Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 578–599
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Global stability of a time-delayed multi-group SIS epidemic model with nonlinear incidence rates and patch structure

Jinliang Wang[a], Yoshiaki Muroya[b], Toshikazu Kuniya[c,*]

[a] *School of Mathematical Science, Heilongjiang University, Harbin 150080, China.*

[b] *Department of Mathematics, Waseda University 3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169- 8555, Japan.*

[c] *Graduate School of System Informatics, Kobe University, 1-1 Rokkodai-cho, Nada-ku, Kobe 657-8501, Japan.*

Abstract

In this paper, we formulate and study a multi-group SIS epidemic model with time-delays, nonlinear incidence rates and patch structure. Two types of delays are incorporated to concern the time-delay of infection and that for population exchange among different groups. Taking into account both of the effects of cross-region infection and the population exchange, we define the basic reproduction number $\mathcal{R}_0$ by the spectral radius of the next generation matrix and prove that it is a threshold value, which determines the global stability of each equilibrium of the model. That is, it is shown that if $\mathcal{R}_0 \leq 1$, the disease-free equilibrium is globally asymptotically stable, while if $\mathcal{R}_0 > 1$, the system is permanent, an endemic equilibrium exists and it is globally asymptotically stable. These global stability results are achieved by constructing Lyapunov functionals and applying LaSalle's invariance principle to a reduced system. Numerical simulation is performed to support our theoretical results.

Keywords: SIS epidemic model, time-delay, nonlinear incidence rate, patch structure.
2010 MSC: 34D23, 34K20, 92D30.

1. Introduction

Based on the framework of Kermack and McKendrick [15], many epidemic models (systems of differential equations) and approximate schemes have been developed in order to understand the underlying phenomena and offer helpful guidance to prevent disease transmission. In particular, time-delayed models (see e.g.,

*Corresponding author
Email addresses: `jinliangwang@hlju.edu.cn` (Jinliang Wang), `ymuroya@waseda.jp` (Yoshiaki Muroya), `tkuniya@port.kobe-u.ac.jp` (Toshikazu Kuniya)

Received 2014-12-18

[3, 22, 28]), multi-group epidemic models (see e.g., [11, 17, 22, 24, 28, 29, 32, 33, 35, 38, 39]), patchy models (see e.g, [5, 12, 26, 36]) and models with general nonlinear incidence rates (see e.g., [8, 28, 29, 38]) play important roles in studying the transmission of disease. In this field, determining threshold conditions for the persistence, extinction of a disease, and global stability of equilibria remains one of the most challenging problems in the analysis of models due to the dimension of the model is higher. Yet such results are necessary for explanation of parameter thresholds for eradication of disease transmission.

The dispersal of species in spatially heterogeneous environment is very interesting topic which have attracted much attention of many scholars (see e.g., [6, 20, 30, 34]). When modeling the spread of infectious diseases in spatially heterogeneous host populations, dispersal among distinct patchy can be interpreted as the exchange that people travel or migrate among cities and regions or countries. By using monotone dynamical systems theory, many authors obtained some dynamical results, which mainly focus on the permanence and extinction of the populations. It should be pointed here that multi-group epidemic models have been formulated to describe the contracts or mixing between heterogeneous groups (different activity levels, sex, age, location etc.), and patchy models focus on the movement or dispersal (immigrate) of the individuals between the discrete spatial patches. In [36], Wang and Zhao proposed an epidemic model in order to simulate the dynamics of disease transmission under the influence of a population dispersal among patches. They established a threshold above which the disease is uniformly persistent and below which disease-free equilibrium is locally attractive, and globally attractive when both susceptible and infective individuals in each patch have the same dispersal rate. In [1], Arino and van den Driessche proposed n-city epidemic models to investigate the effects of inter-city travel on the spatial spread of infectious diseases among cities. In [14], Jin and Wang showed that the n-patch SIS model can be reduced to a monotone system, and the uniqueness and global stability of the endemic equilibrium can be achieved by assuming the dispersal rates of susceptible and infectious individuals are the same. In [21], Li and Shuai investigated an SIR compartmental epidemic model in a patchy environment where individuals in each compartment can travel among n patches. The global stability of equilibria is determined by threshold parameter R_0.

Communicable diseases such as influenza and sexual diseases can be easily transmitted from one country (or one region) to other countries (or other regions). Thus, it is important to consider the effect of population dispersal on spread of a disease [36]. This applies particularly to models involving nonlinearity and delays. Whereas there has been little discussion about how the combinations of time delays, nonlinear incidence rates and population dispersal affects the disease transmission dynamics in higher dimensional system of differential equations. It is, however, not well understood some problems on the mathematical properties (e.g., existence, uniqueness and stability of equilibria) of such models. From this point of view, we are interested in the work of Nakata and Röst [26]. For biological reason and mathematical viewpoint, to clarify such properties is always thought to be an important work. This motivates us to derive a more realistic delayed multi-group model that not only contains dispersal of humans but also incorporates nonlinear incidence rates.

The aim of this paper is threefold. First, we will investigate that under threshold condition, the model we will study is permanence. In the proof, we use a technique based on Muroya *et al.* [25]. Second, we will prove the existence of endemic equilibrium, which is proved by means of a monotone iterative technique proposed by Ortega and Rheinboldt [27] and Muroya [23]. Third, by constructing suitable Lyapunov functionals and applying LaSalle's invariance principle, we will prove that the threshold parameter (basic reproduction number) determines the global stability of equilibria in a sense that if $\mathcal{R}_0 \leq 1$ the disease-free equilibrium $\mathbf{E}^0$ of system (1.1) is globally asymptotically stable, while if $\mathcal{R}_0 > 1$ an endemic equilibrium $\mathbf{E}^*$ exists and it is globally asymptotically stable.

In this paper, we construct a time-delayed multi-group model which can be regarded as a generalization of the model studied in Lajmanovich and Yorke [18]. Based on above considerations, we propose the following time-delayed multi-group SIS epidemic model with nonlinear incidence rates and patch structure (that is, individuals in each patch can move to another patch):

$$\begin{cases} \dfrac{d}{dt}S_k(t) = b_k - \mu_k S_k(t) - \displaystyle\sum_{j=1}^{n} \beta_{kj} \int_0^{+\infty} K_{kj}(s) f_{kj}(S_k(t), I_j(t-s))ds \\ \qquad\qquad + \gamma_k I_k(t) + \displaystyle\sum_{j=1}^{n} \Big(\alpha_{kj} \int_0^{+\infty} L_{kj}(s) S_j(t-s)ds - \alpha_{jk} S_k(t) \Big), \\ \dfrac{d}{dt}I_k(t) = \displaystyle\sum_{j=1}^{n} \beta_{kj} \int_0^{+\infty} K_{kj}(s) f_{kj}(S_k(t), I_j(t-s))ds - (\mu_k + \gamma_k) I_k(t) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{n} \Big(\alpha_{kj} \int_0^{+\infty} L_{kj}(s) I_j(t-s)ds - \alpha_{jk} I_k(t) \Big), \quad k = 1, 2, \ldots, n. \end{cases} \tag{1.1}$$

In system (1.1), $S_k(t)$ and $I_k(t)$ denote the population densities of susceptible and infective individuals at time $t \in \mathbb{R}$ in group $k \in \{1, 2, \ldots, n\}$, respectively. For instance, to model a sexually transmitted disease, we can set $k = 1$ to be the subscript for female and $k = 2$ to be that for male. We list the parameters and their biological explanation as follows:

- $b_k > 0$, $\mu_k > 0$ and $\gamma_k > 0$ denote the birth rate, mortality rate and recovery rate for individuals in group k, respectively.
- $\beta_{kj} \geq 0$ is the coefficient of disease transmission from an infective individual in group j to a susceptible individual in group k.
- $\alpha_{kj} \geq 0$ is the rate of transfer of an individual from group j to group k.

It is advocated in [3] that we should incorporate time delays to investigate the spread of an infectious disease transmitted by a vector (e.g. mosquitoes, rats, etc.). In [3], under the assumptions that time-delay is determined by distribution kernels $f(s)$ and the vector population is proportional to that of infective humans at time $t - s$, the force of infection was given by $\beta I(t - s)$ and it was generalized to a distributed form

$$\beta \int_0^{+\infty} f(s) I(t-s) ds. \tag{1.2}$$

In system (1.1), we introduce an integral kernel $K_{kj}(s)$ to denote the probability a susceptible individual in group k infected by individuals in group j at time $t - s$ and becomes infective at time t. Then the force of infection to a susceptible individual in group k at time t is given by

$$\sum_{j=1}^{n} \beta_{kj} \int_0^{+\infty} K_{kj}(s) G(I_j(t-s)) ds. \tag{1.3}$$

The time delay used here represent the time during which the infectious agents develop in the vector. We assume that transfer of an individual from group j to group k can be affected by the time-delay and determined by distribution kernels $L_{kj}(s) \geq 0$ of past time $s \in \mathbb{R}_+$. The above force of infection (1.3) can be regarded as a further generalization of (1.2) by introducing nonlinear function of infective individuals.

Before going into details, we present some assumptions on these coefficients.

Assumption 1.1. **(i)** *The nonnegative matrices*

$$[\alpha_{kj}]_{1 \leq k,j \leq n} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \quad \text{and} \quad [\beta_{kj}]_{1 \leq k,j \leq n} = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \cdots & \beta_{nn} \end{bmatrix}$$

are irreducible (for the definition of irreducibility, see Berman and Plemmons [4] or Fiedler [10]);

(ii) *For each $k, j \in \{1, 2, \ldots, n\}$,*

$$\int_0^{+\infty} K_{kj}(s)\, ds = \int_0^{+\infty} L_{kj}(s)\, ds = 1. \tag{1.4}$$

Moreover, for simplicity (for more general settings, see Faria [9, Section 2]), we assume that

(iii) *There exists a positive constant M_0 such that*

$$\int_0^{+\infty} sK_{kj}(s)\, ds \leq M_0, \quad \text{for any} \ \ k, j \in \{1, 2, \ldots, n\}. \tag{1.5}$$

(i) of Assumption 1.1 implies that there exists a transportation (or infection) path from one group to every other groups. (ii) implies that these functions K_{kj} and L_{kj} are distributions in $\mathbb{R}_+$. (iii) is used to prove the uniform stability of the disease-free equilibrium and endemic equilibrium, respectively. In addition to these settings, and consider more various types of disease transmission, we assume that the transmission function in system (1.1) is given by a general nonlinear function $f_{kj}(\cdot, \cdot) \geq 0$. That is, we assume that the force of infection to a susceptible individual $S_k(t)$ in group k at time t is given by

$$\sum_{j=1}^{n} \beta_{kj} \int_0^{+\infty} K_{kj}(s) f_{kj}(S_k(t), I_j(t-s)) ds.$$

For a special case of this type of force of infection, see Beretta and Takeuchi [3], Enatsu *et al.* [8] and Xu and Ma [37].

Since system (1.1) contains an infinite delay, its associated initial condition needs to be restricted in an appropriate fading memory space. For any $\lambda_k \in (0, \mu_k + \gamma_k + \sum_{j=1}^{n} \alpha_{jk}), j = 1, 2, \ldots, n$, define the following Banach space of fading memory type (see e.g., [2] and references therein)

$$C_k = \ \{\phi_k \in C((-\infty, 0], \mathbb{R}_+) : \phi_k(s)e^{\lambda_k s} \text{ is uniformly continuous on } (-\infty, 0], \ \sup_{s \leq 0} |\phi_k(s)|e^{\lambda_k s} < \infty\}$$

and

$$Y_\Delta = \{\phi_k \in C_k : \phi_k(s) \geq 0 \text{ for } \ \text{all } s \leq 0\}$$

with norm $\|\phi\|_k = \sup_{s\leq 0} |\phi(s)|e^{\lambda_k s}$. Let $\phi_t \in C_k$ and $t > 0$ be such that $\phi_t(s) = \phi(t + s)$, $s \in (-\infty, 0]$. Let $\varphi_k, \psi_k \in C_k$ such that $\varphi_k(s), \psi_k(s) \geq 0$ for all $s \in (-\infty, 0]$. Throughout the paper, we consider solutions of system (1.1), $(S_1(t), I_1(t), S_2(t), I_2(t), \ldots, S_n(t), I_n(t))$, with initial conditions

$$(S_1(t), I_1(t), \ldots, S_n(t), I_n(t)) = (\varphi_1(t), \psi_1(t), \ldots, \varphi_n(t), \psi_n(t)), \ t \leq 0. \tag{1.6}$$

From the standard theory of functional differential equations (see e.g., [13]), we see that

$$(S_1(t), I_1(t), \ldots, S_n(t), I_n(t)) \in C_k$$

for all $t > 0$. We study system (1.1) in the following phase space

$$\mathbb{X}_{\mathfrak{d}} = \prod_{k=1}^{n} (C_k \times C_k).$$

For f_{kj}, we make the following assumption.

Assumption 1.2. *For each $k, j \in \{1, 2, \ldots, n\}$, f_{kj} belongs to $C^1\left(\mathbb{R}_+^2; \mathbb{R}_+\right)$ and satisfies the following conditions.*

(i) $f_{kj}(0, y) = f_{kj}(x, 0) = 0$ *for any* $(x, y) \in \mathbb{R}_+^2$*;*

(ii) *For any fixed* $y > 0$*,* $f_{kj}(x, y)$ *is strictly monotone increasing with respect to* $x \in \mathbb{R}_+$*;*

(iii) *For any fixed* $x > 0$*,* $f_{kj}(x, y)$ *is monotone nondecreasing with respect to* $y \in \mathbb{R}_+$*;*

(iv) *For any fixed* $x \geq 0$*,* $\frac{f_{kj}(x,y)}{y}$ *is monotone decreasing with respect to* $y \in \mathbb{R}_+ \setminus \{0\}$*;*

(v) *For any fixed* $x \geq 0$*, there exists a limit* $C_{kj}(x) := \lim\limits_{y \to +0} \dfrac{f_{kj}(x, y)}{y}$*;*

(vi) *For any fixed* $x > 0$*,* $\lim\limits_{y \to +\infty} \dfrac{f_{kj}(x - y, y)}{y} = -\infty$*.*

For instance, bilinear incidence rate $f_{kj}(x, y) := xy$ and saturated incidence rate $f_{kj}(x, y) := xy/(1 + a_{kj} y^p)$, where $a_{kj} > 0,\ k, j = 1, 2, \ldots, n$ and $0 < p < 1$, are well-known examples which satisfy Assumption 1.2 (see e.g., Enatsu *et al.* [8, (H1) and (H2)] for similar assumptions).

It is easy to see that the trivial equilibrium $\mathbf{E}^0 = (S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$ of system (1.1) always exists, which is called *disease-free equilibrium*. Here $S_k^0 > 0$, $k = 1, 2, \ldots, n$ is given by the solution of

$$b_k = (\mu_k + \tilde{\alpha}_{kk}) S_k^0 - \sum_{j=1}^{n} (1 - \delta_{kj}) \alpha_{kj} S_j^0, \quad k = 1, 2, \ldots, n, \tag{1.7}$$

where $\tilde{\alpha}_{kk} = \sum_{j=1}^{n} (1 - \delta_{jk}) \alpha_{jk}$ and δ_{kj} denotes the Dirac delta which equals one if $k = j$ and zero otherwise. Under Assumption 1.1, the existence and uniqueness of S_k^0, $k = 1, 2, \ldots, n$ are easily verified (see e.g., [16]). Using this S_k^0, we define the following matrix,

$$\mathbf{M}^0 := \left[\frac{\beta_{kj} C_{kj}(S_k^0) + (1 - \delta_{kj}) \alpha_{kj}}{\mu_k + \gamma_k + \tilde{\alpha}_{kk}} \right]_{1 \leq k,j \leq n}. \tag{1.8}$$

In fact, $\mathbf{M}^0 = \mathbf{V}^{-1} \mathbf{F}(\mathbf{S}^0)$, where $\mathbf{S} = (S_1^0, S_2^0, \cdots, S_n^0)^T$,

$$\mathbf{V} = \begin{bmatrix} \mu_1 + \gamma_1 + \tilde{\alpha}_{11} & 0 & \cdots & 0 \\ 0 & \mu_2 + \gamma_2 + \tilde{\alpha}_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \mu_n + \gamma_n + \tilde{\alpha}_{nn} \end{bmatrix}$$

and

$$\mathbf{F}(\mathbf{S}) = \begin{bmatrix} C_{11}(S_1)\beta_{11} & C_{12}(S_1)\beta_{12} + \alpha_{12} & \cdots & C_{1n}(S_1)\beta_{1n} + \alpha_{1n} \\ C_{21}(S_2)\beta_{21} + \alpha_{21} & C_{22}(S_2)\beta_{22} & \cdots & C_{2n}(S_2)\beta_{2n} + \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1}(S_n)\beta_{n1} + \alpha_{n1} & C_{n2}(S_n)\beta_{n2} + \alpha_{n2} & \cdots & C_{nn}(S_n)\beta_{nn} \end{bmatrix}.$$

It is easy to see that this matrix corresponds to the next generation matrix (see e.g., van den Driessche and Watmough [31]). Hence, we can obtain a threshold value

$$\mathcal{R}_0 = \rho\left(\mathbf{M}^0\right), \tag{1.9}$$

which corresponds to the well-known basic reproduction number R_0 (see e.g., Diekmann *et al.* [7]). Here $\rho(\cdot)$ denotes the spectral radius of a matrix.

The main theorem of this paper is as follows.

Theorem 1.1. *Let* $\mathcal{R}_0$ *be defined by* (1.9) *and* $\mathbf{\Gamma}$ *be a state space for system* (1.1) *defined by*

$$\mathbf{\Gamma} = \left\{ (S_1, I_1, S_2, I_2, \ldots, S_n, I_n) \in \mathbb{R}_+^{2n} \mid S_k + I_k \leq S_k^0,\ k = 1, 2, \ldots, n \right\}. \tag{1.10}$$

(i) *If* $\mathcal{R}_0 \leq 1$, *then the disease-free equilibrium* $\mathbf{E}^0 = (S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$ *of system* (1.1) *is globally asymptotically stable in* $\mathbf{\Gamma}$.

(ii) *If* $\mathcal{R}_0 > 1$, *then an endemic equilibrium* $\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$ *of system* (1.1) *exists in the interior* $\mathbf{\Gamma}^0$ *of* $\mathbf{\Gamma}$. *It is unique and globally asymptotically stable in* $\mathbf{\Gamma}^0$.

Here we emphasize that this theorem is an extension of the previous result obtained by Kuniya and Muroya [16] to the model with time-delays and nonlinear incidence rates.

This paper is organized as follows. In Section 2, we show the positivity of the solution of system (1.1) and the convergence of total population. In Section 3, we prove the global asymptotic stability of the disease-free equilibrium $\mathbf{E}^0$ for $\mathcal{R}_0 \leq 1$. In Section 4, we prove the uniform persistence of system (1.1), existence of endemic equilibrium $\mathbf{E}^*$ and global stability of it for $\mathcal{R}_0 > 1$. In Section 5, numerical simulation is performed to support our theoretical results.

2. Preliminaries

For the positivity of the solution of system (1.1), we have the following proposition.

Proposition 2.1. *Consider system*(1.1), *the solutions remain positive for* $t \geq 0$. *That is,*

$$S_k(t) > 0, \quad I_k(t) > 0, \quad k = 1, 2, \ldots, n$$

for $t \geq 0$.

Proof. By (1.1), we have that $\lim_{S_k \to +0} \frac{d}{dt} S_k \geq b_k > 0$ and $S_k(0) \geq 0$ for any $k = 1, 2, \ldots, n$, which imply that there exist positive constants t_{k0}, $k = 1, 2, \ldots, n$ such that $S_k(t) > 0$ for any $0 < t < t_{k0}$, $k = 1, 2, \ldots, n$. First, we prove that $S_k(t) > 0$ for any $0 < t < +\infty$ and $k = 1, 2, \ldots, n$. On the contrary, suppose that there exist a positive t_1 and a positive integer $k_1 \in \{1, 2, \ldots, n\}$ such that $S_{k_1}(t_1) = 0$ and $S_{k_1}(t) > 0$ for any $0 < t < t_1$. But by (1.1), we have that $\frac{d}{dt} S_{k_1}(t_1) \geq b_{k_1} > 0$ which is a contradiction to the fact that $S_{k_1}(t) > 0 = S_{k_1}(t_1)$ for any $0 < t < t_1$. Hence, we obtain that $S_k(t) > 0$ for any $0 < t < +\infty$ and $k = 1, 2, \ldots, n$.

Moreover, by (1.1), we have that

$$\begin{aligned} I_k(t) = {} & e^{-(\mu_k + \gamma_k + \tilde{\alpha}_{kk} + \alpha_{kk})t} I_k(0) + e^{-(\mu_k + \gamma_k + \tilde{\alpha}_{kk} + \alpha_{kk})t} \int_0^t e^{(\mu_k + \gamma_k + \tilde{\alpha}_{kk} + \alpha_{kk})u} \\ & \times \Bigg\{ \sum_{j=1}^n \beta_{kj} \int_0^{+\infty} K_{kj}(s) f_{kj}(S_k(u), I_j(u-s)) ds \\ & \quad + \sum_{j=1}^n \alpha_{kj} \int_0^{+\infty} L_{kj}(s) I_j(u-s) ds \Bigg\} du, \quad \text{for } k = 1, 2, \ldots, n \text{ and } t > 0, \end{aligned}$$

from which it follows that $I_k(t) > 0$ for any $k = 1, 2, \ldots, n$ and $t > 0$. □

Put $N_k(t) = S_k(t) + I_k(t)$ and $N_k^* = S_k^0$, $k = 1, 2, \ldots, n$. Adding the two equations in (1.1), we have that

$$\begin{aligned} \frac{d}{dt}\{S_k(t) + I_k(t)\} = {} & b_k - (\mu_k + \tilde{\alpha}_{kk} + \alpha_{kk})\{S_k(t) + I_k(t)\} \\ & + \sum_{j=1}^n \alpha_{kj} \int_0^{+\infty} L_{kj}(s)\{S_j(t-s) + I_j(t-s)\} ds, \end{aligned}$$

which implies

$$\frac{dN_k(t)}{dt} = b_k - (\mu_k + \tilde{\alpha}_{kk} + \alpha_{kk}) N_k(t) + \sum_{j=1}^n \alpha_{kj} \int_0^{+\infty} L_{kj}(s) N_j(t-s) ds. \tag{2.1}$$

In what follows, we show that this $N_k(t)$ converges to the steady state N_k^*. For the proof, we use the following Lyapunov function.

$$U_N(t) = \sum_{k=1}^{n}\left\{N_k^* g\left(n_k(t)\right) + \sum_{j=1}^{n}\alpha_{kj}N_j^*\int_0^{+\infty}L_{kj}(s)\int_{t-s}^{t}g(n_j(u))duds\right\}, \tag{2.2}$$

where $g(x) = x - 1 - \ln x \geq g(1) = 0$ for $x > 0$ and $n_k(t) = N_k(t)/N_k^*$, $k = 1, 2, \ldots, n$.

Lemma 2.2. *For the derivative of the Lyapunov function* (2.2)*, the following estimate holds.*

$$\frac{dU_N(t)}{dt} \leq -\sum_{k=1}^{n}\left\{\mu_k N_k^* g(n_k(t)) + b_k g\left(\frac{1}{n_k(t)}\right)\right\} \leq 0, \tag{2.3}$$

and thus, the solution of (2.1) *satisfies*

$$\lim_{t\to+\infty} N_k(t) = N_k^*, \quad k = 1, 2, \ldots, n. \tag{2.4}$$

Proof. Differentiating $U_N(t)$ along the solutions of (1.1) yields

$$\frac{dU_N(t)}{dt} = \sum_{k=1}^{n}\left\{\left(1 - \frac{N_k^*}{N_k(t)}\right)\frac{dN_k(t)}{dt} + \sum_{j=1}^{n}\alpha_{kj}N_j^*\int_0^{+\infty}L_{kj}(s)\{g(n_j(t)) - g(n_j(t-s))ds\right\}.$$

Using $b_k = (\mu_k + \tilde{\alpha}_{kk} + \alpha_{kk})N_k^* - \sum_{j=1}^{n}\alpha_{kj}N_j^*$, $k = 1, 2, \ldots, n$, we can arrange the first term in the right-hand side of the above equation as

$$\begin{aligned}
\left(1 - \frac{N_k^*}{N_k(t)}\right)\frac{dN_k(t)}{dt} &= \left(1 - \frac{N_k^*}{N_k(t)}\right)\left\{b_k - (\mu_k + \tilde{\alpha}_{kk} + \alpha_{kk})N_k(t) + \sum_{j=1}^{n}\alpha_{kj}\int_0^{+\infty}L_{kj}(s)N_j(t-s)ds\right\} \\
&= \left(1 - \frac{N_k^*}{N_k(t)}\right)\left\{-(\mu_k + \tilde{\alpha}_{kk} + \alpha_{kk})\{N_k(t) - N_k^*\}\right. \\
&\quad \left. + \sum_{j=1}^{n}\alpha_{kj}\int_0^{+\infty}L_{kj}(s)\{N_j(t-s) - N_j^*\}ds\right\} \\
&= \left(1 - \frac{1}{n_k(t)}\right)\left\{-(\mu_k + \tilde{\alpha}_{kk} + \alpha_{kk})N_k^*\{n_k(t) - 1\}\right. \\
&\quad \left. + \sum_{j=1}^{n}\alpha_{kj}N_j^*\int_0^{+\infty}L_{kj}(s)\{n_j(t-s) - 1\}ds\right\}.
\end{aligned}$$

It is easy to check that the following relations hold:

$$\begin{aligned}
&\left(1 - \frac{1}{n_k(t)}\right)\{n_k(t) - 1\} = g(n_k(t)) + g\left(\frac{1}{n_k(t)}\right), \\
&\left(1 - \frac{1}{n_k(t)}\right)\{n_j(t-s) - 1\} = g(n_j(t-s)) - g\left(\frac{n_j(t-s)}{n_k(t)}\right) + g\left(\frac{1}{n_k(t)}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\left(1 - \frac{N_k^*}{N_k(t)}\right)\frac{dN_k(t)}{dt} &= -(\mu_k + \tilde{\alpha}_{kk} + \alpha_{kk})N_k^*\left\{g(n_k(t)) + g\left(\frac{1}{n_k(t)}\right)\right\} \\
&\quad + \sum_{j=1}^{n}\alpha_{kj}N_j^*\int_0^{+\infty}L_{kj}(s)\left\{g(n_j(t-s)) - g\left(\frac{n_j(t-s)}{n_k(t)}\right) + g\left(\frac{1}{n_k(t)}\right)\right\}ds.
\end{aligned}$$

Thus, we have

$$\begin{aligned}\frac{dU_N(t)}{dt} = \sum_{k=1}^{n}\Bigg[&-(\mu_k+\tilde{\alpha}_{kk}+\alpha_{kk})N_k^*\left\{g(n_k(t))+g\left(\frac{1}{n_k(t)}\right)\right\}\\ &+\sum_{j=1}^{n}\alpha_{kj}N_j^*\int_0^{+\infty}L_{kj}(s)\left\{g(n_j(t-s))-g\left(\frac{n_j(t-s)}{n_k(t)}\right)+g\left(\frac{1}{n_k(t)}\right)\right\}ds\\ &+\sum_{j=1}^{n}\alpha_{kj}N_j^*\int_0^{+\infty}L_{kj}(s)\{g(n_j(t))-g(n_j(t-s))ds\}\Bigg]\\ = \sum_{k=1}^{n}\Bigg[&-(\mu_k+\tilde{\alpha}_{kk}+\alpha_{kk})N_k^*\left\{g(n_k(t))+g\left(\frac{1}{n_k(t)}\right)\right\}\\ &+\sum_{j=1}^{n}\alpha_{kj}N_j^*\left\{g(n_j(t))-\int_0^{+\infty}L_{kj}(s)g\left(\frac{n_j(t-s)}{n_k(t)}\right)ds+g\left(\frac{1}{n_k(t)}\right)\right\}\Bigg].\end{aligned}$$

It follows from (1.7) that $\sum_{j=1}^{n}\alpha_{kj}N_j^* = (\mu_k+\tilde{\alpha}_{kk}+\alpha_{kk})N_k^*-b_k,\ k=1,2,\dots,n$. Furthermore, we have

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{j=1}^{n}\alpha_{kj}N_j^*\left\{g(n_j(t))+g\left(\frac{1}{n_k(t)}\right)\right\}\\ &=\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\alpha_{jk}\right)N_k^*g(n_k(t))+\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\alpha_{kj}N_j^*\right)g\left(\frac{1}{n_k(t)}\right)\\ &=\sum_{k=1}^{n}(\tilde{\alpha}_{kk}+\alpha_{kk})N_k^*g(n_k(t))+\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\alpha_{kj}N_j^*\right)g\left(\frac{1}{n_k(t)}\right)\\ &=\sum_{k=1}^{n}(\tilde{\alpha}_{kk}+\alpha_{kk})N_k^*g(n_k(t))+\sum_{k=1}^{n}\{(\mu_k+\tilde{\alpha}_{kk}+\alpha_{kk})N_k^*-b_k\}g\left(\frac{1}{n_k(t)}\right).\end{aligned}$$

Hence, we obtain (2.3), which implies that (2.4) holds. □

Lemma 2.3. *If there exist positive constants* $\underline{u}$, $\bar{u}$ *and* u^* *such that*

$$0<\underline{u}\le\liminf_{t\to+\infty}u(t)\le\limsup_{t\to+\infty}u(t)\le\bar{u},\quad \text{and}\quad \underline{u}\le u^*\le\bar{u}, \tag{2.5}$$

then for the function $g(x)=x-1-\ln x$ *for* $x>0$,

$$\frac{1}{\bar{u}^2}|u(t)-u^*|^2\le g\left(\frac{u(t)}{u^*}\right)\le\frac{1}{\underline{u}^2}|u(t)-u^*|^2. \tag{2.6}$$

This lemma is easily obtained by Taylor's series expansion, and together with the assumption (1.5), we use to ensure the uniform stability of the disease-free equilibrium and the endemic equilibrium of (1.1).

3. Global stability of the disease-free equilibrium

In this section, we prove the global asymptotic stability of the disease-free equilibrium $\mathbf{E}^0$ of system (1.1) for $\mathcal{R}_0\le 1$. Under Lemma 2.2, without loss of generality, it is natural to assume that $S_k(t)+I_k(t)\equiv N_k^*$, for $k=1,2,\dots,n$. Since $N_k^*=S_k^0$, we can rewrite system (1.1) by substituting $S_k(t)=S_k^0-I_k(t)$ into the second equation of it.

$$\begin{cases} \dfrac{d}{dt}I_k(t) = \sum_{j=1}^{n}\beta_{kj}\int_0^{+\infty}K_{kj}(s)f_{kj}(S_k^0 - I_k(t), I_j(t-s))ds - (\mu_k+\gamma_k)I_k(t) \\ \qquad + \sum_{j=1}^{n}\Big(\alpha_{kj}\int_0^{+\infty}L_{kj}(s)I_j(t-s)ds - \alpha_{jk}I_k(t)\Big), \ k = 1,2,\ldots,n. \end{cases} \tag{3.1}$$

Using this reduced system, we are in the position to state and prove the main theorem of this section.

Theorem 3.1. *If $\mathcal{R}_0 \le 1$, then the disease-free equilibrium $\mathbf{E}^0$ of system* (1.1) *is globally asymptotically stable in $\mathbf{\Gamma}$.*

Proof. It is sufficient to show that the trivial equilibrium $I_k \equiv 0$, $k = 1,2,\ldots,n$ of system (3.1) is globally asymptotically stable. Now, under Assumptions 1.1-1.2, matrix $\mathbf{M}^0$ is nonnegative and irreducible. Hence, it follows from the Perron-Frobenius theorem (see e.g., Berman and Plemmons [4]) that $\mathcal{R}_0 = \rho(\mathbf{M}^0) \le 1$ is a left eigenvalue of $\mathbf{M}^0$ corresponding to a positive left eigenvector $\omega = (\omega_1,\omega_2,\ldots,\omega_n)$, $\omega_k > 0$, $k = 1,2,\ldots,n$. That is,

$$\omega\mathbf{M}^0 \ = \ \rho\left(\mathbf{M}^0\right)\omega \ \le \ \omega \tag{3.2}$$

holds. For this $\omega = (\omega_1,\omega_2,\ldots,\omega_n)$, we set

$$v_k = \frac{\omega_k}{\mu_k+\gamma_k+\tilde{\alpha}_{kk}}, \quad k = 1,2,\ldots,n. \tag{3.3}$$

Using these coefficients $v_k > 0$, $k = 1,2,\ldots,n$, we construct the following Lyapunov functional.

$$\begin{aligned} W(t) = \sum_{k=1}^{n} v_k \Bigg\{ & I_k(t) + \sum_{j=1}^{n}\beta_{kj}\int_0^{+\infty}K_{kj}(s)\int_{t-s}^{t} f_{kj}\left(S_k^0 - I_k(u+s), I_j(u)\right)duds \\ & + \sum_{j=1}^{n}\alpha_{kj}\int_0^{+\infty}L_{kj}(s)\int_{t-s}^{t} I_j(u)duds \Bigg\}. \end{aligned} \tag{3.4}$$

By (3.1), the derivative of the first term in the right-hand side of (3.4) is calculated as

$$\begin{aligned} \frac{d}{dt}\left(\sum_{k=1}^{n} v_k I_k(t)\right) \ = \ & \sum_{k=1}^{n} v_k \Bigg\{ \sum_{j=1}^{n}\beta_{kj}\int_0^{+\infty}K_{kj}(s)f_{kj}(S_k^0 - I_k(t), I_j(t-s))ds \\ & -(\mu_k+\gamma_k+\tilde{\alpha}_{kk})I_k(t) + \sum_{j=1}^{n}\alpha_{kj}\int_0^{+\infty}L_{kj}(s)I_j(t-s)ds - \alpha_{kk}I_k(t) \Bigg\}. \end{aligned} \tag{3.5}$$

The derivative of the second term in the right-hand side of (3.4) is

$$\begin{aligned} & \frac{d}{dt}\left(\sum_{k=1}^{n} v_k \sum_{j=1}^{n}\beta_{kj}\int_0^{+\infty}K_{kj}(s)\int_{t-s}^{t} f_{kj}\left(S_k^0 - I_k(u+s), I_j(u)\right)duds\right) \\ & \qquad = \sum_{k=1}^{n} v_k \Bigg\{ \sum_{j=1}^{n}\beta_{kj}\int_0^{+\infty}K_{kj}(s)f_{kj}(S_k^0 - I_k(t+s), I_j(t))ds \\ & \qquad\quad - \sum_{j=1}^{n}\beta_{kj}\int_0^{+\infty}K_{kj}(s)f_{kj}(S_k^0 - I_k(t), I_j(t-s))ds \Bigg\}. \end{aligned} \tag{3.6}$$

The derivative of the last term in the right-hand side of (3.4) is

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{k=1}^{n} v_k \sum_{j=1}^{n} \alpha_{kj} \int_0^{+\infty} L_{kj}(s) \int_{t-s}^{t} I_j(u)duds\right)\\
&\qquad = \sum_{k=1}^{n} v_k \left\{\sum_{j=1}^{n} \alpha_{kj} \int_0^{+\infty} L_{kj}(s) I_j(t) ds - \sum_{j=1}^{n} \alpha_{kj} \int_0^{+\infty} L_{kj}(s) I_j(t-s) ds\right\}\\
&\qquad = \sum_{k=1}^{n} v_k \left\{\sum_{j=1}^{n} \alpha_{kj} I_j(t) - \sum_{j=1}^{n} \alpha_{kj} \int_0^{+\infty} L_{kj}(s) I_j(t-s) ds\right\}.
\end{aligned}$$

Thus, combining with (3.5)-(3.7), we obtain the following estimate for the derivative of functional $W(t)$ along the trajectories of system (3.1).

$$\begin{aligned}
\frac{dW(t)}{dt} &= \sum_{k=1}^{n} v_k \left\{\sum_{j=1}^{n} \beta_{kj} \int_0^{+\infty} K_{kj}(s) f_{kj}(S_k^0 - I_k(t+s), I_j(t)) ds\right.\\
&\qquad \left. -(\mu_k + \gamma_k + \tilde{\alpha}_{kk}) I_k(t) + \sum_{j \neq k} \alpha_{kj} I_j(t)\right\}\\
&\leq \sum_{k=1}^{n} v_k \left\{\sum_{j=1}^{n} \beta_{kj} \int_0^{+\infty} K_{kj}(s) f_{kj}(S_k^0, I_j(t)) ds\right.\\
&\qquad \left. -(\mu_k + \gamma_k + \tilde{\alpha}_{kk}) I_k(t) + \sum_{j \neq k} \alpha_{kj} I_j(t)\right\}\\
&\leq \sum_{k=1}^{n} v_k \left\{\sum_{j=1}^{n} \beta_{kj} \frac{f_{kj}(S_k^0, I_j(t))}{I_j(t)} I_j(t) - (\mu_k + \gamma_k + \tilde{\alpha}_{kk}) I_k(t) + \sum_{j \neq k} \alpha_{kj} I_j(t)\right\}\\
&\leq \sum_{k=1}^{n} v_k \left\{\sum_{j=1}^{n} \left\{\beta_{kj} C_{kj}(S_k^0) + (1-\delta_{kj}) \alpha_{kj}\right\} I_j(t) - (\mu_k + \gamma_k + \tilde{\alpha}_{kk}) I_k(t)\right\}\\
&= \omega \left\{\mathbf{M}^0 \mathbf{I}(t) - \mathbf{I}(t)\right\} = \omega \left(\mathcal{R}_0 - 1\right) \mathbf{I}(t) \leq 0.
\end{aligned} \tag{3.7}$$

Here we used Assumption 1.2 and (3.2)-(3.3).

It is obvious from (3.7) that $\mathcal{R}_0 < 1$ if and only if $I_k(t) \equiv 0$, $k = 1, 2, \ldots, n$. If $\mathcal{R}_0 = 1$, then it follows from the first equation of (3.7) that $W'(t) \equiv 0$ implies

$$\sum_{k=1}^{n} v_k (\mu_k + \gamma_k + \tilde{\alpha}_{kk}) I_k(t) = \sum_{k=1}^{n} v_k \left\{\sum_{j=1}^{n} \beta_{kj} \int_0^{+\infty} K_{kj}(s) f_{kj}(S_k^0 - I_k(t+s), I_j(t)) ds + \sum_{j \neq k} \alpha_{kj} I_j(t)\right\}.$$

By (3.3), the left-hand side of this equation is $\omega \mathbf{I}(t)$ and by the first equation of (3.2) and $\mathcal{R}_0 = \rho(\mathbf{M}^0) = 1$, (3.7) implies

$$\begin{aligned}
0 &= \sum_{k=1}^{n} v_k \left\{\sum_{j=1}^{n} \left\{\beta_{kj} C_{kj}(S_k^0) + (1-\delta_{kj}) \alpha_{kj}\right\} I_j(t)\right\}\\
&= \sum_{k=1}^{n} v_k \left\{\sum_{j=1}^{n} \left\{\beta_{kj} \int_0^{+\infty} K_{kj}(s) \frac{f_{kj}(S_k^0 - I_k(t+s), I_j(t))}{I_j(t)} ds + (1-\delta_{kj}) \alpha_{kj}\right\} I_j(t)\right\}.
\end{aligned}$$

It can be seen from Assumption 1.2 that this equality holds if and only if $I_k(t) \equiv 0$, $k = 1, 2, \ldots, n$. Consequently, we conclude that $W'(t) = 0$ if and only if $I_k(t) \equiv 0$, $k = 1, 2, \ldots, n$. Thus, it follows from the classical LaSalle's invariance principle (see [19]), $\mathbf{E}^0$ is global attractive. Moreover, by Lemmas 2.2 and 2.3 with (3.3) and (1.5) and $\frac{dW}{dt} \leq 0$ with (3.4), we can easily prove that there exist positive constants c_1 and c_2 such that c_1 and c_2 do not depend on the initial condition (1.6) and

$$I_k(t) \leq c_1 W(t) \leq c_1 W(0) \leq c_1 c_2 \max_{1\leq j\leq n} I_j(0), \quad k = 1, 2, \ldots, n,$$

which implies that $\mathbf{E}^0$ of (3.1) is uniformly stable. Hence, the disease-free equilibrium $\mathbf{E}^0$ of the original system (1.1) is so. □

4. Global stability of the endemic equilibrium

4.1. Permanence

In this subsection, we prove the permanence (uniform persistence) of system (1.1) for $\mathcal{R}_0 > 1$. As in the previous section, for simplicity, we consider the reduced system (3.1).

Under Assumption 1.1, matrix $\mathbf{M}^0$ is nonnegative and irreducible and hence, as in the previous section, it follows that $\mathcal{R}_0 = \rho(\mathbf{M}^0) > 1$ is an eigenvalue of $\mathbf{M}^0$ and there exists an associated eigenvector $\mathbf{r} = (r_1, r_2, \ldots, r_n)^T$, $r_k > 0$, $k = 1, 2, \ldots, n$ such that

$$\mathbf{M}^0 \mathbf{r} = \rho\left(\mathbf{M}^0\right) \mathbf{r} > \mathbf{r}, \tag{4.1}$$

From this inequality, we obtain the following inequality (cf. (3.2)).

$$\sum_{j=1}^{n} \{C_{kj}(S_k^0)\beta_{kj} + (1 - \delta_{kj})\alpha_{kj}\} r_j - (\mu_k + \gamma_k + \tilde{\alpha}_{kk}) r_k > 0, \; k = 1, 2, \ldots, n. \tag{4.2}$$

We prove the following proposition.

Proposition 4.1. *If* $\mathcal{R}_0 > 1$, *then system* (3.1) *is permanent, that is, there exist positive constants* $m, M > 0$ *such that*

$$m \leq \min_{1\leq k\leq n} \liminf_{t\to+\infty} I_k(t) \leq \max_{1\leq k\leq n} \limsup_{t\to+\infty} I_k(t) \leq M. \tag{4.3}$$

Here m *and* M *are independent from the choice of initial condition.*

Proof. Under Lemma 2.2, the existence of the upper bound is obvious and hence, we show the existence of the lower bound. Let $i \in \{1, 2, \ldots, n\}$ be a positive integer such that

$$\liminf_{t\to+\infty} \frac{I_i(t)}{r_i} = \min_{1\leq k\leq n} \liminf_{t\to+\infty} \frac{I_k(t)}{r_k} =: \underline{I}.$$

We first show $\underline{I} > 0$. To this end, we assume $\underline{I} = 0$ and show a contradiction. In this case, there exists an increasing sequence $0 \leq t_1 < t_2 < \cdots$ and $t_k \to +\infty$ such that

(i) $I_i'(t_p) \leq 0$, $p = 1, 2, \ldots$ and $\lim_{p\to+\infty} I(t_p) = 0$.

(ii) For all $t \in [0, t_p]$, $p = 1, 2, \ldots$,

$$\frac{I_j(t)}{r_j} \geq \frac{I_i(t_p)}{r_i} > 0, \quad j = 1, 2, \ldots, n.$$

Then, it follows from Assumption 1.2 and (3.1) that

$$\begin{aligned}
0 &\geq I_i'(t_p)\\
&= \sum_{j=1}^{n}\beta_{ij}r_j\int_0^{+\infty}K_{ij}(s)\frac{I_i(t_p)}{r_i}\frac{f_{ij}(S_i^0-I_i(t_p),I_j(t_p-s))}{\frac{r_j}{r_i}I_i(t_p)}ds\\
&\quad -(\mu_i+\gamma_i)r_i\frac{I_i(t_p)}{r_i}+\sum_{j=1}^{n}\left(\alpha_{ij}r_j\int_0^{+\infty}L_{ij}(s)\frac{I_j(t_p-s)}{r_j}ds-\alpha_{ji}r_i\frac{I_i(t_p)}{r_i}\right)\\
&\geq \sum_{j=1}^{n}\beta_{ij}r_j\frac{I_i(t_p)}{r_i}\frac{f_{ij}(S_i^0-I_i(t_p),\frac{r_j}{r_i}I_i(t_p))}{\frac{r_j}{r_i}I_i(t_p)}-(\mu_i+\gamma_i)r_i\frac{I_i(t_p)}{r_i}+\sum_{j=1}^{n}\left(\alpha_{ij}r_j\frac{I_i(t_p)}{r_i}-\alpha_{ji}r_i\frac{I_i(t_p)}{r_i}\right)\\
&= \left(\sum_{j=1}^{n}\beta_{ij}r_j\frac{f_{ij}(S_i^0-I_i(t_p),\frac{r_j}{r_i}I_i(t_p))}{\frac{r_j}{r_i}I_i(t_p)}-(\mu_i+\gamma_i)r_i+\sum_{j=1}^{n}(1-\delta_{ij})(\alpha_{ij}r_j-\alpha_{ji}r_i)\right)\frac{I_i(t_p)}{r_i}.
\end{aligned}$$

Then, since $I_i(t_p)>0$, we have

$$\sum_{j=1}^{n}\beta_{ij}r_j\frac{f_{ij}(S_i^0-I_i(t_p),\frac{r_j}{r_i}I_i(t_p))}{\frac{r_j}{r_i}I_i(t_p)}-(\mu_i+\gamma_i)r_i+\sum_{j=1}^{n}(1-\delta_{ij})(\alpha_{ij}r_j-\alpha_{ji}r_i)\leq 0.$$

Then, by virtue of $\lim_{p\to+\infty}I(t_p)=0$ and Assumption 1.2, $p\to+\infty$ leads to

$$0 \;\geq\; \sum_{j=1}^{n}\{C_{ij}(S_i^0)\beta_{ij}+(1-\delta_{ij})\alpha_{ij}\}r_j-(\mu_i+\gamma_i+\tilde{\alpha}_{ii})r_i.$$

However this contradicts with (4.2). Consequently, $\underline{I}>0$.

Next we show that there exists a positive constant $\underline{\hat{I}}>0$ such that $\underline{I}>\underline{\hat{I}}$. Here, $\underline{\hat{I}}$ is a positive constant such that $H(\underline{\hat{I}})>0$ holds, where $H(\cdot)$ is a monotone decreasing function on $\mathbb{R}_+$ defined by

$$H(I):=\sum_{j=1}^{n}\left(\beta_{ij}\frac{f_{ij}(S_i^0-r_iI,r_jI)}{r_jI}+(1-\delta_{ij})\alpha_{ij}\right)r_j-(\mu_i+\gamma_i+\tilde{\alpha}_{ii})r_i.$$

In fact, it follows from (4.2) that $H(\underline{\hat{I}})>0$ holds for sufficiently small $\underline{\hat{I}}>0$. Now, the definition of $\underline{I}$ ensures that for a sufficiently small $\varepsilon>0$ and a sufficiently large $T_0>0$,

$$\frac{I_j(t)}{r_j} \;>\; \underline{I}-\varepsilon \;>\; 0,\quad j=1,2,\dots,n$$

holds for all $t\geq T_0$. Moreover, by (1.4), there exists a sufficiently large positive constant $T_1\geq T_0$ such that

$$\begin{cases}\displaystyle\int_0^{T_1}K_{kj}(s)ds>1-\varepsilon \quad\text{and}\quad 0\leq\int_{T_1}^{+\infty}K_{kj}(s)ds<\varepsilon,\\ \displaystyle\int_0^{T_1}L_{kj}(s)ds>1-\varepsilon \quad\text{and}\quad 0\leq\int_{T_1}^{+\infty}L_{kj}(s)ds<\varepsilon,\quad k,j=1,2,\dots,n.\end{cases}$$

It follows that, for $t\geq T_0+T_1$,

$$\frac{I_j(t-s)}{r_j} \;>\; \underline{I}-\varepsilon,\quad j=1,2,\dots,n,\quad s\in[0,T_1], \tag{4.4}$$

$$\begin{aligned}
\int_0^{+\infty}K_{ij}(s)f_{ij}(S_i(t),I_j(t-s))ds \;&=\; \int_0^{T_1}K_{ij}(s)f_{ij}(S_i(t),I_j(t-s))ds+\int_{T_1}^{+\infty}K_{ij}(s)f_{ij}(S_i(t),I_j(t-s))ds\\
&>\; (1-\varepsilon)f_{ij}(S_k(t),r_j(\underline{I}-\epsilon)),\quad j=1,2,\dots,n,
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\int_0^{+\infty} L_{ij}(s)(I_j(t-s)/r_j)ds &= \int_0^{T_1} L_{ij}(s)(I_j(t-s)/r_j)ds + \int_{T_1}^{+\infty} L_{ij}(s)(I_j(t-s)/r_j)ds \\ &> (1-\varepsilon)(\underline{I}-\varepsilon), \quad j=1,2,\ldots,n. \end{aligned} \tag{4.6}$$

Combining inequalities (4.4)-(4.6) yields

$$\begin{aligned} I_i'(t) &= \sum_{j=1}^n \beta_{ij} r_j \int_0^{+\infty} K_{ij}(s)(\underline{I}-\varepsilon)\frac{f_{ij}(S_i^0 - I_i(t), I_j(t-s))}{r_j(\underline{I}-\varepsilon)} ds - (\mu_i+\gamma_i)I_i(t) \\ &\quad + \sum_{j=1}^n \left\{\alpha_{ij}\int_0^{+\infty} L_{ij}(s)I_j(t-s)ds - \alpha_{ji}I_i(t)\right\} \\ &\geq \sum_{j=1}^n \beta_{ij} r_j (1-\varepsilon)(\underline{I}-\varepsilon)\frac{f_{ij}(S_i^0 - I_i(t), r_j(\underline{I}-\varepsilon))}{r_j(\underline{I}-\varepsilon)} - (\mu_i+\gamma_i)r_i(\underline{I}-\varepsilon) \\ &\quad + \sum_{j=1}^n \{\alpha_{ij}r_j(1-\varepsilon)(\underline{I}-\varepsilon) - \alpha_{ji}r_i(\underline{I}-\varepsilon)\} \\ &= \left(\sum_{j=1}^n \beta_{ij} r_j (1-\varepsilon)\frac{f_{ij}(S_i^0 - I_i(t), r_j(\underline{I}-\varepsilon))}{r_j(\underline{I}-\varepsilon)} - (\mu_i+\gamma_i)r_i + \sum_{j=1}^n\{\alpha_{ij}r_j(1-\varepsilon)-\alpha_{ji}r_i\}\right)(\underline{I}-\varepsilon). \end{aligned} \tag{4.7}$$

In the case that $I_i(t)$ is eventually monotone increasing, the existence of the lower bound is obvious. Hence, it remains to consider the case that $I_i(t)$ is eventually monotone decreasing. In this case, there exists a monotone increasing sequence $0 \leq t_1 < t_2 < \cdots$ and $t_p \to +\infty$ such that

$$I_i'(t_p) \leq 0, \ p=1,2,\ldots, \quad \text{and} \quad \lim_{p\to+\infty}\frac{I_i(t_p)}{r_i} = \underline{I}.$$

Then, it follows from (4.7) that

$$\begin{aligned} 0 \geq I_i'(t_p) \geq \Bigg(&\sum_{j=1}^n \beta_{ij} r_j (1-\varepsilon)\frac{f_{ij}(S_i^0 - I_i(t_p), r_j(\underline{I}-\varepsilon))}{r_j(\underline{I}-\varepsilon)} \\ &-(\mu_i+\gamma_i)r_i + \sum_{j=1}^n\{\alpha_{ij}r_j(1-\varepsilon)-\alpha_{ji}r_i\}\Bigg)(\underline{I}-\varepsilon) \end{aligned}$$

and hence, letting $p \to +\infty$, we have inequality

$$0 \geq \left(\sum_{j=1}^n \beta_{ij} r_j (1-\varepsilon)\frac{f_{ij}(S_i^0 - r_i\underline{I}, r_j(\underline{I}-\varepsilon))}{r_j(\underline{I}-\varepsilon)} - (\mu_i+\gamma_i)r_i + \sum_{j=1}^n\{\alpha_{ij}r_j(1-\varepsilon)-\alpha_{ji}r_i\}\right)(\underline{I}-\varepsilon).$$

Then, letting $\varepsilon \to +0$, we have inequality

$$0 \ \geq \ \sum_{j=1}^n \left(\beta_{ij}\frac{f_{ij}(S_i^0 - r_i\underline{I}, r_j\underline{I})}{r_j\underline{I}} + (1-\delta_{ij})\alpha_{ij}\right) r_j - (\mu_i+\gamma_i+\tilde{\alpha}_{ii})r_i \ = \ H(\underline{I}).$$

Since $H(I)$ is monotone decreasing with respect to I, this inequality implies $\underline{I} \geq \hat{\underline{I}}$ for $\hat{\underline{I}}$ such that $H(\hat{\underline{I}}) > 0$. This completes the proof. □

The permanence of system (1.1) for $\mathcal{R}_0 > 1$ follows from Lemma 2.2 and Proposition 4.1.

4.2. Existence of an endemic equilibrium

Next, we prove the existence of an endemic equilibrium of (1.1) for $\mathcal{R}_0 > 1$. As in the previous sections, we consider the reduced system (3.1). The components of the endemic equilibrium $\mathbf{E}^*$ must satisfy the following equation.

$$\sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*) - (\mu_k + \gamma_k + \tilde{\alpha}_{kk}) I_k^* + \sum_{j=1}^{n} (1-\delta_{kj}) \alpha_{kj} I_j^* = 0, \quad k = 1, 2, \ldots, n. \tag{4.8}$$

In the proof of the subsequent proposition, we use the following function $\mathbf{F}$ on $\mathbb{R}^n_+$.

$$\begin{cases} \mathbf{F}(\mathbf{x}) := (F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_n(\mathbf{x}))^T, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n_+, \\ F_k(\mathbf{x}) := -\Big(\sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^0 - x_k, x_j) - (\mu_k + \gamma_k + \tilde{\alpha}_{kk}) x_k + \sum_{j=1}^{n} (1-\delta_{kj}) \alpha_{kj} x_j \Big), \quad k = 1, 2, \ldots, n. \end{cases} \tag{4.9}$$

Proposition 4.2. *If $\mathcal{R}_0 > 1$, then system* (1.1) *has an endemic equilibrium*

$$\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*) \in \mathbf{\Gamma}^0.$$

Proof. It is enough to show the existence of a nontrivial equilibrium $I_1^*, I_2^*, \ldots, I_n^*$ of the reduced system (3.1) satisfying (4.8). To this end, we seek a root $\mathbf{x}$ of system $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ such that $0 < x_k < S_k^0$, $k = 1, 2, \ldots, n$. Let us define the following two matrices.

$$\mathbf{F}^0 := [C(S_k^0)\beta_{kj} + (1-\delta_{kj})\alpha_{kj}]_{1 \le k,j \le n} \quad \text{and} \quad \mathbf{V} := \operatorname*{diag}_{1 \le k \le n} (\mu_k + \gamma_k + \tilde{\alpha}_{kk})$$

It is easy to see from (1.8) that $\mathbf{M}^0 = \mathbf{V}^{-1}\mathbf{F}^0$. Now, since $\mathcal{R}_0 > 1$, we see that there exists a positive eigenvector $\mathbf{r} = (r_1, r_2, \ldots, r_n)^T$ of matrix $\mathbf{M}^0$ satisfying (4.1). Then, the following relations hold.

$$\mathbf{F}(\mathbf{r}) = -\Big(\mathbf{F}^0\mathbf{r} - \mathbf{V}\mathbf{r}\Big) + \left[\beta_{kj}\left(C_{kj}(S_k^0) - \frac{f_{kj}(S_k^0 - r_k, r_j)}{r_j}\right)\right]_{1 \le k,j \le n} \mathbf{r} \tag{4.10}$$

and

$$-\Big(\mathbf{F}^0\mathbf{r} - \mathbf{V}\mathbf{r}\Big) \;<\; -\Big(\mathbf{F}^0\mathbf{r} - \rho(\mathbf{M}^0)\mathbf{V}\mathbf{r}\Big) \;=\; \mathbf{0}. \tag{4.11}$$

Here the order of vectors in $\mathbb{R}^n$ implies the usual element-wise one in $\mathbb{R}^n$. By (4.10), it holds for any $\alpha > 0$ that

$$\mathbf{F}(\alpha\mathbf{r}) = -\Big(\mathbf{F}^0\alpha\mathbf{r} - \mathbf{V}\alpha\mathbf{r}\Big) + \left[\beta_{kj}\left(C_{kj}(S_k^0) - \frac{f_{kj}(S_k^0 - \alpha r_k, \alpha r_j)}{\alpha r_j}\right)\right]_{1 \le k,j \le n} \alpha\mathbf{r}. \tag{4.12}$$

Noting that under Assumption 1.2 it holds that

$$\lim_{\alpha \to +0} \frac{f_{kj}(S_k^0 - \alpha r_k, \alpha r_j)}{\alpha r_j} = C_{kj}(S_k^0), \quad k, j = 1, 2, \ldots, n,$$

we see from (4.11) and (4.12) that there exists a sufficiently small positive constant $\underline{\alpha} > 0$ such that

$$\mathbf{F}(\underline{\alpha}\mathbf{r}) \le \mathbf{0}. \tag{4.13}$$

Moreover, noting that under Assumption 1.2 it holds that

$$\lim_{y \to +\infty} \frac{f_{kj}(S_k^0 - y, y)}{y} = -\infty, \quad k, j = 1, 2, \ldots, n,$$

it follows from (4.12) that there exists a sufficiently large positive constant vector $\mathbf{k} = (k, k, \ldots, k)^T \in \mathbb{R}^n$ such that

$$\mathbf{F}(\mathbf{k}) \geq \mathbf{0}. \tag{4.14}$$

Hence the Fréchet derivative $\mathbf{F}'(\mathbf{x}) = [\frac{\partial F_k(\mathbf{x})}{\partial x_j}]_{1\leq k,j\leq n}$ of $\mathbf{F}(\mathbf{x})$ is calculated as follows.

$$\frac{\partial F_k(\mathbf{x})}{\partial x_j} = \begin{cases} \sum_{j=1}^{n} \left\{ \beta_{kj} \frac{\partial}{\partial x} f_{kj}(S_k^0 - x_k, x_j) - \beta_{kk} \frac{\partial}{\partial y} f_{kj}(N_k^* - x_k, x_k) \right\} + (\mu_k + \gamma_k + \tilde{\alpha}_{kk}), & k = j, \\ -\beta_{kj} \frac{\partial}{\partial y} f_{kj}(S_k^0 - x_k, x_j) - \alpha_{kj} \leq 0, \quad k \neq j, \quad k, j = 1, 2, \ldots, n. \end{cases}$$

Note here that $f_{kj}(x, y)$ is a two variable function of $x, y \in \mathbb{R}_+$ and $\partial f_{kj}/\partial x$ denotes the partial derivative with respect to the first variable and $\partial f_{kj}/\partial y$ denotes the partial derivative with respect to the second variable.

Under these settings, we consider a sufficiently large positive constant $l > 0$ such that

$$l > \max_{1\leq k\leq n} \sum_{j=1}^{n} \left\{ \beta_{kj} \frac{\partial}{\partial x} f_{kj}(S_k^0 - x_k, x_j) - \beta_{kk} \frac{\partial}{\partial y} f_{kj}(S_k^0 - x_k, x_k) \right\} + (\mu_k + \gamma_k + \tilde{\alpha}_{kk}). \tag{4.15}$$

Then, we see that n-diagonal matrix $\mathbf{B} := \mathrm{diag}(l^{-1}, l^{-1}, \ldots, l^{-1})$ is nonnegative, non-singular and *subinverse* of $\mathbf{F}'(\mathbf{x})$, that is, $\mathbf{B}\mathbf{F}'(\mathbf{x}) \leq I$ and $\mathbf{F}'(\mathbf{x})\mathbf{B} \leq I$ hold for any $\mathbf{x}$ with $0 < x_k < S_k^0,\ k = 1, 2, \ldots, n$, where I denotes the identity matrix (for subinverse matrices, see e.g., Ortega and Rheinboldt [27] or Muroya [23]). Now it follows from (4.13) and (4.14) that $\mathbf{F}(\mathbf{x}_0) \leq \mathbf{0} \leq \mathbf{F}(\mathbf{y}_0)$ with $\mathbf{x}_0 := \underline{\alpha}\mathbf{r}$ and $\mathbf{y}_0 := \bar{\alpha}\mathbf{r}$. Thus, from the property of matrix $\mathbf{B}$, $\mathbf{x}_{p+1} := \mathbf{x}_p - \mathbf{B}\mathbf{F}(\mathbf{x}_p),\ p = 1, 2, \ldots, n$ becomes a monotone increasing sequence and hence, $\mathbf{x}_0 \leq \mathbf{x}_1 \leq \ldots \leq \mathbf{x}_p \leq \mathbf{x}_{p+1} \leq \mathbf{y}_0$ holds (see for similar approaches, Ortega and Rheinboldt [27, Theorem 4.1 and Corollary 4.1] or Muroya [23, Theorem 3.1]). Since this sequence is bounded, there exists a limit $\lim_{p\to+\infty} \mathbf{x}_p = \mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$. This is a nontrivial root of $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, that is, the desired nontrivial equilibrium (see Zhao and Jing [40]). □

4.3. Global asymptotic stability of the endemic equilibrium

In this subsection, we investigate the global asymptotic stability of the endemic equilibrium $\mathbf{E}^*$ of system (1.1). Propositions 4.1-4.2 imply that for $\mathcal{R}_0 > 1$, the disease-free equilibrium $\mathbf{E}^0 = (S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$ of (1.1) becomes unstable and a positive equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, \ldots, S_n^*, I_n^*)$ of (1.1) exists. From (4.8), the components of such E^* satisfy the following equations.

$$(\mu_k + \gamma_k + \tilde{\alpha}_{kk} + \alpha_{kk}) I_k^* = \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*) + \sum_{j=1}^{n} \alpha_{kj} I_j^*, \quad k = 1, 2, \ldots, n. \tag{4.16}$$

We prove the following helpful lemma, which play an important role in computation and estimation of derivative of Lyapunov functional. Similar argument can be found in Guo *et al.* [11, Lemma 2.1 and Proof of Theorem 3.3].

Lemma 4.3. *There exists a positive solution* $(v_1, v_2, \ldots, v_n)$ *such that*

$$\sum_{j=1}^{n} v_j \left(\beta_{jk} \frac{f_{jk}(S_j^0 - I_j^*, I_k^*)}{I_k^*} + \alpha_{jk} \right) = v_k(\mu_k + \gamma_k + \tilde{\alpha}_{kk} + \alpha_{kk}), \quad k = 1, 2, \ldots, n, \tag{4.17}$$

and it is expressed by

$$(v_1, v_2, \cdots, v_n) = (C_{11}, C_{22}, \ldots, C_{nn}), \tag{4.18}$$

where C_{kk} is the cofactor of the k-th diagonal entry of matrix

$$\tilde{\mathbf{B}} = \begin{bmatrix} \sum_{j\neq 1}\tilde{\beta}_{1j} & -\tilde{\beta}_{21} & \cdots & -\tilde{\beta}_{n1} \\ -\tilde{\beta}_{12} & \sum_{j\neq 2}\tilde{\beta}_{2j} & \cdots & -\tilde{\beta}_{n2} \\ \vdots & \vdots & & \vdots \\ -\tilde{\beta}_{1n} & -\tilde{\beta}_{2n} & \cdots & \sum_{j\neq n}\tilde{\beta}_{nj} \end{bmatrix}, \quad \begin{array}{r} \tilde{\beta}_{kj} = \beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*) + \alpha_{kj} I_j^*, \\ 1 \le k, j \le n. \end{array}$$

The proof is omitted since it is similar to that of [16, Lemma 4.1].

Proof of Theorem 1.1 If $\mathcal{R}_0 \le 1$, then by Theorem 3.1, we can obtain the first part $\mathcal{R}_0 \le 1$ of Theorem 1.1.

Now, consider the case $\mathcal{R}_0 > 1$. Then, by Proposition 4.1, system (1.1) is permanent in $\mathbf{\Gamma}^0$, and by Proposition 4.2, there exists at least one endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$.

As in the previous sections, we focus on the reduced system (3.1). It can be rewritten as follows.

$$\left\{ \begin{aligned} \frac{d}{dt} I_k(t) = &\sum_{j=1}^n \beta_{kj} \int_0^{+\infty} K_{kj}(s) f_{kj}(S_k^0 - I_k(t), I_j(t-s)) ds \\ &- (\mu_k + \gamma_k + \tilde{\alpha}_{kk} + \alpha_{kk}) I_k(t) + \sum_{j=1}^n \alpha_{kj} \int_0^{+\infty} L_{kj}(s) I_j(t-s) ds. \end{aligned} \right. \tag{4.19}$$

Put

$$y_k(t) := \frac{I_k(t)}{I_k^*}, \quad z_{kj}(t,s) := \frac{f_{kj}\left(S_k^0 - I_k(t), I_j(t-s)\right)}{f_{kj}\left(S_k^0 - I_k^*, I_j^*\right)}, \quad \tilde{z}_{kj}(t) := \frac{f_{kj}\left(S_k^0 - I_k^*, I_j(t)\right)}{f_{kj}\left(S_k^0 - I_k^*, I_j^*\right)} \tag{4.20}$$

and consider the following Lyapunov function.

$$U(t) := \sum_{k=1}^n v_k \left\{ I_k^* g\left(y_k(t)\right) + U_1(t) + U_2(t) \right\}, \tag{4.21}$$

where

$$U_1(t) := \sum_{j=1}^n \beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*) \int_0^{+\infty} K_{kj}(s) \int_{t-s}^t g\left(\tilde{z}_{kj}(u)\right) du ds \tag{4.22}$$

and

$$U_2(t) := \sum_{j=1}^n \alpha_{kj} I_j^* \int_0^{+\infty} L_{kj}(s) \int_{t-s}^t g(y_j(u)) du ds \tag{4.23}$$

and $v_1, v_2, \cdots, v_n$ are chosen as in (4.18).

First we consider the derivative of the first term in (4.21). It follows from (4.16), (4.19) and (4.20) that

$$\begin{aligned} \left(\sum_{k=1}^n v_k I_k^* g\left(y_k(t)\right)\right)' &= \sum_{k=1}^n v_k \left(1 - \frac{1}{y_k(t)}\right) \frac{d}{dt} I_k(t) \\ &= \sum_{k=1}^n v_k \left(1 - \frac{1}{y_k(t)}\right) \left\{ \sum_{j=1}^n \beta_{kj} \int_0^{+\infty} K_{kj}(s) f_{kj}(S_k^0 - I_k(t), I_j(t-s)) ds \right. \\ &\qquad \left. - (\mu_k + \gamma_k + \tilde{\alpha}_{kk} + \alpha_{kk}) I_k^* y_k(t) + \sum_{j=1}^n \alpha_{kj} \int_0^{+\infty} L_{kj}(s) I_j(t-s) ds \right\} \\ &= \sum_{k=1}^n v_k \left(1 - \frac{1}{y_k(t)}\right) \left\{ \sum_{j=1}^n \beta_{kj} \int_0^{+\infty} K_{kj}(s) f_{kj}(S_k^0 - I_k^*, I_j^*) \left\{ z_{kj}(t,s) - y_k(t) \right\} ds \right. \end{aligned}$$

$$
\begin{aligned}
&\left. + \sum_{j=1}^{n} \alpha_{kj} \int_0^{+\infty} L_{kj}(s) I_j^* \left\{ y_j(t-s) - y_k(t) \right\} ds \right\} \\
= &\sum_{k=1}^{n} v_k \left\{ \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*) \int_0^{+\infty} K_{kj}(s) \left(1 - \frac{1}{y_k(t)} \right) \left\{ z_{kj}(t,s) - y_k(t) \right\} ds \right. \\
&\left. + \sum_{j=1}^{n} \alpha_{kj} I_j^* \int_0^{+\infty} L_{kj}(s) \left(1 - \frac{1}{y_k(t)} \right) \left\{ y_j(t-s) - y_k(t) \right\} ds \right\}. \qquad (4.24)
\end{aligned}
$$

Now we prove the following relations.

$$
\left(1 - \frac{1}{y_k(t)} \right) \left\{ z_{kj}(t,s) - y_k(t) \right\} \leq g\left(\tilde{z}_{kj}(t-s) \right) - g\left(y_k(t) \right) - g\left(\frac{\tilde{z}_{kj}(t-s)}{y_k(t)} \right) \qquad (4.25)
$$

$$
\left(1 - \frac{1}{y_k(t)} \right) \left\{ y_j(t-s) - y_k(t) \right\} = g\left(y_j(t-s) \right) - g\left(y_k(t) \right) - g\left(\frac{y_j(t-s)}{y_k(t)} \right). \qquad (4.26)
$$

Since (4.26) is obtained by performing a simple calculation, we omit the proof. We next show (4.25). We first rewrite (4.25) as

$$
\begin{aligned}
\left(1 - \frac{1}{y_k(t)} \right) \left\{ z_{kj}(t,s) - y_k(t) \right\} = &\left(1 - \frac{1}{y_k(t)} \right) \left\{ z_{kj}(t,s) - \tilde{z}_{kj}(t-s) \right\} \\
&+ \left(1 - \frac{1}{y_k(t)} \right) \left\{ \tilde{z}_{kj}(t-s) - y_k(t) \right\}
\end{aligned} \qquad (4.27)
$$

and the first term in the right-hand side of this equation is non-positive.

In fact, if $I_k(t) \geq I_k^*$, then it follows from (ii) of Assumption 1.2 that

$$
\left(1 - \frac{1}{y_k(t)} \right) \geq 0 \ \text{ and } \ z_{kj}(t,s) - \tilde{z}_{kj}(t-s) = \frac{f_{kj}\left(S_k^0 - I_k(t), I_j(t-s) \right) - f_{kj}\left(S_k^0 - I_k^*, I_j(t-s) \right)}{f_{kj}\left(S_k^0 - I_k^*, I_j^* \right)} \leq 0,
$$

and if $I_k(t) \leq I_k^*$, then it follows also from (ii) of Assumption 1.2 that

$$
\left(1 - \frac{1}{y_k(t)} \right) \leq 0 \ \text{ and } \ z_{kj}(t,s) - \tilde{z}_{kj}(t-s) = \frac{f_{kj}\left(S_k^0 - I_k(t), I_j(t-s) \right) - f_{kj}\left(S_k^0 - I_k^*, I_j(t-s) \right)}{f_{kj}\left(S_k^0 - I_k^*, I_j^* \right)} \geq 0.
$$

Thus, from (4.27), we have

$$
\begin{aligned}
\left(1 - \frac{1}{y_k(t)} \right) \left\{ z_{kj}(t,s) - y_k(t) \right\} &\leq \left(1 - \frac{1}{y_k(t)} \right) \left\{ \tilde{z}_{kj}(t-s) - y_k(t) \right\} \\
&= g\left(\tilde{z}_{kj}(t-s) \right) - g\left(y_k(t) \right) - g\left(\frac{\tilde{z}_{kj}(t-s)}{y_k(t)} \right)
\end{aligned}
$$

and (4.25) is proved. Using (4.25) and (4.26), we can evaluate (4.24) as follows.

$$
\begin{aligned}
&\left(\sum_{k=1}^{n} v_k I_k^* g\left(y_k(t) \right) \right)' \\
&\leq \sum_{k=1}^{n} v_k \left[\sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*) \int_0^{+\infty} K_{kj}(s) \left\{ g\left(\tilde{z}_{kj}(t-s) \right) - g\left(y_k(t) \right) - g\left(\frac{\tilde{z}_{kj}(t-s)}{y_k(t)} \right) \right\} ds \right. \\
&\left. + \sum_{j=1}^{n} \alpha_{kj} I_j^* \int_0^{+\infty} L_{kj}(s) \left\{ g\left(y_j(t-s) \right) - g\left(y_k(t) \right) - g\left(\frac{y_j(t-s)}{y_k(t)} \right) \right\} ds \right]
\end{aligned}
$$

$$\leq \sum_{k=1}^{n} v_k \left[\sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*) \int_0^{+\infty} K_{kj}(s) \left\{ g\left(\tilde{z}_{kj}(t-s)\right) - g\left(y_k(t)\right) \right\} ds \right.$$
$$\left. + \sum_{j=1}^{n} \alpha_{kj} I_j^* \int_0^{+\infty} L_{kj}(s) \left\{ g\left(y_j(t-s)\right) - g\left(y_k(t)\right) \right\} ds \right]. \tag{4.28}$$

On the other hand, calculating the derivative of (4.22) yields

$$U_1'(t) = \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*) \int_0^{+\infty} K_{kj}(s) \left\{ g\left(\tilde{z}_{kj}(t)\right) - g\left(\tilde{z}_{kj}(t-s)\right) \right\} ds \tag{4.29}$$

and calculating the derivative of (4.23) yields

$$U_2'(t) = \sum_{j=1}^{n} \alpha_{kj} I_j^* \int_0^{+\infty} L_{kj}(s) \left\{ g(y_j(t)) - g(y_j(t-s)) \right\} ds. \tag{4.30}$$

Hence, using (4.28), (4.29) and (4.30), the derivative of Lyapunov functional (4.21) can be evaluated as follows.

$$\begin{aligned} U'(t) \quad \leq \quad & \sum_{k=1}^{n} v_k \left[\sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*) \int_0^{+\infty} K_{kj}(s) \left\{ g\left(\tilde{z}_{kj}(t)\right) - g\left(y_k(t)\right) \right\} ds \right. \\ & \left. + \sum_{j=1}^{n} \alpha_{kj} I_j^* \int_0^{+\infty} L_{kj}(s) \left\{ g\left(y_j(t)\right) - g\left(y_k(t)\right) \right\} ds \right]. \end{aligned} \tag{4.31}$$

Now we are in the position to prove the following inequality.

$$g\left(\tilde{z}_{kj}(t)\right) \leq g\left(y_j(t)\right). \tag{4.32}$$

In fact, if $\tilde{z}_{kj}(t) \geq 1$, then it follows from (4.20) and (iii) of Assumption 1.2 that $I_j(t) \geq I_j^*$. Then $y_j(t) = I_j(t)/I_j^* \geq 1$ and hence, to prove (4.32), it suffices to show that $\tilde{z}_{kj}(t) \leq y_j(t)$ (note that $g(x)$ is monotone increasing for $x \geq 1$). In fact,

$$\begin{aligned} \tilde{z}_{kj}(t) - y_j(t) \quad &= \quad \frac{f_{kj}\left(S_k^0 - I_k^*, I_j(t)\right)}{f_{kj}\left(S_k^0 - I_k^*, I_j^*\right)} - \frac{I_j(t)}{I_j^*} \\ &= \quad \frac{f_{kj}\left(S_k^0 - I_k^*, I_j(t)\right)}{f_{kj}\left(S_k^0 - I_k^*, I_j^*\right)} \left\{ 1 - \frac{f_{kj}\left(S_k^0 - I_k^*, I_j^*\right)}{I_j^*} \Big/ \frac{f_{kj}\left(S_k^0 - I_k^*, I_j(t)\right)}{I_j(t)} \right\} \ \leq \ 0 \end{aligned}$$

follows from (iv) of Assumption 1.2 and $I_j(t) \geq I_j^*$. Thus, $g\left(\tilde{z}_{kj}(t)\right) \leq g\left(y_j(t)\right)$ is shown.

If $\tilde{z}_{kj}(t) \leq 1$, then it follow again from (4.20) and (iii) of Assumption 1.2 that $I_j(t) \leq I_j^*$. Then, $y_j(t) = I_j(t)/I_j^* \leq 1$ and hence, to prove (4.32), it suffices to show that $\tilde{z}_{kj}(t) \geq y_j(t)$ (note that $g(x)$ is monotone decreasing for $x \leq 1$). In fact,

$$\begin{aligned} \tilde{z}_{kj}(t) - y_j(t) \quad &= \quad \frac{f_{kj}\left(S_k^0 - I_k^*, I_j(t)\right)}{f_{kj}\left(S_k^0 - I_k^*, I_j^*\right)} - \frac{I_j(t)}{I_j^*} \\ &= \quad \frac{f_{kj}\left(S_k^0 - I_k^*, I_j(t)\right)}{f_{kj}\left(S_k^0 - I_k^*, I_j^*\right)} \left\{ 1 - \frac{f_{kj}\left(S_k^0 - I_k^*, I_j^*\right)}{I_j^*} \Big/ \frac{f_{kj}\left(S_k^0 - I_k^*, I_j(t)\right)}{I_j(t)} \right\} \ \geq \ 0 \end{aligned}$$

follows from (iv) of Assumption 1.2 and $I_j(t) \le I_j^*$. Consequently, (4.32) is proved.

Using (4.31), (4.32) and (4.17) in Lemma 4.3, we can evaluate the derivative of Lyapunov functional (4.21) as follows.

$$\begin{aligned} U'(t) \le\ & \sum_{k=1}^n v_k \left[\sum_{j=1}^n \beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*) \int_0^{+\infty} K_{kj}(s) \left\{ g\left(y_j(t)\right) - g\left(y_k(t)\right) \right\} ds \right. \\ & \left. + \sum_{j=1}^n \alpha_{kj} I_j^* \int_0^{+\infty} L_{kj}(s) \left\{ g\left(y_j(t)\right) - g\left(y_k(t)\right) \right\} ds \right] \\ =\ & \sum_{k=1}^n v_k \sum_{j=1}^n \left\{ \beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*) + \alpha_{kj} I_j^* \right\} \left\{ g\left(y_j(t)\right) - g\left(y_k(t)\right) \right\} \\ =\ & \sum_{k=1}^n v_k \sum_{j=1}^n \left\{ \frac{\beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*)}{I_j^*} + \alpha_{kj} \right\} I_j^* \left\{ g\left(y_j(t)\right) - g\left(y_k(t)\right) \right\} \\ =\ & \sum_{k=1}^n v_k \sum_{j=1}^n \left\{ \frac{\beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*)}{I_j^*} + \alpha_{kj} \right\} I_j^* g\left(y_j(t)\right) \\ & - \sum_{k=1}^n v_k \sum_{j=1}^n \left\{ \frac{\beta_{kj} f_{kj}(S_k^0 - I_k^*, I_j^*)}{I_j^*} + \alpha_{kj} \right\} I_j^* g\left(y_k(t)\right) \\ =\ & \sum_{k=1}^n \sum_{j=1}^n v_j \left\{ \frac{\beta_{jk} f_{jk}(S_j^0 - I_j^*, I_k^*)}{I_k^*} + \alpha_{jk} \right\} I_k^* g\left(y_k(t)\right) \\ & - \sum_{k=1}^n v_k \left(\mu_k + \gamma_k + \tilde{\alpha}_{kk} + \alpha_{kk}\right) I_k^* g\left(y_k(t)\right) \\ =\ & \sum_{k=1}^n \left\{ \sum_{j=1}^n v_j \left\{ \frac{\beta_{jk} f_{jk}(S_j^0 - I_j^*, I_k^*)}{I_k^*} + \alpha_{jk} \right\} - v_k \left(\mu_k + \gamma_k + \tilde{\alpha}_{kk} + \alpha_{kk}\right) \right\} I_k^* g\left(y_k(t)\right) \\ =\ & 0. \end{aligned} \tag{4.33}$$

It is easy to see that the equality holds only if the equality holds in (4.25), that is, the first term in the right-hand side of (4.27) is equal to zero, that is,

$$\left(1 - \frac{1}{y_k(t)}\right) \left\{ z_{kj}(t,s) - \tilde{z}_{kj}(t-s) \right\} = 0, \quad k, j = 1, 2, \ldots, n.$$

Then, by (4.33) and the above discussion, we see that $\frac{dU(t)}{dt} \le 0$ and from (ii) of Assumption 1.2, this equality holds if and only if

$$S_k(t) = S_k^* \quad \text{and} \quad I_k(t) = I_k^*, \quad k = 1, 2, \ldots, n,$$

Therefore, it follows from the LaSalle's invariance principle that for $\mathcal{R}_0 > 1$, the endemic equilibrium $\mathbf{E}^*$ is globally attractive. Moreover, by Lemmas 2.2 and 2.3 with (4.18) and (1.5) and $\frac{dU}{dt} \le 0$ with (4.21), we can easily prove that there exist positive constants c_3 and c_4 such that c_3 and c_4 do not depend on the initial condition (1.6) and

$$|I_k(t) - I_k^*| \le c_3 W(t) \le c_3 W(0) \le c_3 c_4 \max_{1 \le j \le n} \left| I_j(0) - I_j^* \right|, \quad k = 1, 2, \ldots, n,$$

which implies that $\mathbf{E}^*$ is uniformly stable. Hence, $\mathbf{E}^*$ is globally asymptotically stable in $\mathbf{\Gamma}^0$. This completes the proof. □

5. Numerical simulation

In this section, we perform simple numerical simulation in order to verify the validity of Theorem 1.1. For simplicity, we let $n = 2$ and consider the case of constant time delay in which the kernel functions $K_{kj}(s)$ and $L_{kj}(s)$ are given by the Dirac delta functions. Moreover, we consider the case of saturated incidence in which f_{kj} is given by $f_{kj}(x, y) = xy/(1 + a_{kj}y)$, where $a_{kj} > 0$, $k, j = 1, 2$. In this case, system (1.1) can be rewritten in the following simplified form.

$$\begin{cases} \dfrac{d}{dt}S_k(t) = b_k - \mu_k S_k(t) - \displaystyle\sum_{j=1}^{2} \beta_{kj} S_k(t) \frac{I_j(t-s)}{1 + a_{kj} I_j(t-s)} + \gamma_k I_k(t) + \sum_{j=1}^{2} \Big(\alpha_{kj} S_j(t-s) - \alpha_{jk} S_k(t) \Big), \\ \dfrac{d}{dt}I_k(t) = \displaystyle\sum_{j=1}^{2} \beta_{kj} S_k(t) \frac{I_j(t-s)}{1 + a_{kj} I_j(t-s)} - (\mu_k + \gamma_k) I_k(t) + \sum_{j=1}^{2} \Big(\alpha_{kj} I_j(t-s) - \alpha_{jk} I_k(t) \Big), \quad k = 1, 2, \end{cases} \tag{5.1}$$

where $s > 0$ denotes the constant time delay. For simplicity, we fix the following parameters (they are chosen only for experimental reason).

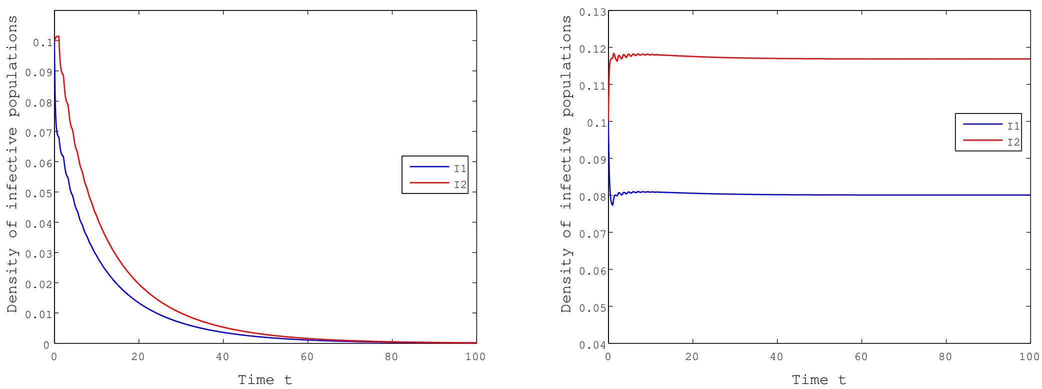
$$\begin{aligned} & a_{11} = 1, \ a_{12} = 2, \ a_{21} = 3, \ a_{22} = 4, \ b_1 = 1, \ b_2 = 2, \ \mu_1 = 1, \ \mu_2 = 2, \\ & \alpha_{11} = 1, \ \alpha_{12} = 2, \ \alpha_{21} = 3, \ \alpha_{22} = 4, \ \gamma_1 = 0.1, \ \gamma_2 = 0.2, \ s = 1, \\ & \varphi_1(\theta) = \varphi_2(\theta) \equiv 0.9, \ \psi_1(\theta) = \psi_2(\theta) \equiv 0.1, \end{aligned}$$

where φ_k and ψ_k, $k = 1, 2$ are initial conditions defined by (1.6).

First we set

$$\beta_{11} = 0.25, \ \beta_{12} = 0.5, \ \beta_{21} = 0.75, \ \beta_{22} = 1.$$

Then, we can calculate $\mathcal{R}_0 \simeq 0.8927 < 1$ and the infective population density converges to zero (see Figure 1 (a)). This corresponds to the situation of Theorem 1.1 (i).



(a) For $R_0 \simeq 0.8927 < 1$ (b) For $R_0 \simeq 1.2020 > 1$

Figure 1: $I_1(t)$ and $I_2(t)$ of system (5.1) versus time t

Next we set

$$\beta_{11} = 0.5, \ \beta_{12} = 1, \ \beta_{21} = 1.5, \ \beta_{22} = 2.$$

Then, we can calculate $\mathcal{R}_0 \simeq 1.2020 > 1$ and the infective population density converges to the positive equilibrium (see Figure 1 (b)). This corresponds to the situation of Theorem 1.1 (ii).

Acknowledgements

The authors would like to thank the editor and the referees for their helpful comments. J. Wang was supported by National Natural Science Foundation of China (Nos. 11401182, 11471089), Science and Technology Innovation Team in Higher Education Institutions of Heilongjiang Province (No. 2014TD005), Overseas studies of Heilongjiang Education Departments (2014), Youth Foundation of Heilongjiang University, and Project funded by China Postdoctoral Science Foundation (No. 2014M552295), Project funded by Chongqing Postdoctoral Foundation (No. Xm2014024). Y. Muroya was supported by Scientifc Research (c), No.15K05010 of Japan Society for the Promotion of Science. T. Kuniya was supported by Grant-in-Aid for Young Scientists (B), No.15K17585 of Japan Society for the Promotion of Science.

References

[1] J. Arino, P. V. D. Driessche, *A multi-city epidemic model*, Math. Popul. Stud., **10** (2003), 175–193. 1

[2] F. V. Atkinson, J. R. Haddock, *On determining phase spaces for functional differential equations*, Funkcial. Ekvac., **31** (1988), 331–347. 1

[3] E. Beretta, Y. Takeuchi, *Global stability of an SIR epidemic model with time delays*, J. Math. Biol., **33** (1995), 250–260. 1, 1, 1

[4] A. Berman, R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York-London (1979). 1.1, 3

[5] F. Brauer, P. V. D. Driessche, *Models for transmission of disease with immigration of infectives*, Math. Biosci., **171** (2001), 143–154. 1

[6] J. Cui, Y. Takeuchi, Z. Lin, *Permanent and extinction for dispersal population systems*, J. Math. Anal. Appl., **298** (2004), 73–93. 1

[7] O. Diekmann, J. A. P. Heesterbeek, J. A. J. Metz, *On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations*, J. Math. Biol., 28 (1990), 365–382. 1

[8] Y. Enatsu, Y. Nakata, Y. Muroya, *Global stability of SIR epidemic models with a wide class of nonlinear incidence rates and distributed delays*, Discrete Contin. Dyn. Syst. B, **15** (2011), 61–74. 1, 1, 1

[9] T. Faria, *Global dynamics for Lotka-Volterra systems with infinite delay and patch structure*, Appl. Math. Comput., **245** (2014), 575-590. 1.1

[10] M. Fiedler, *Special matrices and their applications in numerical mathematics*, Martinus Nijhoff Publishers, Dordrecht (1986). 1.1

[11] H. Guo, M. Y. Li, Z. S. Shuai, *Global stability of the endemic equilibrium of multigroup SIR epidemic models*, Can. Appl. Math. Quart., **14** (2006), 259–284. 1, 4.3

[12] H. W. Hethcote, *Qualitative analyses of communicable disease models*, Math Biosci., **28** (1976), 335–356. 1

[13] Y. Hino, S. Murakami, T. Naito, *Functional Differential Equations with Infinite Delay*, Springer, Berlin, (1991). 1

[14] Y. Jin, W. Wang, *The effect of population dispersal on the spread of a disease*, J. Math. Anal. Appl., **308** (2005), 343–364. 1

[15] W. O. Kermack, A. G. McKendrick, *Contributions to the mathematical theory of epidemics I*, Bull. Math. Bio., **53** (1991), 33–55. 1

[16] T. Kuniya, Y. Muroya, *Global stability of a multi-group SIS epidemic model for population migration*, Disc. Cont. Dyn. Syst. B, **19** (2014), 1105–1118. 1, 1, 4.3

[17] T. Kuniya, Y. Muroya, *Global stability of a multi-group SIS epidemic model with varying total population size*, Appl. Math. Comput., accepted. 1

[18] A. Lajmanovich, J. A. Yorke, *A deterministic model for gonorrhea in a nonhomogeneous population*, Math. Biosci., **28** (1976), 221–236. 1

[19] J. P. LaSalle, *The stability of dynamical systems*, SIAM, Philadelphia, (1976). 3

[20] S. Levin, *Dispersion and population interactions*, Amer. Nat., **108** (1974), 207–228. 1

[21] M. Y. Li, Z. S. Shuai, *Global stability of an epidemic model in a patchy environment*, Can. Appl. Math. Q, **17** (2009), 175–187. 1

[22] M. Y. Li, Z. S. Shuai, *Global-stability problem for coupled systems of differential equations on networks*, J. Diff. Equ., **284** (2010), 1–20. 1

[23] Y. Muroya, *Practical monotonous iterations for nonlinear equations*, Mem. Fac. Sci. Kyushu Univ. A, **22** (1968), 56–73. 1, 4.2

[24] Y. Muroya, Y. Enatsu, T. Kuniya, *Global stability of extended multi-group SIR epidemic models with patches through migration and cross patch infection*, Acta Math. Sci., **33** (2013), 341–361. 1

[25] Y. Muroya, H. Li, T. Kuniya, *Complete global analysis of an SIRS epidemic model with graded cure rate and incomplete recovery rate*, J. Math. Anal. Appl., **410** (2014), 719–732. 1

[26] Y. Nakata, G. Röst, *Global analysis for spread of infectious diseases via transportation networks*, J. Math. Biol., **70** (2015), 1411–1456. 1

[27] J. Ortega, W. Rheinboldt, *Monotone iterations for nonlinear equations with application to Gauss-Seidel methods*, SIAM J. Numer. Anal., **4** (1967), 171–190. 1, 4.2

[28] H. Shu, D. Fan, J. Wei, *Global stability of multi-group SEIR epidemic models with distributed delays and nonlinear transmission*, Nonlinear Anal. Rral World Appl., **13** (2012), 1581–1592. 1

[29] R. Sun, *Global stability of the endemic equilibrium of multigroup SIR models with nonlinear incidence*, Comput. Math. Appl., **60** (2010), 2286–2291. 1

[30] Y. Takeuchi, *Cooperative system theory and global stability of diffusion models*, Acta. Appl. Math., **14** (1989), 49–57. 1

[31] P. V. D. Driessche, J. Watmough, *Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission*, Math. Biosci., **180** (2002), 29–48. 1

[32] J. Wang, Y. Takeuchi, S. Liu, *A multi-group SVEIR epidemic model with distributed delay and vaccination*, Int. J. Biomath., **2012** (2012), 18 pages. 1

[33] J. Wang, J. Zu, X. Liu, G. Huang, J. Zhang, *Global dynamics of a multi-group epidemic model with general relapse distribution and nonlinear incidence rate*, J. Biol. Syst., **20** (2012), 235–258. 1

[34] W. Wang, L. Chen, Z. Lu, *Global stability of a population dispersal in a two-patch environment*, Dynam. Systems Appl., **6** (1997), 207–216. 1

[35] J. Wang, X. Liu, J. Pang, D. Hou, *Global dynamics of a multi-group epidemic model with general exposed distribution and relapse*, Osaka J. Math., **52** (2015), 117–138. 1

[36] W. Wang, X. Q. Zhao, *An epidemic model in a patchy environment*, Math. Biosci., **190** (2004), 97–112. 1

[37] R. Xu, Z. Ma, *Global stability of a SIR epidemic model with nonlinear incidence rate and time delay*, Nonlinear Anal. Real World Appl., **10** (2009), 3175–3189. 1

[38] Z. Yuan, L. Wang, *Global stability of epidemiological models with group mixing and nonlinear incidence rates*, Nonlinear Anal. Real World Appl., **11** (2010), 995–1004. 1

[39] Z. Yuan, X. Zou, *Global threshold property in an epidemic model for disease with latency spreading in a heterogeneous host population*, Nonlinear Anal. Real World Appl., **11** (2010), 3479–3490. 1

[40] X. Q. Zhao, Z. J. Jing, *Global asymptotic behavior in some cooperative systems of functional differential equations*, Canad. Appl. Math. Quart., **4** (1996), 421–444. 4.2