have

$$x_t=tf(x_t)+(1-t)Tx_t.$$

Therefore, from Lemma 2.2, we know that, as $s \to 0$, $\{x_s\}$ converges strongly to $p \in F()T$ which solves the variational inequality

$$\langle (I-f)p, j(x-p)\rangle \geq 0, \text{ for all } \quad x \in F(T).$$

This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a q-uniformly smooth Banach space E. Let $f : C \to C$ be a ρ-contraction. Let $T : C \to C$ be a strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Define a mapping $S : C \to C$ by $Sx = (1-\delta)x + \delta Tx$ for all $x \in C$, where $\delta = (1-\sigma)\eta$ for any $\sigma \in (0,1)$ and $\eta \in (0, \min\{1, (\frac{q\lambda}{c_q})^{\frac{1}{q-1}}\})$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ which satisfies the following conditions:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$;

$(C2)$ $\sum \alpha_n = \infty$.

For arbitrary $x_0 \in C$, define a sequence $\{x_n\}$ in C by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Sx_n, \quad \forall n \geq 0. \tag{3.2}$$

Then $\{x_n\}$ converges strongly to $p \in F(T)$ which solves the variational inequality

$$\langle (I-f)p, j(x-p)\rangle \geq 0, \textit{ for all } \quad x \in F(T).$$

Proof. We first show that the sequence $\{x_n\}$ is bounded.

We note that $\delta < \eta \in (0, \min\{1, (\frac{q\lambda}{c_q})^{\frac{1}{q-1}}\})$. Hence, S is a nonexpansive mapping. At the same time, it is clear that $F(T) = F(S)$.

Take $x^* \in F(T)$. From (3.2), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(Sx_n - x*)\| \\
&\leq \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\|Sx_n - x^*\| \\
&\leq [1-(1-\rho)\alpha_n]\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| \\
&\leq \max\{\frac{\|f(x^*) - x^*\|}{1-\rho}, \|x_n - x^*\|\}.
\end{aligned}$$

By induction, we obtain, for all $n \geq 0$,

$$\|x_n - x^*\| \leq \max\{\frac{\|f(x^*) - x^*\|}{1-\rho}, \|x_0 - x^*\|\}.$$

Hence $\{x_n\}$ is bounded and so is $\{Sx_n\}$. From (3.2), we observe that

$$\begin{aligned}
Sx_n - \sigma x_n &= [(1-\eta+\sigma\eta)x_n + (\eta - \sigma\eta)Tx_n] - \sigma x_n \\
&= (1-\sigma)[(1-\eta)x_n + \eta Tx_n].
\end{aligned}$$

Define a sequence $\{x_n\}$ in C by $x_{n+1} = \sigma x_n + (1-\sigma)y_n$ for all $n \geq 0$. Then we obtain

$$\begin{aligned}
y_n &= \frac{\alpha_n f(x_n) + (1-\alpha_n)Sx_n - \sigma x_n}{1-\sigma} \\
&= \frac{\alpha_n(f(x_n) - Sx_n)}{1-\sigma} + \frac{Sx_n - \sigma x_n}{1-\sigma} \\
&= \frac{\alpha_n(f(x_n) - Sx_n)}{1-\sigma} + (1-\eta)x_n + \eta Tx_n
\end{aligned}$$

and so

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\leq \frac{\alpha_{n+1}(\|f(x_n)\| + \|Sx_{n+1}\|) + \alpha_n(\|f(x_n)\| + \|Sx_n\|)}{1-\sigma} \\
&\quad + \|(1-\eta)(x_{n+1} - x_n) + \eta(Tx_{n+1} - Tx_n)\| \\
&\leq \frac{\alpha_{n+1}(\|f(x_n)\| + \|Sx_{n+1}\|) + \alpha_n(\|f(x_n)\| + \|Sx_n\|)}{1-\sigma} + \|x_{n+1} - x_n\|,
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence, by Lemma 2.3, $\|y_n - x_n\| \to 0$ and so $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, which implies that $\lim_{n\to\infty}\|x_n - Sx_n\| = \lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Next, we show that

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_n-p)\rangle \le 0,$$

where $p = \lim_{t\to 0} x_t$ and $x_t = tf(x_t) + (1-t)Tx_t$.

We note that $x_t - x_n = t(f(x_t) - x_n) + (1-t)(Tx_t - x_n)$. It follows that

$$\begin{aligned}\|x_t-x_n\|^2 &= t\langle f(x_t)-x_n, j(x_t-x_n)\rangle + (1-t)\langle Tx_t-x_n, j(x_t-x_n)\rangle\\ &= t\langle f(x_t)-x_t, j(x_t-x_n)\rangle + t\langle x_t-x_n, j(x_t-x_n)\rangle\\ &\quad + (1-t)\langle Tx_t-Tx_n, j(x_t-x_n)\rangle + (1-t)\langle Tx_n-x_n, j(x_t-x_n)\rangle\\ &\le \|x_t-x_n\|^2 + \|Tx_n-x_n\|\|x_t-x_n\| + t\langle f(x_t)-x_t, j(x_t-x_n)\rangle.\end{aligned}$$

It follows that

$$\langle f(x_t)-x_t, j(x_n-x_t)\rangle \le \frac{\|Tx_n-x_n\|\|x_t-x_n\|}{t},$$

which implies that

$$\limsup_{n\to\infty}\langle f(x_t)-x_t, j(x_n-x_t)\rangle \le 0.$$

It follows that

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_n-p)\rangle \le 0. \tag{3.3}$$

Finally, we prove that $x_n \to p$. From (3.2), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(f(x_n)-p) + (1-\alpha_n)(Sx_n-p)\|^2\\ &\le (1-\alpha_n)^2\|Sx_n-p\|^2 + 2\alpha_n\langle f(x_n)-f(p), j(x_{n+1}-p)\rangle\\ &\quad + 2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle\\ &\le (1-\alpha_n)^2\|x_n-p\|^2 + 2\alpha_n\rho\|x_n-p\|\|x_{n+1}-p\|\\ &\quad + 2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle\\ &\le (1-\alpha_n)^2\|x_n-p\|^2 + \alpha_n\rho(\|x_n-p\|^2 + \|x_{n+1}-p\|^2)\\ &\quad + 2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle,\end{aligned}$$

that is,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \frac{1-(2-\rho)\alpha_n+\alpha_n^2}{1-\rho\alpha_n}\|x_n-p\|^2\\ &\quad + \frac{2-\alpha_n}{1-\rho\alpha_n}\langle f(p)-p, j(x_{n+1}-p)\rangle\\ &= [1-\frac{2(1-\rho)\alpha_n}{1-\rho\alpha_n}]\|x_n-p\|^2 + \frac{\alpha_n^2}{1-\rho\alpha_n}\|x_n-p\|^2\\ &\quad + \frac{2\alpha_n}{1-\rho\alpha_n}\langle f(p)-p, j(x_{n+1}-p)\rangle.\end{aligned} \tag{3.4}$$

From (3.3), (3.4) and Lemma 2.4, we deduce immediately the desired result. This completes the proof. □

Remark 3.3. We correct the gap in the proof of Theorem 1.1 and, at the same time, we drop the boundedness assumption on C.

Remark 3.4. It is worth of mentioning that our proof is very simpler than that of Theorem 1.1.

Remark 3.5. We would like to point out that we prove a strong convergence result on pseudocontractive mappings which solves some variational inequality under conditions (C1) and (C2) on algorithm parameters $\{\alpha_n\}$.

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References

[1] F. E. Browder, W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl., **20** (1967), 197–228. 1

[2] C. O. Chidume, G. De Souza, *Convergence of a Halpern-type iteration algorithm for a class of pseudocontractive mappings*, Nonliear Anal., **69** (2008), 2286–2292. 1, 1

[3] P. Li, S. M. Kang, L. Zhu, *Visco-resolvent algorithms for monotone operators and nonexpansive mappings*, J. Nonlinear Sci. Appl., **7** (2014), 325–344. 1

[4] G. Marino, H. K. Xu, *Weak and strong convergence theorems for strictly pseudocontractions in Hilbert spaces*, J. Math. Anal. Appl., **329** (2007), 336–346. 1

[5] C. H. Morales, J. S. Jung, *Convergence of paths for pseudocontractive mappings in Banach spaces*, Proc. Amer. Math. Soc., **128** (2000), 3411–3419. 1

[6] M. A. Noor, *General variational inequalities*, Appl. Math. Lett., **1** (1988), 119–121. 1

[7] M. A. Noor, *Some developments in general variational inequalities*, Appl. Math. Comput., **152** (2004), 199–277. 1

[8] M. A. Noor, *Differentiable nonconvex functions and general variational inequalities*, Appl. Math. Comput., **199** (2008), 623–630. 1

[9] M. A. Noor, K. I. Noor, *Self-adaptive projection algorithms for general variational inequalities*, Appl. Math. Comput., **151** (2004), 659–670. 1

[10] M. A. Noor, K. I. Noor, Th. M. Rassias, *Some aspects of variational inequalities*, J. Comput. Appl. Math., **47** (1993), 285–312. 1

[11] M. O. Osilike, A. Udomene, *Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn Type*, J. Math. Anal. Appl., **256** (2001), 431–445. 1

[12] P. Shi, *Equivalence of variational inequalities with Wiener-Hopf equations*, Proc. Amer. Math. Soc., **111** (1991), 339–346. 1

[13] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes*, C.R. Acad. Sci. Paris, **258** (1964), 4413–4416. 1

[14] T. Suzuki, *Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces*, Fixed Point Theory Appl., **2005** (2005), 103–123. 2.3

[15] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16** (1991), 1127–1138. 2.1

[16] H. K. Xu, *Viscosity approximation methods for non-expansive mappings*, J. Math. Anal. Appl., **298** (2004), 279–291. 2.2, 2.4

[17] J. C. Yao, *Variational inequalities with generalized monotone operators*, Math. Operations Research, **19** (1994), 691–705. 1

[18] Y. Yao, Y. J. Cho, Y. C. Liou, R. P. Agarwal, *Constructed nets with perturbations for equilibrium and fixed point problems*, J. Inequal. Appl., **2014** (2014), 14 pages. 1

[19] Y. Yao, Y. C. Liou, C. C. Chyu, *Fixed points of pseudocontractive mappings by a projection method in Hilbert spaces*, J. Nonlinear Convex Anal., **14** (2013), 785–794. 1

[20] Y. Yao, Y. C. Liou, S. M. Kang, *Coupling extragradient methods with CQ mathods for equilibrium points, pseudomontone variational inequalities and fixed points*, Fixed Point Theory, **15** (2014), 311–324. 1

[21] Y. Yao, M. Postolache, S. M. Kang, *Strong convergence of approximated iterations for asymptotically pseudocontractive mappings*, Fixed Point Theory Appl., **2014** (2014) 13 pages. 1

[22] H. Y. Zhou, *Convergence theorems of fixed points for k-strict pseudocontractions in Hilbert spaces*, Nonlinear Anal., **69** (2008), 456–462. 1