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Dynamics of a harvested logistic type model with delay and piecewise constant argument

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Abstract

In this paper, a harvested logistic equation with delay and piecewise constant argument of generalized type is addressed. Both discrete and piecewise constant delays are incorporated into the logistic equation for investigation. Existence, boundedness of positive solutions and permanence are studied for the proposed logistic model.

Keywords: Delayed logistic equation, piecewise constant argument of generalized type, boundedness, permanence, harvesting.
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1. Introduction

Mathematical modeling of population dynamics is one of the important parts of applied mathematics. In the literature, various mathematical models have been proposed to study population dynamics, and logistic equation is among the most popular ones. Time delays exist in various biological processes [15], especially in single population models like logistic equation. Delayed logistic equation has been investigated in numerous papers where delays are incorporated in different forms [14, 18]. There are also several papers that are interested in the logistic equation with piecewise constant arguments [6, 8, 17, 20]. Biologically, insertion of piecewise constant arguments into a population model means that the rate of the population depends both on the present size and the memorized values of the population.

Since introduced in [13], differential equations with piecewise constant arguments have been intensively developed and used for modeling of real processes [19, 21, 22]. In the literature, most of the results for differential equations with piecewise constant argument are obtained by reducing them into discrete equations

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and by applying numerical methods. The concept of differential equations with piecewise constant arguments has been generalized by considering arbitrary piecewise constant functions as arguments by Akhmet in [1, 2, 3, 5] and a new approach based on the construction of an equivalent integral equation has been used in the investigation of these equations, called as differential equations with piecewise constant argument of generalized type. Later, this new class of differential equations and its applications have been considered in [4, 6, 7, 8, 10, 11]. Akhmet's book [9] covers many theoretical and application problems in the theory of differential equations with piecewise constant argument of generalized type.

Since most populations in ecology are subject to external forces, it is significant to study population models with harvesting. Mathematically, harvesting term leads to various challenging issues especially in the study of existence of positive solutions. It is probable that the harvesting term has a time delay [14]. It is also meaningful to consider a delay differential equation with both continuous and piecewise constant arguments [10, 11, 12, 19].

In this study, we extend the logistic population model by using both discrete and piecewise constant delays and also using a harvesting term. We shall consider time delays present both in the logistic and the harvesting parts. Then, we establish sufficient conditions for several qualitative properties such as existence, boundedness of positive solutions and permanence for the proposed model. For the existence and boundedness of positive solutions, we adapt the previously obtained results [14] to differential equations with piecewise constant arguments.

2. Preliminaries

Let $\mathbb{Z}$ and $\mathbb{R}^+$ be the sets of integers and nonnegative real numbers, respectively. Fix a real-valued, strictly ordered sequence $\{\theta_i\}$, $i \in \mathbb{Z}$, with $|\theta_i| \to \infty$ as $|i| \to \infty$.

In this paper, we consider the following logistic-type equation with a discrete delay, piecewise constant arguments of generalized type and a deviated linear harvesting term:

$$x'(t) = rx(t)(1 - ax(t-\tau) - bx(\beta(t))) - E(t)x(\beta(t)), \quad t \geq 0 \tag{2.1}$$

where $x(t)$ is the population density at time t; r, a, b are positive constants; $E(t)$ denotes the rate at which individuals are harvested; $\tau > 0$ is a single discrete delay; $\beta(t) = \theta_i$ if $\theta_i \leq t < \theta_{i+1}$, $i \in \mathbb{Z}$. We note that (2.1) is a special case of the quasilinear retarded differential equation with functional dependence on piecewise constant argument, which has been initiated in [11] by Akhmet.

We consider the equation (2.1) with the following initial function, initial condition and the assumptions (A1)-(A3):

$$x(t) = \phi_0(t),\ t \in [-\tau, 0],\ x(0) = x_0, \tag{2.2}$$

(A1) there exists a constant $\theta \in (0, \tau)$ such that $\theta_{i+1} - \theta_i \leq \theta$, $i \in \mathbb{Z}$;

(A2) $E : \mathbb{R}^+ \to (0, \infty)$ is a bounded, continuous function with the possible exception of the points $\theta_i \geq 0$, where $\lim\limits_{t \to \theta_i{}^+} E(t) = E(\theta_i)$ and $\int_0^\infty E(s)ds = \infty$;

(A3) $\phi_0 : [-\tau, 0] \to [0, x_0]$ is a continuous function with $x_0 > 0$.

For convenience, we define $\kappa := \dfrac{1}{a+b}$ and $\mu := \max\{1, \dfrac{x_0}{\kappa}\}$.

Note that there exists at least one and at most a finite number of discontinuity points, θ_i, $i \in \mathbb{Z}$, in any interval $[n\tau, (n+1)\tau]$, $n \in \mathbb{Z}$. Moreover, we have $\theta_{k_0} \leq 0 < \theta_{k_0+1}$ for some $k_0 \in \mathbb{Z}$.

Below, we give an example in order to give an idea about the considered strategy.

Example 2.1. (*Metapopulation model*) One of the metapopulation models considering the immigration of some species such as birds from a continent to islands in the ocean was initiated in [16]. Let us consider (2.1) as a metapopulation model with delay and piecewise constant argument. Let $x(t)$ denote the density of the species colonized on the islands at time t; r be the colonization rate and $E(t)$ be the extinction rate due to the immigration. According to the model (2.1), the rate of change of the population depends not only on the time t at which they are determined, but somehow depends on its past history denoted by $t-\tau$ as well as on constant values given by the function $\beta(t)$. For a logistic type equation with a harvesting term where we do not know the population at the exact time, we need it to determine the harvesting rate. Hence, there is always a delay in getting the information of the population. We take this effect into account by the term $E(t)x(\beta(t))$. To illustrate the effect, we may think the population which meet at the beginning of immigration season with their instinctive evaluations of the population state, environment and implicitly decide which living conditions to prefer and where to go in line with group hierarchy, communications, dynamics and then adapt to those conditions [8].

In particular, if we set $\theta_i = \dfrac{2i-1}{10}$ for $i \in \mathbb{Z}$, $\phi_0(t) = 1$ on $[-\tau, 0]$ and $E(t) = \begin{cases} 1 & \text{if} \quad \theta_{2i} \le t < \theta_{2i+1}, \\ 2 & \text{if} \quad \theta_{2i+1} \le t < \theta_{2i+2}, \end{cases}$ we can apply the results obtained below for appropriate parameters r, a and b of the model.

Definition 2.2. A function $x(t)$ is a solution of (2.1) and (2.2) on $[-\tau, \infty)$ if:

(i) $x(t)$ is continuous on $[-\tau, \infty)$ and $x(t) = \phi_0(t)$, $t \in [-\tau, 0]$,
(ii) the derivative $x'(t)$ exists for $t \in \mathbb{R}^+$ with the possible exception of the points θ_i, $i \ge k_0 + 1$, where one-sided derivatives exist. (We understand $x'(0)$ to mean the right hand derivative)
(iii) equation (2.1) is satisfied by $x(t)$ on each interval $[0, \theta_{k_0+1})$, $[\theta_i, \theta_{i+1})$, $i \ge k_0 + 1$.

Theorem 2.3. *Let (A1)-(A3) be satisfied. Then* (2.1)-(2.2) *has a unique solution defined for all* $t \in [-\tau, \infty)$ *in the sense of Definition 2.2.*

Proof. We shall apply the method of steps. We can find $k_n \in \mathbb{Z}$ such that $\theta_{k_n} \le n\tau < \theta_{k_n+1}$for $n = 0, 1, 2, \cdots$. We have $x(t) = \phi_0(t)$ for all $t \in [-\tau, 0]$. First, we consider the interval $[0, \tau]$. For $t \in [0, \tau]$, we obtain

$$x'(t) = rx(t)(1 - a\phi_0(t-\tau) - bx(\beta(t))) - E(t)x(\beta(t)),$$

which is a differential equation with piecewise constant argument. For $t \in [0, \theta_{k_0+1})$, the solution $x(t)$ satisfies the following initial value problem

$$\begin{aligned} &y'(t) = ry(t)(1 - a\phi_0(t-\tau) - b\phi_0(\theta_{k_0})) - E(t)\phi_0(\theta_{k_0}), \\ &y(0) = x_0, \end{aligned}$$

which is a linear ordinary differential equation. Since the functions $\phi_0(t-\tau)$ and $E(t)$ are continuous on $[0, \theta_{k_0+1})$, it follows that $x(t)$ exists uniquely on this interval. From the continuity of solutions, we have $x(\theta_{k_0+1}) = \lim\limits_{t \to \theta_{k_0+1}^-} x(t)$. Continuing the process on each interval $t \in [\theta_j, \omega]$, where $\omega = \theta_{j+1}$ if $k_0 + 1 \le j \le k_1 - 1$, and $\omega = \tau$ if $j = k_1$, we can see that $x(t)$ satisfies the initial value problem

$$\begin{aligned} &y'(t) = ry(t)(1 - a\phi_0(t-\tau) - by(\theta_j)) - E(t)y(\theta_j), \\ &y(\theta_j) = x(\theta_j). \end{aligned}$$

For the same reason as that behind the existence and uniqueness of the solution of linear ordinary differential equations with continuous coefficients, we obtain that the solution is uniquely defined on $[\theta_j, \omega]$. Therefore, there exists a unique solution $x(t) = \phi_1(t)$ of (2.1)-(2.2) on $t \in [0, \tau]$.

Next, consider the interval $[\tau, 2\tau]$. For $t \in [\tau, 2\tau]$, we have

$$x'(t) = rx(t)(1 - a\phi_1(t-\tau) - bx(\beta(t))) - E(t)x(\beta(t)),$$

a differential equation with piecewise constant argument. If we apply a similar technique used for the interval $[0, \tau]$, we see that the solution $x(t) = \phi_2(t)$ exists and it is unique on the interval $[\tau, 2\tau]$. In fact, if we proceed for $t \in [n\tau, (n+1)\tau]$, $n = 2, 3, \cdots$, similarly, we find a unique solution $x(t) = \phi_{n+1}(t)$. □

In the next result, we shall construct an equivalent integral equation for the system (2.1)-(2.2) by applying a similar technique considered in the book [9].

Theorem 2.4. *Let (A1)-(A3) be fulfilled. Then finding a solution of equation* (2.1) *through* (2.2) *is equivalent to solving the following integral equation*

$$x(t) = x_0 + \int_0^t \{rx(s)(1 - ax(s-\tau) - bx(\beta(s))) - E(s)x(\beta(s))\}ds, \tag{2.3}$$

for $t \geq 0$ *where* $x(t) = \phi_0(t),\ t \in [-\tau, 0]$.

Proof. Necessity. Let $x(t)$ be a solution of (2.1) satisfying (2.2). Then $x(t)$ satisfies the conditions listed in Definition 2.2. We define a function ψ such that $\psi(t) = \phi_0(t)$, $t \in [-\tau, 0]$ and

$$\psi(t) = x_0 + \int_0^t \{rx(s)(1 - ax(s-\tau) - bx(\beta(s))) - E(s)x(\beta(s))\}ds,\ t \geq 0.$$

We can see by direct computation that the integral on the right side of the last equation exists.

Denote $z(t) = \psi(t) - x(t)$ for $t \geq -\tau$. It is clear that $z(t) = 0$ on $[-\tau, 0]$. For $t \neq \theta_j$, $j \geq k_0 + 1$, we have

$$\psi'(t) = rx(t)(1 - ax(t-\tau) - bx(\beta(t))) - E(t)x(\beta(t))$$

and

$$x'(t) = rx(t)(1 - ax(t-\tau) - bx(\beta(t))) - E(t)x(\beta(t)).$$

Thus, we obtain $z'(t) = 0$ for $t \neq \theta_j$, $j \geq k_0 + 1$. Moreover, it can be seen by straightforward evaluation that $\lim_{t\to\theta_i{}^-} z'(t) = \lim_{t\to\theta_i{}^+} z'(t)$. Hence, $z'(t) = 0$ for all $t \geq 0$ and $z(0) = 0$. Consequently, we derive $z(t) = 0$, $t \in [-\tau, 0]$, i.e., $\psi(t) = x(t)$ for $t \geq -\tau$.

Sufficiency. Assume that (2.3) holds true. Consider the interval $[0, \theta_{k_0+1})$. If we differentiate (2.3) for $t \in (0, \theta_{k_0+1})$, we can see that $x(t)$ satisfies (2.1). Since $x(\beta(t))$ and $E(t)$ are right continuous functions, we find by taking the limit as $t \to \theta_i{}^+$ that $x(t)$ satisfies (2.1) on $[0, \theta_{k_0+1})$. Now, fix $i \in \mathbb{Z}$, $i \geq k_0 + 1$ and differentiate (2.3) for $t \in (\theta_i, \theta_{i+1})$. Then, we derive that $x(t)$ satisfies (2.1). In fact, $x(t)$ satisfies (2.1) on $t \in [\theta_i, \theta_{i+1})$ due to a similar discussion made for the interval $[0, \theta_{k_0+1})$. □

3. Existence and boundedness of positive solutions

First, let us consider the following linear differential equation with piecewise constant argument of generalized type

$$X'(t) = -E(t)X(\beta(t)),\ t \geq 0, \tag{3.1}$$

and the corresponding differential inequality with piecewise constant argument of generalized type

$$Y'(t) \leq -E(t)Y(\beta(t)),\ t \geq 0, \tag{3.2}$$

where assumption (A2) holds for the function $E(t)$.

Lemma 3.1. *Let* $Y(t)$ *be a positive continuous solution of* (3.2) *for* $t \geq \theta_i$ *for some* $i \geq k_0 + 1$. *Suppose that* $X(t)$ *is a solution of* (3.1) *with* $X(\theta_i) = Y(\theta_i)$. *Then* $Y(t) \leq X(t)$ *for* $t \geq \theta_i$.

Proof. Let $t \in [\theta_i, \theta_{i+1})$. For $\theta_i \leq t_1 < t_2 < \theta_{i+1}$, we get

$$Y(t_2) - Y(t_1) \leq -\int_{t_1}^{t_2} E(s)Y(\theta_i)ds = -\int_{t_1}^{t_2} E(s)X(\theta_i)ds = X(t_2) - X(t_1).$$

If we set $t_1 = \theta_i$ and $t_2 = t$ in the last inequality, it follows from the assumption $X(\theta_i) = Y(\theta_i)$ that

$$Y(t) \leq X(t), \qquad t \in [\theta_i, \theta_{i+1}). \tag{3.3}$$

From the continuity of solutions, it follows that

$$Y(\theta_{i+1}) \leq X(\theta_{i+1}). \tag{3.4}$$

Letting $t_1 = t$ and $t_2 \to \theta_{i+1}$ in the inequality $Y(t_2) - Y(t_1) \leq X(t_2) - X(t_1)$, we find that

$$X(t) - X(\theta_{i+1}) \leq Y(t) - Y(\theta_{i+1}), \quad t \in [\theta_i, \theta_{i+1}). \tag{3.5}$$

The inequalities (3.4) and (3.5) lead us to $\dfrac{X(t) - X(\theta_{i+1})}{X(\theta_{i+1})} \leq \dfrac{Y(t) - Y(\theta_{i+1})}{Y(\theta_{i+1})}$ or equivalently,

$$X(t) \leq \frac{X(\theta_{i+1})}{Y(\theta_{i+1})} Y(t), \quad t \in [\theta_i, \theta_{i+1}].$$

Define $Y_1(t) = \dfrac{X(\theta_{i+1})}{Y(\theta_{i+1})} Y(t)$. Then, we obtain that

$$X(t) \leq Y_1(t), \quad t \in [\theta_i, \theta_{i+1}) \quad \text{and} \quad Y_1(\theta_{i+1}) = X(\theta_{i+1}).$$

Next, we shall consider the interval $[\theta_{i+1}, \theta_{i+2})$. For $\theta_{i+1} \leq t_1 < t_2 < \theta_{i+2}$, inequality (3.2) leads to

$$Y_1(t_2) - Y_1(t_1) \leq X(t_2) - X(t_1).$$

Taking $t_1 = \theta_{i+1}$ and $t_2 = t$, and remembering the fact that $Y_1(\theta_{i+1}) = X(\theta_{i+1})$, we find that

$$Y_1(t) \leq X(t), \qquad t \in [\theta_{i+1}, \theta_{i+2}).$$

Since $Y(t) \leq Y_1(t)$, it follows that

$$Y(t) \leq X(t), \qquad t \in [\theta_{i+1}, \theta_{i+2}), \tag{3.6}$$

and in particular $Y(\theta_{i+2}) \leq X(\theta_{i+2})$. If we combine the results given by (3.3) and (3.6), we see that $Y(t) \leq X(t)$, $t \in [\theta_i, \theta_{i+2}]$. Continuing the process on each interval $[\theta_k, \theta_{k+1})$, $k = i+2, i+3, \cdots$, in a similar manner, we get $Y(t) \leq X(t)$ for all $t \geq \theta_i$. □

Definition 3.2. A solution $X(t)$ of (3.1) is said to be eventually positive (or eventually negative) if there exists $T \geq 0$ such that $X(t) > 0$ (or $X(t) < 0$) for all $t \geq T$. A solution of $X(t)$ of (3.1) is said to be nonoscillatory if it is either eventually positive or eventually negative; otherwise, it is oscillatory.

Lemma 3.3. *If $X(t)$ is a nonoscillatory solution of (3.1), then $\lim_{t\to\infty} X(t) = 0$.*

Proof. Suppose that $X(t)$ is a nonoscillatory solution of (3.1). Without loss of generality, we may assume that $X(t) > 0$ for $t \geq T \geq 0$. It is clear that there exists $i \geq k_0$ such that $T \in [\theta_i, \theta_{i+1})$. Thus, $X(\beta(t)) > 0$ for $t \geq \theta_{i+1}$. This means that $X'(t) \leq 0$ for $t \geq \theta_{i+1}$ and hence there exists $L \geq 0$ such that $\lim_{t\to\infty} X(t) = L$.

Suppose $L > 0$. Then for any $\varepsilon > 0$, we can find $t^* \geq \theta_{i+1}$ such that $L < X(t) < L + \varepsilon$ for $t \geq t^*$. Then $-E(t)(L+\varepsilon) \leq X'(t) \leq -E(t)L$ for $t \geq \beta(t^*) + \theta$. Integrating the last inequality from $\beta(t^*) + \theta$ to t and using the condition (A2), we see that $\lim_{t\to\infty} X(t) = -\infty$, a contradiction. This shows that $L = 0$. □

Lemma 3.4. *Suppose that assumptions (A1)-(A3) hold. If $\sup_{t>0} \int_{\beta(t)}^{t} E(u)du < 1$, then the solution $X(t)$ of (3.1) satisfying $X(t) = \phi_0(t)$ for $t \in [-\tau, 0]$ and $X(0) = x_0$ is positive for all $t \geq 0$, and thus it is nonoscillatory.*

Proof. First, consider the interval $[0, \theta_{k_0+1})$. For $t \in [0, \theta_{k_0+1})$, we have $X'(t) = -E(t)\phi_0(\theta_{k_0})$ with $X(0) = x_0$. It follows from the hypotheses of the statement that the solution of this initial value problem satisfies the inequality $X(t) \geq x_0(1 - \int_0^t E(u)du)$ and hence it is positive. Continuity of solutions implies that $X(\theta_{k_0+1}) > 0$.

Suppose that solution of (3.1) with $X(t) = \phi_0(t)$, $t \in [-\tau, 0]$ and $X(0) = x_0$ is positive on $[0, \theta_j]$ for some $j \geq k_0 + 1$. If $t \in [\theta_j, \theta_{j+1}]$, then we have $X(t) = X(\theta_j)(1 - \int_{\theta_1}^t E(u)du) > 0$. Therefore, solution of (3.1) with $X(t) = \phi_0(t)$, $t \in [-\tau, 0]$ and $X(0) = x_0$ is positive for all $t \geq 0$, and thus it is nonoscillatory. □

Lemma 3.5. *If there exists $\hat{t} \geq 0$ such that $x(\hat{t}) > \kappa$ for any solution $x(t)$ of (2.1), then $x(\tilde{t}) = \kappa$ for some $\tilde{t} > \hat{t}$.*

Proof. Assume on the contrary that $x(t) > \kappa$ for all $t \geq \hat{t}$. We can find an $i \in \mathbb{Z}$ such that $\hat{t} + \tau \in [\theta_i, \theta_{i+1})$. For $t \geq \theta_{i+1}$, we have $1 - ax(t-\tau) - bx(\beta(t)) \leq 0$. Hence, it follows from the equation (2.1) that

$$x'(t) \leq -E(t)x(\beta(t)), \ t \geq \theta_{i+1}.$$

Let $X(t)$ be the solution of the equation

$$X'(t) = -E(t)X(\beta(t)), \ t \geq \theta_{i+1},$$

with $X(\theta_{i+1}) = x(\theta_{i+1})$.

Lemma 3.1 and Lemma 3.3 imply that $x(t) \leq X(t)$ for $t \geq \theta_{i+1}$ and $\lim_{t\to\infty} X(t) = 0$. Hence, we obtain $\lim_{t\to\infty} x(t) = 0$, a contradiction proving the lemma. □

Lemma 3.6. *Let $x(t)$ be a solution of (2.1) satisfying (2.2) and $x_0 > \kappa$. Then there exists $\hat{t} > 0$ such that $x(\hat{t}) = \kappa$ and $0 < x(t) \leq x_0 \mathrm{e}^{r\tau}$, $t \in [0, \hat{t}]$.*

Proof. Define $\mathcal{A} = \{t > 0 : x(t) = \kappa\}$. Since $x_0 > \kappa$, Lemma 3.5 implies that $\mathcal{A}$ is not empty. Let $\hat{t}$ be the smallest element of $\mathcal{A}$. Then, $x(t) > \kappa$ for $t \in [0, \hat{t})$ and $x(\hat{t}) = \kappa$. Being continuous on $[0, \hat{t}]$, $x(t)$ has a maximum at a point belonging to the interval $[0, \hat{t})$. Let $\tilde{t} \in [0, \hat{t})$ denote the smallest member such that $x(\tilde{t})$ is maximum.

Now, we claim that $\tilde{t} \in [0, \tau]$. In order to prove this assertion, let us consider the cases $\hat{t} \leq \tau$ and $\hat{t} > \tau$. Suppose $\hat{t} \leq \tau$. Then, it is obviuos that $\tilde{t} \in [0, \tau]$ holds true.

Suppose $\hat{t} > \tau$. Then, $x(t-\tau) > \kappa$ and $x(\beta(t)) \geq \kappa$ for $t \in [\tau, \hat{t}]$. Hence, it can be inferred from (2.1) that $x'(t) \leq 0$ for $t \in [\tau, \hat{t}]$. This means that $\tilde{t}$ should be located in the interval $[0, \tau]$.

Note that $x(t)$ is nonnegative for $t \in [-\tau, \tilde{t}]$. Hence, we can derive from (2.1) that $x'(t) \leq rx(t)$ for all $t \in [0, \tilde{t}]$. If we integrate the last inequality over the interval $[0, \tilde{t}]$, we obtain

$$x(\tilde{t}) \leq x_0 \mathrm{e}^{r\tilde{t}} \leq x_0 \mathrm{e}^{r\tau}.$$

That is, $0 < x(t) \leq x_0 \mathrm{e}^{r\tau}$, $t \in [0, \hat{t})$, as desired. □

Lemma 3.7. *For a solution $x(t)$ of (2.1) and (2.2), if there exists $\hat{t} \geq 0$ such that $x(\hat{t}) = \kappa$ and $x(t) \geq 0$ on $[0, \tilde{t})$ for some $\tilde{t} > \hat{t}$, then $x(\tilde{t}) \leq \kappa \mathrm{e}^{r\tau}$.*

Proof. Assume that $x(\tilde{t}) > \kappa \mathrm{e}^{r\tau}$ and consider the set

$$S = \{t \geq \hat{t} : x(t-\tau) = \kappa\}.$$

Since $\hat{t} + \tau \in S$, $S \neq \emptyset$. By Lemma 3.5, there exists a $T > \tilde{t}$ such that $x(T) = \kappa$. Hence, $x(t)$ has a local maximum on $[\tilde{t}, \infty)$. Let t^*, $t^* \geq \tilde{t}$, be the first point at which these local maximums occur. Thus, we have $x(t^*) \geq x(\tilde{t})$. Set

$$S^* = \{t \in S : t - \tau < t^*\}.$$

We can see that the set S^* is not empty since it contains the element $\hat{t}+\tau$. We denote $\alpha = \sup S^*$ and claim that $t^* \in [\alpha-\tau, \alpha]$. It is clear from the definition of α that $\alpha - \tau \leq t^*$. In order to verify the inequality $t^* \leq \alpha$, we consider two cases:

In the first case, assume that there exist $t_0 \in (\alpha-\tau, \alpha]$ such that $x(t_0) \leq \kappa$. It will be sufficient to show that $t^* < t_0$. On the contrary, assume that $t^* > t_0$. We note that $t_0 \neq t^*$ due to the fact that $x(t_0) \leq \kappa < x(t^*)$. Additionally, this fact together with the continuity of solutions and intermediate value theorem leads to the existence of $w \in [t_0, t^*)$ such that $x(w) = \kappa$. Thus, $w+\tau \in S^*$. By the definition of α, we should have $w + \tau \leq \alpha$, which implies in turn that $w \leq \alpha - \tau < t_0$. However, $t_0 \in (\alpha-\tau, \alpha]$ and $w \in [t_0, t^*)$ give us $w + \tau > \alpha$, which is a contradiction. Thus, for that case the assertion is valid.

Secondly, we assume that $x(t) > \kappa$ for all $t \in (\alpha - \tau, \alpha]$. Lemma 3.5 implies that we can find a point $t > \alpha$ such that $x(t) = \kappa$. Denote by α^* the smallest member of these values. Then, $x'(t) \leq 0$ on the interval $[\alpha, \alpha^*]$. It follows from the definition of t^* that $t^* < \alpha$, conforming the assertion for the second case.

By the discussions made above, we see that $t^* \in [\alpha-\tau, \alpha]$ holds true. Now, we will show that $\alpha - \tau < \tilde{t}$. In order to prove the last inequality, let us assume that $\alpha-\tau > \tilde{t}$ noting that $\alpha - \tau \neq \tilde{t}$ since $x(\alpha-\tau) = \kappa < x(\tilde{t}\,)$. This assumption implies the existence of a point of local maximum in $[\tilde{t}, \alpha-\tau)$, which is a contradiction to the definition of t^* and to the fact that $t^* \in [\alpha-\tau, \alpha]$. Hence, we have $\alpha - \tau < \tilde{t}$ and thus $x(t) \geq 0$ for $t \leq t^*$. All these discussions lead us to the inequality $x'(t) \leq r x(t)$ on $[\alpha-\tau, t^*]$. Integrating this differential inequality over the interval $[\alpha-\tau, t^*]$, we obtain that

$$x(t^*) \leq x(\alpha-\tau)\mathrm{e}^{r(t^*-(\alpha-\tau))} \leq \kappa \mathrm{e}^{r\tau} < x(\tilde{t}\,),$$

which contradicts that $x(t^*) \geq x(\tilde{t}\,)$. Thus, the proof is completed. □

Intuitively, one expects the existence of a positive solution, i.e., the survival of the species only for small harvesting. Our results given below supports this expectation.

Theorem 3.8. *Let (A1)-(A3) be fulfilled and* $\mathrm{e}^{r\theta(\mu \mathrm{e}^{r\tau}-1)} \sup\limits_{t>0} \displaystyle\int_{\beta(t)}^{t} E(u)du < 1$. *If* $x(t)$ *is a solution of* (2.1) *and* (2.2) *with* $\phi_0(t) \leq x_0 \leq \kappa$ *on* $[-\tau, 0]$, *then*

$$0 < x(t) \leq \kappa \mathrm{e}^{r\tau}. \tag{3.7}$$

Proof. Assume, on the contrary, that (3.7) does not hold. Then, exactly one of the following two possibilities must occur:

(i) there exists $T > 0$ such that $x(t) > 0$ on $[0, T)$ and $x(T) > \kappa \mathrm{e}^{r\tau}$, or
(ii) there exists $T > 0$ such that $0 < x(t) \leq \kappa \mathrm{e}^{r\tau}$ on $[0, T)$ and $x(T) = 0$.

Suppose that the first possibility (i) holds for a solution $x(t)$ of (2.1) and (2.2). Then, we have $x(0) = x_0 \leq \kappa < x(T)$. So, the equality $x(\bar{t}) = \kappa$ is satisfied for some $\bar{t} \in [0, T)$. Now, it follows from Lemma 3.7 that $x(T) \leq \kappa \mathrm{e}^{r\tau}$, which contradicts the hypothesis. Hence, (i) can never hold.

Now let us take the second possibility (ii) into consideration. Suppose that (ii) is fulfilled. If we write

$$x(t) = \exp\left(r\int_0^t \left(1 - ax(s-\tau) - bx(\beta(s))\right) ds\right) X(t) \tag{3.8}$$

in (2.1) and (2.2) and assume that $r = 0$ for $t \in [-\tau, 0)$, we obtain the following system

$$\begin{aligned} &X'(t) = -P(t)X(\beta(t)),\ t \geq 0,\\ &X(t) = \phi_0(t),\ t \in [-\tau, 0],\ X(0) = x_0, \end{aligned} \tag{3.9}$$

where $P(t) = E(t)\exp\left(-r\displaystyle\int_{\beta(t)}^t \left(1 - ax(s-\tau) - bx(\beta(s))\right) ds\right)$.

Since $x_0 \le \kappa$, we have $\mu = 1$. For $t \in [0, T)$, we get

$$\begin{aligned}\int_{\beta(t)}^{t} P(u)du &= \int_{\beta(t)}^{t} E(u)\exp\left(-r\int_{\beta(u)}^{u}(1-ax(s-\tau)-bx(\beta(s)))\,ds\right)du \\ &\le \mathrm{e}^{r\theta(\mu \mathrm{e}^{r\tau}-1)}\sup_{t>0}\int_{\beta(t)}^{t} E(u)du < 1.\end{aligned}$$

Since $X(0) = x(0) > 0$, it follows from Lemma 3.4 that $X(T) > 0$. Therefore, we find $x(T) > 0$, which contradicts (ii). This completes the proof. □

Theorem 3.9. *Let (A1)-(A3) be fulfilled and* $\mathrm{e}^{r\theta(\mu \mathrm{e}^{r\tau}-1)}\sup_{t>0}\int_{\beta(t)}^{t} E(u)du < 1$. *If* $x(t)$ *is a solution of* (2.1) *and* (2.2) *with* $\phi_0(t) \le x_0$ *for* $t \in [-\tau, 0]$ *and* $x_0 > \kappa$, *then there exists* $\hat{t} > 0$ *such that*

$$0 < x(t) \le x_0 \mathrm{e}^{r\tau}, \quad t \in [0, \hat{t}\,) \tag{3.10}$$

and

$$0 < x(t) \le \kappa \mathrm{e}^{r\tau}, \quad t \ge \hat{t}. \tag{3.11}$$

Proof. Suppose the contrary. We know from Lemma 3.6 that there exists $\hat{t} > 0$ such that $0 < x(t) \le x_0\mathrm{e}^{r\tau}$ on $[0, \hat{t}\,)$ and $x(\hat{t}\,) = \kappa$, i.e., (3.10) holds true. Thus, we assume that (3.11) is not true. Then, exactly one of the following two possibilities must occur:

(i) there exists $T > 0$ such that $x(t) > 0$ on $[\hat{t}, T)$ and $x(T) > \kappa\mathrm{e}^{r\tau}$, or
(ii) there exists $T > 0$ such that $0 < x(t) \le \kappa\mathrm{e}^{r\tau}$ on $[\hat{t}, T)$ and $x(T) = 0$.

It can be shown easily that the first possibility (i) can not be true due to Lemma 3.7. Suppose that (ii) is valid. Similar to the proof of Theorem 3.8, when (3.8) is substituted in (2.1) and (2.2), (3.9) is obtained. Note that $\mu = (a + b)x_0$. Then for $t \in [\hat{t}, T)$, we have

$$\begin{aligned}\int_{\beta(t)}^{t} P(u)du &= \int_{\beta(t)}^{t} E(u)\exp\left(-r\int_{\beta(u)}^{u}(1-ax(s-\tau)-bx(\beta(s)))\,ds\right)du \\ &\le \mathrm{e}^{r\theta(\mu \mathrm{e}^{r\tau}-1)}\sup_{t>0}\int_{\beta(t)}^{t} E(u)du < 1.\end{aligned}$$

Then, it follows from Lemma 3.4 that $X(T) > 0$, which implies in turn that $x(T) > 0$. Since the last inequality contradicts (ii), the proof is done. □

4. Permanence results

Let us review a few mathematical terms which will be used in this section.

Definition 4.1. A set $\mathcal{V} \subset \mathbb{R}$ is called a positively invariant set of (2.1) if for all $s \in [-\tau, 0]$, $\phi_0(s) \subset \mathcal{V}$ implies $x(t, 0, \phi_0) \subset \mathcal{V}$, $t \ge 0$.

Definition 4.2. The solution of (2.1) is said to be ultimately bounded if there exists $B > 0$ such that for every solution $x(t)$ of (2.1), there exists $T > 0$ such that $|x(t)| \le B$, for all $t \ge T$, where B is independent of particular solution while T may depend on the solution.

Definition 4.3. Equation (2.1) is said to be permanent if there exist positive constants ρ and ν such that

$$\rho \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le \nu$$

for all solutions of (2.1) with the initial condition (2.2).

Definition 4.4. Equation (2.1) is said to be non-persistent if there exists a positive solution $x(t)$ satisfying

$$\liminf_{t\to\infty} x(t) = 0.$$

Now, let us consider the permanence results for (2.1). Unless otherwise stated, we shall assume the following inequality in addition to (A1)-(A3):

(A4) $\mathrm{e}^{r\theta(\mu \mathrm{e}^{r\tau}-1)} \sup_{t>0} \int_{\beta(t)}^{t} E(u)du < 1,$

which ensures that a solution $x(t)$ of (2.1)-(2.2) is positive.

From Theorem 3.8 and Theorem 3.9, we can conclude the following result.

Theorem 4.5. *The set* $(0, \mu\kappa \mathrm{e}^{r\tau})$ *is positively invariant for system* (2.1).

Denote $E^l := \inf_{t\in\mathbb{R}^+} E(t)$, $E^u := \sup_{t\in\mathbb{R}^+} E(t)$.

Lemma 4.6. *If* $r - E^l \mathrm{e}^{-r\theta} > 0$, *then* $\limsup_{t\to+\infty} x(t) \le K = \dfrac{r - E^l \mathrm{e}^{-r\theta}}{r(a\mathrm{e}^{-r\tau} + b\mathrm{e}^{-r\theta})}$.

Proof. From equation (2.1) and the positivity of solutions of this equation, it follows that

$$x'(t) \le rx(t) \text{ for all } t \ge 0.$$

This inequality leads us to

$$x(t) \le x(\theta_i)\mathrm{e}^{r(t-\theta_i)} \le x(\beta(t))\mathrm{e}^{r\theta}$$

on each interval $[0, \theta_{k_0+1})$ and $[\theta_i, \theta_{i+1})$, $i \ge k_0 + 1$. Since the solution $x(t)$ is continuous, we can state that

$$x(t) \le x(\beta(t))\mathrm{e}^{r\theta} \text{ for all } t \ge 0,$$

which is equivalent to $x(\beta(t)) \ge x(t)\mathrm{e}^{-r\theta}$ for all $t \ge 0$. We can also find that

$$x(t) \le x(t-\tau)\mathrm{e}^{r\tau} \text{ for all } t \ge \tau.$$

Then for $t \ge \tau$, we have

$$\begin{aligned} x'(t) &\le rx(t)\left(1 - a\mathrm{e}^{-r\tau}x(t) - b\mathrm{e}^{-r\theta}x(t)\right) - E^l\mathrm{e}^{-r\theta}x(t) \\ &= x(t)\left[r - E^l\mathrm{e}^{-r\theta} - \left(ra\mathrm{e}^{-r\tau} + rb\mathrm{e}^{-r\theta}\right)x(t)\right]. \end{aligned}$$

A comparison argument [4] shows that

$$\limsup_{t\to+\infty} x(t) \le K = \frac{r - E^l\mathrm{e}^{-r\theta}}{r(a\mathrm{e}^{-r\tau} + b\mathrm{e}^{-r\theta})}.$$

□

For convenience, let us adopt the following notations.

(N1) $A = r(1 - aK - bK)$ and $B = E^u K$;

(N2) $L_1 = -r(a\mathrm{e}^{-A\tau} + b\mathrm{e}^{-A\theta})$;

(N3) $L_2 = r\left[1 - a\frac{B}{A}(1 - \mathrm{e}^{-A\tau}) - b\frac{B}{A}(1 - \mathrm{e}^{-A\theta}) - \dfrac{E^u\mathrm{e}^{-A\theta}}{r}\right]$;

(N4) $L_3 = -E^u\frac{B}{A}(1 - \mathrm{e}^{-A\theta})$.

Lemma 4.7. *If* $\kappa < K$, $L_2 > 0$ *and* ${L_2}^2 - 4L_1L_3 \geq 0$ *then*

$$\liminf_{t\to+\infty} x(t) \geq k = \frac{-L_2 + \sqrt{{L_2}^2 - 4L_1L_3}}{2L_1}.$$

Proof. Since $\limsup_{t\to+\infty} x(t) \leq K$, for any sufficiently small $\varepsilon > 0$, there is some $T > 0$ such that for $t \geq T$, $x(t) < K_\varepsilon$ where $K_\varepsilon = K + \varepsilon$. We can find a $j \in \mathbb{Z}$ such that $\theta_j \leq T + \tau < \theta_{j+1}$. We derive from the equation (2.1) for $t \geq \theta_{j+1}$ that

$$x'(t) \geq rx(t)\left(1 - aK_\varepsilon - bK_\varepsilon\right) - E^u K_\varepsilon.$$

We denote $A_\varepsilon = r\left(1 - aK_\varepsilon - bK_\varepsilon\right)$ and $B_\varepsilon = E^u K_\varepsilon$. Note that $A_\varepsilon < 0$ and $B_\varepsilon > 0$. Then for $t \geq \theta_{j+1}$, we have

$$x(\beta(t)) \leq \frac{B_\varepsilon}{A_\varepsilon}(1 - \mathrm{e}^{-A_\varepsilon\theta}) + \mathrm{e}^{-A_\varepsilon\theta}x(t) \text{ and } x(t-\tau) \leq \frac{B_\varepsilon}{A_\varepsilon}(1 - \mathrm{e}^{-A_\varepsilon\tau}) + \mathrm{e}^{-A_\varepsilon\tau}x(t),$$

which imply in turn that

$$x'(t) \geq L_1(\varepsilon)x^2(t) + L_2(\varepsilon)x(t) + L_3(\varepsilon), \tag{4.1}$$

where

$L_1(\varepsilon) = -r(a\mathrm{e}^{-A_\varepsilon\tau} + b\mathrm{e}^{-A_\varepsilon\theta}) < 0,$

$L_2(\varepsilon) = r\left[1 - a\frac{B_\varepsilon}{A_\varepsilon}(1 - \mathrm{e}^{-A_\varepsilon\tau}) - b\frac{B_\varepsilon}{A_\varepsilon}(1 - \mathrm{e}^{-A_\varepsilon\theta}) - \dfrac{E^u\mathrm{e}^{-A_\varepsilon\theta}}{r}\right]$ and

$L_3(\varepsilon) = -E^u\frac{B_\varepsilon}{A_\varepsilon}(1 - \mathrm{e}^{-A_\varepsilon\theta}) < 0.$

Pick $\varepsilon > 0$ so small that $L_2(\varepsilon) > 0$ and $L_2(\varepsilon)^2 - 4L_1(\varepsilon)L_3(\varepsilon) \geq 0$. Then $L_1(\varepsilon)x^2(t) + L_2(\varepsilon)x(t) + L_3(\varepsilon) = 0$ has two real roots.

Using a comparison argument for the differential inequality (4.1) in a similar manner in the proof of Lemma 4.6 and considering $\varepsilon \to 0$, we find $\liminf_{t\to+\infty} x(t) \geq k$, where

$$k = \frac{-L_2 + \sqrt{{L_2}^2 - 4L_1L_3}}{2L_1}.$$

□

The following assertions on the ultimate boundedness, permanence and persistence follow directly from the proofs of Lemma 4.6 and Lemma 4.7.

Theorem 4.8. *Suppose all of the conditions of Lemma 4.6 and Lemma 4.7 are satisfied. Then the set defined by*

$$\Sigma = \{x \in \mathbb{R} \mid k \leq x \leq K\}$$

is an ultimately bounded region for the system (2.1)-(2.2).

Theorem 4.9. *Suppose all of the conditions of Lemma 4.6 and Lemma 4.7 are satisfied. Then equation* (2.1) *with the initial condition* (2.2) *is permanent.*

Theorem 4.10. *Suppose all of the conditions of Lemma 4.6 and Lemma 4.7 are satisfied. Then equation* (2.1) *with the initial condition* (2.2) *is persistent.*

5. Conclusion

We conclude with a brief discussion of our results. In ecology, we can observe some perturbations such as time delays that are not suitable to be treated classically. Time delays can be introduced into a model in several forms; including discrete delays, infinite delays or piecewise constant delays. It is reasonable, especially for ecological applications, to have a system with both constant (discrete) and piecewise constant delays as evolution of the system may depend not only on the past history but also on some previous constant values of the unknown function that corresponds to the fundamental information in memory. In this light, it is important to discuss the qualitative behaviors such as existence and uniqueness of solutions, existence and boundedness of positive solutions, permanence, persistence of such systems. Based on the realistic nature of the problem, we study the dynamics of a logistic equation which consists of both discrete and piecewise constant delays as well as a delayed harvesting term. These results can be worthy of future investigations.

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